An η-Intuitionistic Fuzzy Rings Structure

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ABSTRACT

In this article, we present the structure of η-intuitionistic fuzzy ring. An η-intuitionistic fuzzy ring is a structure which is built with combinating the definition of fuzzy ring, intuitionistic fuzzy set, and η-intuitionistic fuzzy set. The η-intuitionistic fuzzy set is characterized by any value $\eta \in [0,1]$, where the degree of membership $\mu_{A^\eta}(k)$ is obtained based on the averaging operator of the degree of membership $\mu_A(k)$ and the value of $\eta \in [0,1]$. While the degree of non membership $\nu_{A^\eta}(k)$ is obtained based on the averaging operator of the degree of non membership $\nu_A(k)$ and the value of $1 - \eta \in [0,1]$. In its development, new concepts were obtained, namely the η-intuitionistic fuzzy ideal and its properties related to the sum and product operation of η-intuitionistic fuzzy ideals. Furthermore, the η-intuitionistic fuzzy ideals concept can be developed into an η-intuitionistic fuzzy quotient ring, η-intuitionistic fuzzy homomorphism, and its properties on the next research.

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A. INTRODUCTION

The development of a set associated with the degree of membership was first introduced where the degree of membership of an element is expressed as a number in closed intervals 0 and 1 (Zadeh, 1965). That theory is developed to the concept of fuzzy sets prompted various studies on modern algebra. One of them is the theory of fuzzy middle coset on cosets and fuzzy normal subgroups (Onasanya & Ilori, 2014). In addition, fuzzy and anti-fuzzy ring theory was built with operators along with ideal properties and fuzzy homomorphism (Alam, 2015). Furthermore, the module theory builds semi-essential submodule fuzzy structures (Abbas & Al-Aeashi, 2012) and a new type of fuzzy module over ring fuzzy (Yetkin & Olgun, 2014).

In 1986, the fuzzy sets developed the fuzzy sets theory into intuitionistic fuzzy sets concept. In this set, each element in addition to being expressed in degrees of membership is also expressed in degrees of non-membership (Atanassov, 1986). Atanassov's theory also encouraged the emergence of new research on intuitionistic fuzzy groups theory, such as the development of intuitionistic fuzzy subgroups in normal multigroups (Adamu et al., 2019). Furthermore, the isomorphism theorems in intuitionistic fuzzy algebra is obtained (Çuvalcıoğlu & Tarsuslu (Yılmaz), 2021).

In the ring concept, the characteristics of an intuitionistic fuzzy subring from an intuitionistic fuzzy ring are introduced and it is proved that the infimum-supremum star family

of characteristic intuitionistic fuzzy sets is a lattice and the sublattice of lattice of intuitionistic fuzzy ideal (Meena, 2017). Then, intuitionistic fuzzy ring equipped with an operator and it is proved that mapping $f: R \rightarrow R/A$ defined by $f(x) = x + A$ is an M-homomorphism from R onto R/A (Yamin & Sharma, 2018). Further, the intuitionistic fuzzy ideal in ring concept is built from the intuitionistic fuzzy subring in a commutative ring and it is obtained in details about algebraic structure of rough intuitionistic fuzzy ideals of intuitionistic fuzzy subrings (Mandal & Ranadive, 2014), and then the properties of intuitionistic fuzzy ideal and prime ideal in a ring is obtained (Bakhadach et al., 2016).

Moreover, several types of ideal algebra are given, including $(\in, \in \vee q)$-intuitionstic fuzzy ideals of BG-algebra and it is proved that the notions (α, β)-fuzzy ideals can define on twelve different types of ideals by three choices of α and four choices of β (Barbhuiya, 2015). Further, it is proved that an intuitionistic fuzzy set A of X is an intuitionistic fuzzy ideal of X with thresholds (s, t) of X if and only if for any $p \in (s, t]$, then A_p is a fuzzy ideal of X (Jana & Pal, 2017), and also every anti-intuitionistic fuzzy soft closed ideal of BCK/BCI-algebra X for all $\alpha \in A$ is an anti-intuitionistic fuzzy soft BCK/BCI-algebra over X for all $\alpha \in A$ (Balamurugan et al., 2019). In modules, any research of algebraic properties of intuitionistic fuzzy soft sets in module structures and its obtained a tensor product of (F, A) and (G, B) is called an intuitionistic fuzzy soft module over $M \otimes N$ (Gunduz & Bayramov, 2011) and intuitionistic fuzzy modules over an intuitionistic fuzzy rings which is proved that an intersection of intuitionistic fuzzy modules over intuitionistic fuzzy rings is also intuitionistic fuzzy module over intuitionistic fuzzy ring (Sharma, 2012).

