

Asymptotic Distribution of an Estimator for Variance Function of a Compound Periodic Poisson Process with Power Function Trend

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ABSTRACT

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In this paper, an asymptotic distribution of the estimator for the variance function of a compound periodic Poisson process with power function trend is discussed. The periodic component of this intensity function is not assumed to have a certain parametric form, except it is a periodic function with known period. The slope of power function trend is assumed to be positive, but its value is unknown. The objectives of this research are to modify the existing variance function estimator and to determine its asymptotic distribution. This research begins by modifying the formulation of the variance function estimator. After the variance function is obtained, the research is continued by determining the asymptotic distribution of the variance function estimator of the compound periodic Poisson process with a power function trend. The first result is modification of existing estimator so that its asymptotic distribution can be determined. The main result is asymptotic normality of the estimator of variance function of a compound periodic Poisson process with power function trend.



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A. INTRODUCTION

In this paper, we discuss asymptotic distribution of an estimator for the variance function of a compound periodic Poisson process with power function trend. The periodic Poisson process is a nonhomogeneous Poisson process whose intensity function is a periodic function (Mangku, 2001). Some fields that apply the periodic Poisson process include the fields of health and medicine (Lewis, 1971), finance (Engle, 2000), and communication (Belitser et al., 2015). A compound Poisson process is a summation of sequence of independent and identically distributed (i.i.d.) random variables having a certain distribution, where the number of variables is a Poisson random variable (Mangku, 2017). Several applications of the compound Poisson process include in the field of physics, namely the application of the theory of the development of electron avalanches (Byrne, 1969), demography, namely the application of death reporting (Kegler, 2007), seismology, which is the application to the sequence of aftershocks (Özel & Inal, 2008), finance and insurance, namely the application of risk (Bening & Korolev, 2002), as well as biology i.e. modeling the number of dicentrics (Puig & Barquinero, 2011).

The scope of the compound inhomogeneous Poisson process is very broad, so the study begins with a special form of the inhomogeneous Poisson process, namely the compound periodic Poisson process. A compound periodic Poisson process is a process in which the Poisson component has a periodic intensity function. Research on estimating the expected value function and variance in a compound periodic Poisson process is reported in (Ruhayat et al., 2013), (Mangku & Purnaba, 2014) and (Makhmudah et al., 2016). After that the study was expanded by considering the presence of trends, in the compound periodic Poisson process has a trend function. Research on estimating the expected value function and variance function in a compound periodic Poisson process with a linear function trend has been reported in (Wibowo et al., 2017) and (Abdullah et al., 2017). Several related studies can be found in (Mangku et al., 2016) and (Prasetya et al., 2017), and estimators of the expected value function and variance function on a compound periodic Poisson process with a power function trend have been reported in (Sari et al., 2016) and (Fajri, 2018). Furthermore, to find the asymptotic distribution of the compound periodic Poisson process variance function estimator with a power function trend, it is necessary to modify the variance function estimator to make it easier to find the distribution.

Research on the asymptotic distribution of estimators of the expected value function and variance function has been reported by several researchers (Fitria, 2017), (Adriani, 2019) and (Safitri, 2022). As for this research, a study was conducted to determine the asymptotic distribution of the variance function estimator in a compound periodic Poisson process with a power function trend. Specifically, this research was conducted to modify the multiple periodic Poisson process variance function estimator with a power function trend, and to find the asymptotic distribution of the compound periodic Poisson process variance function estimator with a power function trend.

B. METHODS

The research method used is the development of the theory to study the variance function estimator of the compound periodic Poisson process with a power function trend and its asymptotic distribution. The steps of the research carried out are as follows:

1. Preliminary research
 - a. Investigating the compound periodic Poisson process as well as the compound periodic Poisson process with the power function trend.
 - b. Investigating mathematical foundations in order to create new models based on existing theories.
2. Main study of research
 - a. Modify the variance function estimator of a compound periodic Poisson process with a power function trend.
 - b. Formulating the asymptotic distribution of the estimator for variance function of compound periodic Poisson process with the power function trend. The stages carried out in this study are described in a flow chart, as shown in Figure 1.

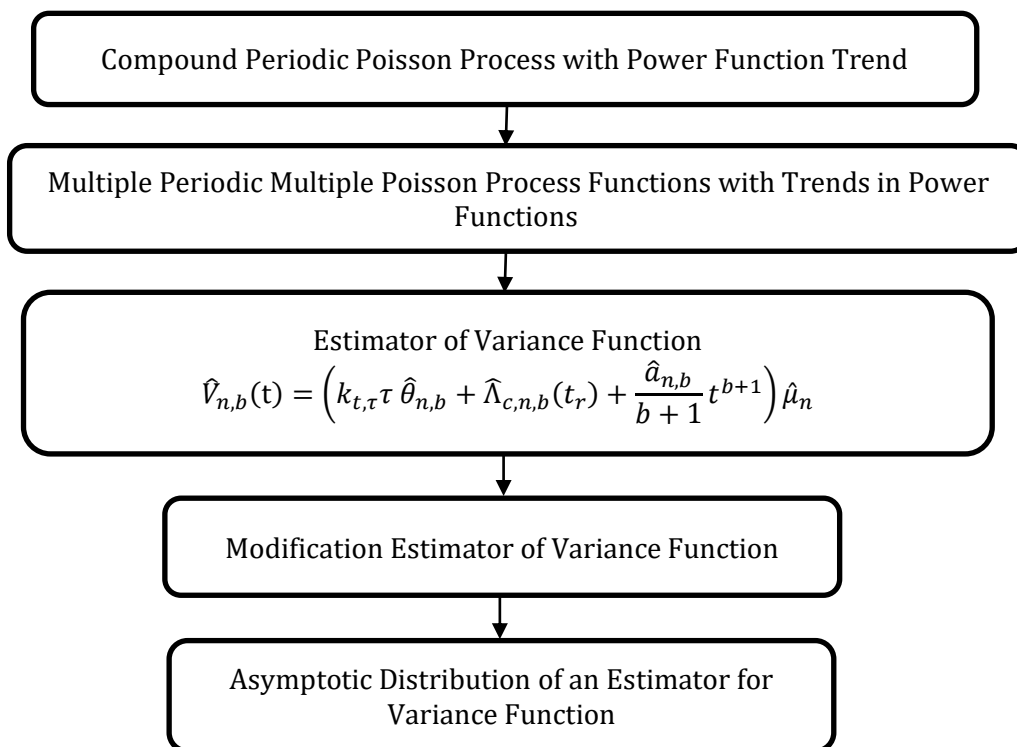


Figure 1. Research Steps

Let $\{N(t), t \geq 0\}$ be a nonhomogeneous Poisson process with an intensity function λ that is locally integrable and unknown. The intensity function λ is assumed to have two components, namely a periodic component λ_c , with a known period $\tau > 0$ and a trend component which is a power function. In other words, for each $s \geq 0$, the intensity function can be written as follows

$$\lambda(s) = \lambda_c(s) + as^b \tag{1}$$

where $\lambda_c(s)$ is a periodic function with period τ and $a > 0$. It is assumed that the value of b is known and $0 < b < 1$. The intensity function λ_c is not assumed to have any parametric form except that it is a periodic function, that is, a function that satisfies the following equation

$$\lambda_c(s) = \lambda_c(s + k\tau) \tag{2}$$

for each $s \geq 0$ and $k \in \mathbb{N}$, where \mathbb{N} represents the set of natural numbers. Let $\{Y(t), t \geq 0\}$ be a process with

$$Y(t) = \sum_{i=1}^{N(t)} X_i \tag{3}$$

where $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d) random variables with expected value $\mu < \infty$ and variance $\sigma^2 < \infty$, and is independent of $\{N(t), t \geq 0\}$. The process $\{Y(t), t \geq 0\}$ is called a compound periodic Poisson process with a power function trend. Let $V(t)$ be the notation for the variance function of $Y(t)$. The notation $E(X_1^2) = \mu_2, V(t)$ can be written as follows

$$V(t) = E[N(t)]E[X_1^2] = \Lambda(t)\mu_2 \tag{4}$$

with

$$\Lambda(t) = \int_0^t \lambda(s)ds. \tag{5}$$

Suppose that $t_r = t - \lfloor \frac{t}{\tau} \rfloor \tau$, where for every real number x , $\lfloor x \rfloor$ represents the largest integer that is less than or equal to x , and let also $k_{t,\tau} = \lfloor \frac{t}{\tau} \rfloor$. For any real number $t \geq 0$, t can be expressed as

$$t = k_{t,\tau}\tau + t_r$$

with $0 \leq t_r < \tau$. Let $\theta = \frac{1}{\tau} \int_0^\tau \lambda_c(s) ds$ represent the global intensity function of the periodic component of the process $\{N(t), t \geq 0\}$, and assume that $\theta > 0$. Then $\int_0^\tau \lambda_c(s) ds$ is written as $\int_0^\tau \lambda_c(s) ds = \Lambda_c(t_r) + \Lambda_c^c(t_r)$, where $\Lambda_c(t_r) = \int_0^{t_r} \lambda_c(s) ds$ and $\Lambda_c^c(t_r) = \int_{t_r}^\tau \lambda_c(s) ds$. Then for every given $t \geq 0$ we get

$$\Lambda(t) = (1 + k_{t,\tau})\Lambda_c(t_r) + k_{t,\tau}\Lambda_c^c(t_r) + \frac{a}{b+1}t^{b+1}. \tag{6}$$

Finally, by substituting equation (6) into equation (4), the variance function of $Y(t)$ can be written as

$$V(t) = \left((1 + k_{t,\tau})\Lambda_c(t_r) + k_{t,\tau}\Lambda_c^c(t_r) + \frac{a}{b+1}t^{b+1} \right) \mu_2. \tag{7}$$

C. RESULT AND DISCUSSION

1. Formulation of the Estimator and Asymptotic Normality

Suppose that for some $\omega \in \Omega$, a single realization $N(\omega)$ of the process $\{N(t), t \geq 0\}$ is observed over a finite interval $[0, m]$. Let's also $m = n^{1+\delta}$, where δ is any small positive real number. The realization of the Poisson process at the observation interval $[0, m]$ is used to estimate the slope of the trend (a). Meanwhile, the estimators of other components use the realization of the Poisson process at the interval $[0, n]$. Using the available data set, estimation of the variance function $V(t)$ in equation (7) can be divided into several estimators, namely

- a. Estimation of the global intensity function θ

In Sari (2016), the following estimator was obtained

$$\hat{\theta}_{n,b} = \frac{1-b}{\tau^b n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau, (k+1)\tau])}{k^b} - \hat{a}_{n,b}(1-b)n^b.$$

Then this estimator was modified by Safitri (2022) and formulated as

$$\tilde{\theta}_n = \frac{1-b}{n^{1-b}\tau^b b^2} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau, (k+1)\tau])}{k^b} - \frac{(1+b)(1-b)n^b N([0, n])}{n^{1+b} b^2}. \tag{8}$$

- b. The estimator for a is $\hat{a}_{m,b}$, which is written as

$$\hat{a}_{m,b} = \frac{(b+1)N([0, m])}{m^{b+1}} - \frac{(b+1)\tilde{\theta}_n}{m^b}. \tag{9}$$

This estimator is taken from Safitri (2022).

- c. The estimator for $\Lambda_c(t_r)$ is $\hat{\Lambda}_{c,n,b}(t_r)$. Estimator of $\Lambda_c(t_r)$ is a modified result of Sari (2016). The estimator of $\Lambda_c(t_r)$ is defined as follows

$$\hat{\Lambda}_{c,n,b}(t_r) = \frac{(1-b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau, k\tau + t_r])}{k^b} - \hat{a}_{m,b}(1-b)n^b t_r \tag{10}$$

with $\hat{a}_{m,b}$ is given by (9).

- d. The estimator for $\Lambda_c^c(t_r)$ is $\hat{\Lambda}_{c,n,b}^c(t_r)$ which is formulated as

$$\hat{\Lambda}_{c,n,b}^c(t_r) = \frac{(1-b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau+t_r, k\tau+\tau])}{k^b} - \hat{a}_{m,b}(1-b)n^b(\tau - t_r). \tag{11}$$

e. The estimator for μ_2 is $\hat{\mu}_{2,n}$. This estimator has been studied in Makhmudah (2016) and is formulated as follows

$$\hat{\mu}_{2,n} = \frac{1}{N([0, n])} \sum_{i=1}^{N([0,n])} X_i^2 \tag{12}$$

if $N([0, n]) > 0$ and $\hat{\mu}_{2,n} = 0$ if $N([0, n]) = 0$.

By using the formulas of the estimators above, an estimator for the variance function of a compound periodic Poisson process with a power function trend can be formulated as follows

$$\hat{V}_{n,b}(t) = \left((1 + k_{t,\tau})\hat{\Lambda}_{c,n,b}(t_r) + k_{t,\tau}\hat{\Lambda}_{c,n,b}^c(t_r) + \frac{\hat{a}_{m,b}}{b+1}t^{b+1} \right) \hat{\mu}_{2,n} \tag{13}$$

and $\hat{V}_{n,b}(t) = 0$, if $N([0, n]) = 0$.

The asymptotic normality of the estimator for the variance function in a compound periodic Poisson process with a power function trend is summarized in the following theorem.

Theorem 1 (Asymptotic Distribution)

Suppose that the intensity function λ satisfies equation (1) and is locally integrable. If $Y(t)$ satisfies equation (3) and estimator $Y(t)$ satisfies equation (13), then

$$\begin{aligned} &\sqrt{n^{1-b}} \left(\hat{V}_{n,b}(t) - V(t) \right) \\ &\xrightarrow{d} Normal \left(0, (1 + k_{t,\tau})^2 a \tau t_r (1 - b) \mu_2^2 + k_{t,\tau}^2 a \tau (1 - b) (\tau - t_r) \mu_2^2 \right) \end{aligned} \tag{14}$$

as $n \rightarrow \infty$.

In the process of proving Theorem 1, several technical lemmas are needed to prove the asymptotic distribution of the estimator for variance function of compound periodic Poisson process with power function trend.

2. Some Technical Lemmas

To prove the asymptotic distribution of the estimator for variance function of a compound periodic Poisson process, we need the following lemmas: First, consider the case that a is known, so that for $t \geq 0$, the estimator in equation (6) can be written as

$$\tilde{\Lambda}_n(t) = (1 + k_{t,\tau})\tilde{\Lambda}_{c,n,b}(t_r) + k_{t,\tau}\tilde{\Lambda}_{c,n,b}^c(t_r) + \frac{a}{b+1}t^{b+1}. \tag{15}$$

By using equation (15), the variance function estimator of $Y(t)$ for case a is known can be written as

$$\tilde{V}_{n,b}(t) = \left((1 + k_{t,\tau})\tilde{\Lambda}_{c,n,b}(t_r) + k_{t,\tau}\tilde{\Lambda}_{c,n,b}^c(t_r) + \frac{a}{b+1}t^{b+1} \right) \hat{\mu}_{2,n} \tag{16}$$

with

$$\begin{aligned} \tilde{\Lambda}_{c,n,b}(t_r) &= \frac{(1-b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau, k\tau + t_r])}{k^b} - a(1-b)n^b t_r \\ \tilde{\Lambda}_{c,n,b}^c(t_r) &= \frac{(1-b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau + t_r, k\tau + \tau])}{k^b} - a(1-b)n^b (\tau - t_r) \\ \hat{\mu}_{2,n} &= \frac{1}{N([0,n])} \sum_{i=1}^{N([0,n])} X_i^2. \end{aligned}$$

Lemma 1

If the intensity function λ satisfies equation (1) and is locally integrable, then for each $t \geq 0$ and $0 < b < \frac{1}{2}$

$$\begin{aligned} \text{Var}(\tilde{\Lambda}_{c,n,b}(t_r)) &= \frac{a\tau t_r(1-b)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \\ \text{Var}(\tilde{\Lambda}_{c,n,b}^c(t_r)) &= \frac{a\tau(1-b)(\tau - t_r)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \end{aligned}$$

as $n \rightarrow \infty$.

Proof:

$$\begin{aligned} \text{Var}(\tilde{\Lambda}_{c,n,b}(t_r)) &= \frac{(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \sum_{k=1}^{k_{n,\tau}} \left(\frac{1}{k^b}\right)^2 \text{Var}(N([k\tau, k\tau + t_r])) \\ &= \frac{(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \left(\sum_{k=1}^{k_{n,\tau}} \frac{1}{k^{2b}}\right) E(N([k\tau, k\tau + t_r])) \\ &= \frac{(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^{2b}} \int_{k\tau}^{k\tau+t_r} (\lambda_c(s) + as^b) ds \\ &= \frac{(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^{2b}} \int_0^{t_r} \lambda_c(s + k\tau) ds + \frac{(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^{2b}} \int_0^{t_r} (a(s + k\tau)^b) ds. \end{aligned}$$

Note that, for $0 < b < 1/2$,

$$\sum_{k=1}^{k_{n,\tau}} \frac{1}{k^{2b}} = \frac{n^{1-2b}}{(1-2b)\tau^{1-2b}} + \mathcal{O}(1)$$

as $n \rightarrow \infty$. Because $(s + k\tau)^b = (k\tau)^b + \mathcal{O}(1)$ as $n \rightarrow \infty$,

$\text{Var}(\tilde{\Lambda}_{c,n,b}(t_r))$ can be written as

$$\begin{aligned} &\frac{(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \left(\frac{n^{1-2b}}{(1-2b)\tau^{1-2b}} + \mathcal{O}(1)\right) \int_0^{t_r} \lambda_c(s + k\tau) ds \\ &\quad + \frac{a(1-b)^2 \tau^{2-2b}}{n^{2-2b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^{2b}} \int_0^{t_r} (k\tau)^b + \mathcal{O}(1) ds \\ &= \frac{(1-b)^2 \tau}{n(1-2b)} \int_0^{t_r} \lambda_c(s + k\tau) ds + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) + \frac{a t_r (1-b)^2 \tau^{2-b}}{n^{2-2b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^b} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \\ &= \frac{\Lambda(t_r)(1-b)^2 \tau}{n(1-2b)} + \frac{a t_r (1-b)^2 \tau^{2-b}}{n^{2-2b}} \left(\frac{n^{1-b}}{(1-b)\tau^{1-b}} + \mathcal{O}(1)\right) + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \\ &= \frac{\Lambda(t_r)(1-b)^2 \tau}{n(1-2b)} + \frac{a \tau t_r (1-b)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \\ &= \frac{a \tau t_r (1-b)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \end{aligned}$$

as $n \rightarrow \infty$.

The following result can also be obtained in the same manner

$$\text{Var}(\tilde{\Lambda}_{c,n,b}^c(t_r)) = \frac{a\tau(1-b)(\tau - t_r)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right)$$

as $n \rightarrow \infty$. The proof for Lemma 1 is complete. ■

Based on Lemma 1, we have

$$\text{Var}(\tilde{\Lambda}_n(t)) = (1 + k_{t,\tau}^2) \text{Var}(\tilde{\Lambda}_{c,n,b}(t_r)) + k_{t,\tau}^2 \text{Var}(\tilde{\Lambda}_{c,n,b}^c(t_r))$$

$$\begin{aligned}
 &= (1 + k_{t,\tau})^2 \left(\frac{\alpha\tau t_r(1-b)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \right) + k_{t,\tau}^2 \left(\frac{\alpha\tau(1-b)(\tau - t_r)}{n^{1-b}} + \mathcal{O}\left(\frac{1}{n^{2-2b}}\right) \right) \\
 &= (1 + k_{t,\tau})^2 \left(\frac{\alpha\tau t_r(1-b)}{n^{1-b}} \right) + k_{t,\tau}^2 \left(\frac{\alpha\tau(1-b)(\tau - t_r)}{n^{1-b}} \right)
 \end{aligned}$$

as $n \rightarrow \infty$.

Lemma 2

If the intensity function λ satisfies equation (1) and is locally integrable, then for each $t \geq 0$, it holds that

$$\begin{aligned}
 &\sqrt{n^{1-b}} \left(\tilde{\Lambda}_n(t) - \Lambda(t) \right) \\
 &\xrightarrow{d} \text{Normal} \left(0, (1 + k_{t,\tau})^2 \alpha\tau t_r(1-b) + k_{t,\tau}^2 \alpha\tau(1-b)(\tau - t_r) \right)
 \end{aligned} \tag{17}$$

as $n \rightarrow \infty$. Proof of this lemma can be found in Safitri (2022).

Lemma 3

If the intensity function λ satisfies equation (1) and is locally integrable, then for $0 < b < \frac{1}{2}$, we have

$$E(\tilde{\theta}_n) = \theta + \mathcal{O}\left(\frac{1}{n^{1-b}}\right) \tag{18}$$

as $n \rightarrow \infty$. Proof of this lemma can be found in Safitri (2022).

Lemma 4

If the intensity function λ satisfies equation (1) and is locally integrable, then for $0 < b < \frac{1}{2}$, we have

$$\text{Var}(\tilde{\theta}_n) = \mathcal{O}\left(\frac{1}{n^{1-b}}\right) \tag{19}$$

as $n \rightarrow \infty$. Proof of this lemma can be found in Safitri (2022).

Lemma 5

If the intensity function λ satisfies equation (1) and is locally integrable, then for each $a > 0$ and $0 < b < \frac{1}{2}$, it holds that

$$n^{b+1} \left(E(\hat{a}_{m,b} - a)^2 \right) \rightarrow 0 \tag{20}$$

as $m \rightarrow \infty$.

Proof:

The expected value of $\hat{a}_{m,b}$ can be calculated as follows

$$\begin{aligned}
 E(\hat{a}_{m,b}) &= E\left(\frac{(b+1)N([0, m])}{m^{b+1}}\right) - E\left(\frac{(b+1)\tilde{\theta}_n}{m^b}\right) \\
 &= \frac{(b+1)}{m^{b+1}} E(N([0, m])) - \frac{(b+1)}{m^b} E(\tilde{\theta}_n).
 \end{aligned}$$

Based on Lemma 3, we obtained $E(\tilde{\theta}_n) = \theta + \mathcal{O}\left(\frac{1}{n^{1-b}}\right)$ as $n \rightarrow \infty$, so that

$$\begin{aligned}
 E(\hat{a}_{m,b}) &= \frac{(b+1)}{m^{b+1}} \left(m\theta + \frac{a}{1+b} m^{1+b} + \mathcal{O}(1) \right) - \frac{(b+1)}{m^b} \left(\theta + \mathcal{O}\left(\frac{1}{n^{1-b}}\right) \right) \\
 &= \frac{(b+1)\theta}{m^b} + a + \mathcal{O}\left(\frac{1}{m^{1+b}}\right) - \frac{(b+1)\theta}{m^b} + \mathcal{O}\left(\frac{1}{m^b n^{1-b}}\right).
 \end{aligned}$$

Since $m = n^{1+\delta}$, where δ is any small positive real number, the above expected value can be written as follows

$$\begin{aligned}
 E(\hat{a}_{m,b}) &= \frac{(b+1)\theta}{n^{(1+\delta)b}} + a + \mathcal{O}\left(\frac{1}{n^{(1+\delta)(1+b)}}\right) - \frac{(b+1)\theta}{n^{(1+\delta)b}} + \mathcal{O}\left(\frac{1}{n^{1-b}n^{(1+\delta)b}}\right) \\
 &= a + \mathcal{O}\left(\frac{1}{n^{1-b}n^{(1+\delta)b}}\right) \\
 &= a + \mathcal{O}\left(\frac{1}{n^{1+b\delta}}\right). \tag{21}
 \end{aligned}$$

Finally we have

$$\text{Bias}(\hat{a}_{m,b}) = \mathcal{O}\left(\frac{1}{n^{1+b\delta}}\right), \tag{22}$$

as $n \rightarrow \infty$. Furthermore, the variance of $\hat{a}_{m,b}$ can be calculated as follows

$$\begin{aligned}
 \text{Var}(\hat{a}_{m,b}) &= \text{Var}\left(\frac{(b+1)N([0, m])}{m^{b+1}}\right) + \text{Var}\left(\frac{(b+1)\tilde{\theta}_n}{m^b}\right) \\
 &\quad + 2\text{Cov}\left(\frac{(b+1)N([0, m])}{m^{b+1}}, \frac{(b+1)\tilde{\theta}_n}{m^b}\right) \\
 &= \frac{(b+1)^2}{m^{2b+2}} \text{Var}(N([0, m])) + \frac{(b+1)^2}{m^{2b}} \text{Var}(\tilde{\theta}_n) + 2\text{Cov}\left(\frac{(b+1)N([0, m])}{m^{b+1}}, \frac{(b+1)\tilde{\theta}_n}{m^b}\right).
 \end{aligned}$$

Based on Lemma 4 we obtained $\text{Var}(\tilde{\theta}_n) = \mathcal{O}\left(\frac{1}{n^{1-b}}\right)$ as $n \rightarrow \infty$, so that

$$\begin{aligned}
 \text{Var}(\hat{a}_{m,b}) &= \frac{(b+1)^2}{m^{2b+2}} E(N([0, m])) + \frac{(b+1)^2}{m^{2b}} \left(\mathcal{O}\left(\frac{1}{n^{1-b}}\right) \right) \\
 &\quad + 2\text{Cov}\left(\frac{(b+1)N([0, m])}{m^{b+1}}, \frac{(b+1)\tilde{\theta}_n}{m^b}\right) \\
 &= \frac{(b+1)^2}{m^{2b+2}} \left(m\theta + \frac{a}{1+b} m^{1+b} + \mathcal{O}(1) \right) + \frac{(b+1)^2}{m^{2b}} \left(\mathcal{O}\left(\frac{1}{n^{1-b}}\right) \right) \\
 &\quad + 2\text{Cov}\left(\frac{(b+1)N([0, m])}{m^{b+1}}, \frac{(b+1)\tilde{\theta}_n}{m^b}\right) \\
 &= \left(\frac{(b+1)^2\theta}{m^{2b+1}} + \frac{a(b+1)}{m^{b+1}} + \mathcal{O}\left(\frac{1}{m^{2b+2}}\right) \right) + \left(\mathcal{O}\left(\frac{1}{m^{2b}n^{1-b}}\right) \right) + 2\text{Cov}\left(\frac{(b+1)N([0, m])}{m^{b+1}}, \frac{(b+1)\tilde{\theta}_n}{m^b}\right).
 \end{aligned}$$

Using the Cauchy Schwarz inequality, we get

$$\text{Cov}\left(\frac{(b+1)N([0, m])}{m^{b+1}}, \frac{(b+1)\tilde{\theta}_n}{m^b}\right) \leq \sqrt{\text{Var}\left(\frac{(b+1)N([0, m])}{m^{b+1}}\right) \text{Var}\left(\frac{(b+1)\tilde{\theta}_n}{m^b}\right)}.$$

Since $m = n^{1+\delta}$, where δ is any small positive real number, the above variance can be written as

$$\begin{aligned}
 \text{Var}(\hat{a}_{m,b}) &\leq \left(\mathcal{O}\left(\frac{1}{n^{(1+\delta)(2b+1)}}\right) + \mathcal{O}\left(\frac{1}{n^{(1+\delta)(b+1)}}\right) + \mathcal{O}\left(\frac{1}{n^{(1+\delta)(2b+2)}}\right) \right) + \left(\mathcal{O}\left(\frac{1}{n^{1+b+2\delta b}}\right) \right) \\
 &\quad + 2 \sqrt{\text{Var}\left(\frac{(b+1)N([0, m])}{m^{b+1}}\right) \text{Var}\left(\frac{(b+1)\tilde{\theta}_n}{m^b}\right)}
 \end{aligned}$$

$$= \left(\mathcal{O} \left(\frac{1}{n^{(1+\delta)(b+1)}} \right) \right) + \left(\mathcal{O} \left(\frac{1}{n^{1+b+2\delta b}} \right) \right) + 2 \sqrt{\left(\mathcal{O} \left(\frac{1}{n^{(1+\delta)(b+1)}} \right) \right) \left(\mathcal{O} \left(\frac{1}{n^{1+b+2\delta b}} \right) \right)}.$$

as $n \rightarrow \infty$. Finally, after simplifying, we get

$$\text{Var}(\hat{a}_{n,b}) = \mathcal{O} \left(\frac{1}{n^{1+b+(1+b)\delta}} \right), \tag{23}$$

as $n \rightarrow \infty$.

From equations (22) and (23), the left side of equation (20) can be written as

$$\begin{aligned} n^{b+1} \left(E(\hat{a}_{m,b} - a)^2 \right) &= n^{b+1} \left(\text{Var}(\hat{a}_{n,b}) + \left(\text{Bias}(\hat{a}_{m,b}) \right)^2 \right) \\ &= n^{b+1} \left(\mathcal{O} \left(\frac{1}{n^{1+b+(1+b)\delta}} \right) + \mathcal{O} \left(\frac{1}{n^{2+2b\delta}} \right) \right) \\ &= \mathcal{O} \left(\frac{1}{n^{(1+b)\delta}} \right) \end{aligned}$$

as $n \rightarrow \infty$. Since $\delta > 0$, then

$$\left(\mathcal{O} \left(\frac{1}{n^{(1+b)\delta}} \right) \right) \rightarrow 0$$

as $n \rightarrow \infty$. Hence we have statement (20). Proof of Lemma 5 is complete. ■

Lemma 6

Suppose that the intensity function λ satisfies equation (1) and is locally integrable. If conditions $a > 0$ and $\theta > 0$ are satisfied, then with probability of 1,

$$N([0, n]) \rightarrow \infty \tag{24}$$

as $n \rightarrow \infty$. Proof of this lemma can be found in Fajri (2018).

Lemma 7

If conditions $a > 0$ and $b > 0$ are satisfied, then

$$\frac{N([0, n])}{n^{b+1}} = \frac{a}{b+1} + \sigma_p(1) \tag{25}$$

as $n \rightarrow \infty$.

Proof:

$$E(N([0, n])) = \int_0^n \lambda(s) ds = \int_0^n \lambda_c(s) ds + \int_0^n as^b ds.$$

The second term on the right hand side above can be calculated as follows

$$\int_0^n as^b ds = \frac{a}{b+1} n^{1+b}.$$

The first term can be calculated as follows

$$\begin{aligned} \int_0^n \lambda_c(s) ds &= \int_0^{k_{n,\tau\tau}} \lambda_c(s) ds + \int_{k_{n,\tau\tau}}^n \lambda_c(s) ds \\ &= \int_0^\tau \lambda_c(s) ds + \int_\tau^{2\tau} \lambda_c(s) ds + \dots + \int_{(k_{n,\tau}-1)\tau}^{k_{n,\tau\tau}} \lambda_c(s) ds + \int_{k_{n,\tau\tau}}^n \lambda_c(s) ds \\ &= \left(\int_0^\tau \lambda_c(s) ds + \int_0^\tau \lambda_c(s) ds + \dots + \int_0^\tau \lambda_c(s) ds \right), k_{n,\tau} \text{ kali} + \int_{k_{n,\tau\tau}}^n \lambda_c(s) ds \\ &= k_{n,\tau} \int_0^\tau \lambda_c(s) ds + \int_{k_{n,\tau\tau}}^n \lambda_c(s) ds \end{aligned}$$

$$\begin{aligned}
 &= k_{n,\tau} \tau \frac{1}{\tau} \int_0^\tau \lambda_c(s) ds + \int_{k_{n,\tau} \tau}^n \lambda_c(s) ds \\
 &= k_{n,\tau} \tau \theta + \int_{k_{n,\tau} \tau}^n \lambda_c(s) ds.
 \end{aligned}$$

Because $\lambda_c(s)$ is a locally integrable and $n - k_{n,\tau} \tau = \mathcal{O}(1)$ as $n \rightarrow \infty$, then

$$\int_{k_{n,\tau} \tau}^n \lambda_c(s) ds = \mathcal{O}(1)$$

as $n \rightarrow \infty$. Hence

$$\begin{aligned}
 E(N([0, n])) &= k_{n,\tau} \tau \theta + \frac{a}{b+1} n^{1+b} + \mathcal{O}(1) \\
 &= (n + \mathcal{O}(1)) \theta + \frac{a}{b+1} n^{b+1} + \mathcal{O}(1) \\
 &= n\theta + \frac{a}{b+1} n^{b+1} + \mathcal{O}(1),
 \end{aligned} \tag{26}$$

as $n \rightarrow \infty$. Based on equation (26), it is obtained that

$$\begin{aligned}
 E\left(\frac{N([0, n])}{n^{b+1}}\right) &= \frac{n\theta + \frac{a}{b+1} n^{b+1} + \mathcal{O}(1)}{n^{b+1}} \\
 &= \frac{a}{b+1} + \mathcal{O}\left(\frac{1}{n^b}\right),
 \end{aligned}$$

so that

$$\text{Bias}\left(\frac{N([0, n])}{n^{b+1}}\right) = \mathcal{O}\left(\frac{1}{n^b}\right) \rightarrow 0$$

as $n \rightarrow \infty$. Then

$$\begin{aligned}
 \text{Var}\left(\frac{N([0, n])}{n^{b+1}}\right) &= \frac{1}{n^{2+2b}} \left(n\theta + \frac{a}{b+1} n^{b+1} + \mathcal{O}(1) \right) \\
 &= \mathcal{O}\left(\frac{1}{n^{b+1}}\right) \rightarrow 0
 \end{aligned}$$

as $n \rightarrow \infty$. Proof of Lemma 7 is complete. ■

Lemma 8

If X_1^2, X_2^2, \dots is a sequence of independent random variables having an identical distribution with the expected value $E(X_1^2) < \infty$ and the variance $\text{Var}(X_1^2) = \sigma_2^2 < \infty$, then

$$\sqrt{n^{b+1}}(\hat{\mu}_{2,n} - \mu_2) \xrightarrow{d} \text{Normal}\left(0, \frac{b+1}{a} \sigma_2^2\right) \tag{27}$$

as $n \rightarrow \infty$.

Proof:

The left hand side of equation (27) can be written as

$$\frac{\sigma_2 \sqrt{n^{b+1}}}{\sqrt{N([0, n])}} \frac{\sqrt{N([0, n])}}{\sigma_2} (\hat{\mu}_{2,n} - \mu_2). \tag{28}$$

Noted that

if $X_n \xrightarrow{P} \alpha$ and $Y_n \xrightarrow{P} \beta$, then

$$\frac{X_n}{Y_n} \xrightarrow{P} \frac{\alpha}{\beta} \tag{29}$$

with $Y_n \neq 0$ for all n and $\beta \neq 0$ (Mangku 2017).

From Lemma 7 and equation (29), then

$$\frac{n^{b+1}}{N([0,n])} = \frac{b+1}{a} + \sigma_p(1).$$

Note that

$$\begin{aligned} \frac{\sigma_2 \sqrt{n^{b+1}}}{\sqrt{N([0,n])}} &= \sigma_2 \sqrt{\frac{n^{b+1}}{N([0,n])}} \\ &= \sigma_2 \sqrt{\frac{b+1}{a}} + \sigma_p(1) \end{aligned} \tag{30}$$

as $n \rightarrow \infty$. Equation (28) can be proved by checking

$$\frac{\sqrt{N([0,n])}}{\sigma_2} (\hat{\mu}_{2,n} - \mu_2) \xrightarrow{d} Normal(0,1) \tag{31}$$

as $n \rightarrow \infty$. Equation (31) can be proven as follows:

$$\begin{aligned} \frac{\sqrt{N([0,n])}}{\sigma_2} (\hat{\mu}_{2,n} - \mu_2) &= \frac{\sqrt{N([0,n])}}{\sigma_2} \left(\frac{\sum_{i=1}^{N([0,n])} X_i^2}{N([0,n])} - \mu_2 \right) \\ &= \sqrt{N([0,n])} \left(\frac{\sum_{i=1}^{N([0,n])} X_i^2}{\sigma_2 N([0,n])} - \frac{N([0,n])\mu_2}{\sigma_2 N([0,n])} \right) \\ &= \frac{X_1^2 + X_2^2 + \dots + X_{N([0,n])}^2 - N([0,n])\mu_2}{\sigma_2 \sqrt{N([0,n])}}. \end{aligned} \tag{32}$$

By the Central Limit Theorem and Lemma 6, equation (31) is obtained. Based on equations (30) and (31), equation (27) is obtained. Proof of Lemma 8 is complete. ■

Lemma 9 (Asymptotic Normality of Variation Function Estimator in Case a Known)

Suppose that the intensity function λ satisfies equation (1) and is locally integrable. If $Y(t)$ satisfies equation (3), then

$$\begin{aligned} \sqrt{n^{1-b}} (\tilde{V}_{n,b}(t) - V(t)) \\ \xrightarrow{d} Normal \left(0, (1 + k_{t,\tau})^2 a\tau t_r(1-b)\mu_2^2 + k_{t,\tau}^2 a\tau(1-b)(\tau - t_r)\mu_2^2 \right) \end{aligned} \tag{33}$$

as $n \rightarrow \infty$.

Proof:

The left hand side of equation (33) can be written as follows

$$\begin{aligned} \sqrt{n^{1-b}} (\tilde{V}_{n,b}(t) - V(t)) &= \sqrt{n^{1-b}} (\tilde{\Lambda}_n(t)\hat{\mu}_{2,n} - \Lambda(t)\mu_2) \\ &= \sqrt{n^{1-b}} (\hat{\mu}_{2,n}(\tilde{\Lambda}_n(t) - \Lambda(t)) + \Lambda(t)(\hat{\mu}_{2,n} - \mu_2)) \\ &= \hat{\mu}_{2,n}\sqrt{n^{1-b}}(\tilde{\Lambda}_n(t) - \Lambda(t)) + \Lambda(t)\sqrt{n^{1-b}}(\hat{\mu}_{2,n} - \mu_2). \end{aligned} \tag{34}$$

Consider the first term on the right hand side of equation (34). By the weak law of large numbers, we have $\hat{\mu}_{2,n} \xrightarrow{P} \mu_2$ as $n \rightarrow \infty$. These results are substituted in Lemma 2 and Slutsky's Theorem so that we get

$$\begin{aligned} & \hat{\mu}_{2,n} \sqrt{n^{1-b}} \left(\tilde{\Lambda}_n(t) - \Lambda(t) \right) \\ & \xrightarrow{d} N \left(0, (1 + k_{t,\tau})^2 a \tau t_r (1 - b) \mu_2^2 + k_{t,\tau}^2 a \tau (1 - b) (\tau - t_r) \mu_2^2 \right) \end{aligned} \tag{35}$$

as $n \rightarrow \infty$. Next, consider the second term on the right hand side of equation (34). This term can be written as

$$\begin{aligned} \sqrt{n^{1-b}} (\hat{\mu}_{2,n} - \mu_2) &= \frac{\sqrt{n^{1-b}}}{\sqrt{n^{1+b}}} \sqrt{n^{1+b}} (\hat{\mu}_{2,n} - \mu_2) \\ &= \frac{1}{\sqrt{n^{2b}}} \sqrt{n^{1+b}} (\hat{\mu}_{2,n} - \mu_2). \end{aligned}$$

Since $\frac{1}{\sqrt{n^{2b}}} \rightarrow 0$ as $n \rightarrow \infty$ and based on Lemma 5, we obtain the second term on the right hand side of equation (34) as follows

$$\begin{aligned} \Lambda(t) \sqrt{n^{1-b}} (\hat{\mu}_{2,n} - \mu_2) &= \Lambda(t) \frac{1}{\sqrt{n^{2b}}} \sqrt{n^{1+b}} (\hat{\mu}_{2,n} - \mu_2) \\ &= \sigma_p(1) N \left[0, (\Lambda(t))^2 \frac{\sigma_2^2 (b + 1)}{a} \right] \\ &= \sigma_p(1) \end{aligned} \tag{36}$$

as $n \rightarrow \infty$. Based on statements (35) and (36), the statement (33) was obtained. Proof of Lemma 9 is complete. ■

Lemma 10 (Convergence of the Difference of Variance Function Estimator in Case a Unknown and Case a Known)

Suppose that the intensity function λ satisfies equation (1) and is locally integrable. If $Y(t)$ satisfies equation (3), then

$$\sqrt{n^{1-b}} \left(\hat{V}_{n,b}(t) - \tilde{V}_{n,b}(t) \right) \rightarrow \sigma_p(1) \tag{37}$$

as $n \rightarrow \infty$.

Proof:

$$\begin{aligned} & \sqrt{n^{1-b}} \left(\hat{V}_{n,b}(t) - \tilde{V}_{n,b}(t) \right) \\ &= \sqrt{n^{1-b}} \left(\left((1 + k_{t,\tau}) \hat{\Lambda}_{c,n,b}(t_r) + k_{t,\tau} \hat{\Lambda}_{c,n,b}^c(t_r) + \frac{\hat{a}_{m,b}}{b+1} t^{b+1} \right) \hat{\mu}_{2,n} - \left((1 + k_{t,\tau}) \tilde{\Lambda}_{c,n,b}(t_r) + \right. \right. \\ & \quad \left. \left. k_{t,\tau} \tilde{\Lambda}_{c,n,b}^c(t_r) + \frac{a}{b+1} t^{b+1} \right) \hat{\mu}_{2,n} \right) \\ &= \sqrt{n^{1-b}} \hat{\mu}_{2,n} \left(\left((1 + k_{t,\tau}) \hat{\Lambda}_{c,n,b}(t_r) + k_{t,\tau} \hat{\Lambda}_{c,n,b}^c(t_r) + \frac{\hat{a}_{m,b}}{b+1} t^{b+1} \right) \right. \\ & \quad \left. - \left((1 + k_{t,\tau}) \tilde{\Lambda}_{c,n,b}(t_r) + k_{t,\tau} \tilde{\Lambda}_{c,n,b}^c(t_r) + \frac{a}{b+1} t^{b+1} \right) \right) \\ &= \sqrt{n^{1-b}} (\hat{a}_{m,b} - a) \\ & \quad \times \left((1 + k_{t,\tau}) a \tau t_r (1 - b) + k_{t,\tau} a \tau (1 - b) (\tau - t_r) + \frac{t^{b+1}}{b+1} \right). \end{aligned} \tag{38}$$

Because $\sqrt{n^{1+b}} (\hat{a}_{m,b} - a) = \sigma_p(1)$, equation (38) becomes

$$\sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - \check{V}_{n,b}(t)) = \sigma_p(1)$$

as $n \rightarrow \infty$. So equation (37) is proven. ■

3. Proof of Theorem 1

Note that the left hand side of equation (14) can be written as follows

$$\begin{aligned} &\sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - V(t)) \\ &= \sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - \check{V}_{n,b}(t) + \check{V}_{n,b}(t) - V(t)) \\ &= \sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - \check{V}_{n,b}(t)) + \sqrt{n^{1-b}} (\check{V}_{n,b}(t) - V(t)). \end{aligned} \tag{39}$$

Based on Lemma 9 and Lemma 10, equation (39) becomes

$$\begin{aligned} &\sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - V_{n,b}(t)) \\ &= N \left(0, (1 + k_{t,\tau})^2 a\tau t_r(1 - b)\mu_2^2 + k_{t,\tau}^2 a\tau(1 - b)(\tau - t_r)\mu_2^2 \right) + \sigma_p(1) \end{aligned}$$

as $n \rightarrow \infty$. Alternatively, equation (39) can be written as follows

$$\sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - V(t)) \xrightarrow{d} N \left(0, (1 + k_{t,\tau})^2 a\tau t_r(1 - b)\mu_2^2 + k_{t,\tau}^2 a\tau(1 - b)(\tau - t_r)\mu_2^2 \right)$$

as $n \rightarrow \infty$. Equation (14) is established. Therefore, Theorem 1 is proven.

D. CONCLUSION AND SUGGESTIONS

The asymptotic distribution of the compound periodic Poisson process variance estimator with the power function trend is obtained with the help of technical equations. Based on the research that has been done, the conclusions are as follows: (1) The formula of the estimator for variance function of a compound periodic Poisson process with a power function trend is

$$\hat{V}_{n,b}(t) = \left((1 + k_{t,\tau})\hat{\Lambda}_{c,n,b}(t_r) + k_{t,\tau}\hat{\Lambda}_{c,n,b}^c(t_r) + \frac{\hat{a}_{m,b}}{b + 1} t^{b+1} \right) \hat{\mu}_{2,n}$$

with

$$\begin{aligned} \hat{a}_{m,b} &= \frac{(b + 1)N([0, m])}{m^{b+1}} - \frac{(b + 1)\tilde{\theta}_n}{m^b} \\ \hat{\Lambda}_{c,n,b}(t_r) &= \frac{(1 - b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau, k\tau + t_r])}{k^b} - \hat{a}_{m,b}(1 - b)n^b t_r \\ \hat{\Lambda}_{c,n,b}^c(t_r) &= \frac{(1 - b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{N([k\tau + t_r, k\tau + \tau])}{k^b} - \hat{a}_{m,b}(1 - b)n^b(\tau - t_r) \\ \hat{\mu}_{2,n} &= \frac{1}{N([0, n])} \sum_{i=1}^{N([0, n])} X_i^2 \end{aligned}$$

and (2) The asymptotic distribution for the variance function estimator $\hat{V}_{n,b}(t)$ is formulated as follows:

$$\sqrt{n^{1-b}} (\hat{V}_{n,b}(t) - V(t)) \xrightarrow{d} Normal \left(0, (1 + k_{t,\tau})^2 a\tau t_r(1 - b)\mu_2^2 + k_{t,\tau}^2 a\tau(1 - b)(\tau - t_r)\mu_2^2 \right)$$

as $n \rightarrow \infty$.

The next researcher can use the asymptotic distribution of the estimator of the periodic Poisson process variance function with the power function trend. So that it can find the confidence interval for the compound periodic Poisson process variance function with a power function trend.

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