The development also includes the concept of graphs, namely operations that apply to intuitionistic fuzzy graph structures, it is proved that the every result of operations of intuitionistic fuzzy graph structures is also an intuitionistic fuzzy graph structure (Akram & Akmal, 2016) and introduced the direct product, semi-strong product, and strong product of two intuitionistic fuzzy graphs then investigated the interesting properties of that concept (Sahoo & Pal, 2015). In other that, it explained the rationality of some defined important notions, then shown categorical goodness of intuitionistic fuzzy graphs by proving that intuitionistic fuzzy group is isomorphic-closed, complete, and co-complete (Rashmanlou et al., 2015). An the last, it is obtained three types of product operations of interval valued intuitionistic (S, T)-fuzzy graphs and also introduced regular and totally regular interval valued intuitionistic (S, T)-fuzzy graphs (Rashmanlou et al., 2016).

In 2020, Shuaib et al. introduced a concept of η-intuitionistic fuzzy subgroup depend on any value $\eta \in [0,1]$ that given in intuitionistic fuzzy set and fuzzy subgroup. Then, based on that research, one develop the structure to fuzzy ring and ideal ring concept become η-intuitionistic fuzzy rings and η-intuitionistic fuzzy ideal.

B. METHODS

An η-intuitionistic fuzzy ring concept is combined of the fuzzy ring, intuitionistic fuzzy set, and η-intuitionistic fuzzy set definitions. So is the η-intuitionistic fuzzy ideal that is built from the fuzzy ideal, intuitionistic fuzzy set, and η-intuitionistic fuzzy set concepts. Therefore, one need to be defined first the fuzzy ring and fuzzy ideal as below.

Definition 2.1. Let $(R, +, \cdot)$ is a ring. A set of ordered pairs $\alpha = \{(k, \mu_\alpha(k)) \mid k \in R\}$ is called fuzzy ring (FR) of R if it admits the following axioms:
1. $\mu_\alpha(k_1 - k_2) \geq \min(\mu_\alpha(k_1), \mu_\alpha(k_2))$
2. $\mu_\alpha(k_1 k_2) \geq \min(\mu_\alpha(k_1), \mu_\alpha(k_2))$
   for every $k_1, k_2 \in R$ (Yan, 2008).

Definition 2.2. Let R is a ring. A set of ordered pairs $\alpha = \{(k, \mu_\alpha(k)) \mid k \in R\}$ over R is called fuzzy ideal (FI) of R if it admits the following axioms:
1. $\mu_\alpha(k_1 - k_2) \geq \min(\mu_\alpha(k_1), \mu_\alpha(k_2))$
2. $\mu_\alpha(k_1 k_2) \geq \max(\mu_\alpha(k_1), \mu_\alpha(k_2))$
   for every $k_1, k_2 \in R$ (Basnet, 2019).

Definition 2.3. Let S an empty set. An Intuitionistic fuzzy set (IFS) α over S is defined as
$$\alpha = \{(k, \mu_\alpha(k), \nu_\alpha(k)) \mid k \in S\} \tag{1}$$
which is $\mu_\alpha(k) : S \to [0,1]$ as the degree of membership, $\nu_\alpha(k) : S \to [0,1]$ as the degree of nonmembership of α, and $0 \leq \mu_\alpha(k) + \nu_\alpha(k) \leq 1$ for all $k \in S$ (Atanassov, 1986).

Definition 2.4. Some basic properties and operations on IFS are given as follows (Ejegwa et al., 2014):
1. IFS is called empty if and only if the degree of membership function is 0 in S.
2. IFS α and β are equal, denoted by $\alpha = \beta$, if and only if $\mu_\alpha(k) = \mu_\beta(k)$ and $\nu_\alpha(k) = \nu_\beta(k)$ for all $k \in S$.
3. The union of IFS's is defined as,
$$\alpha \cup \beta = \left\{ \left(k, \max\left(\mu_\alpha(k), \mu_\beta(k) \right), \min\left(\nu_\alpha(k), \nu_\beta(k) \right) \right) \mid k \in S \right\} \tag{2}$$
4. The intersection of IFS's is defined as,
$$\alpha \cap \beta = \left\{ \left(k, \min\left(\mu_\alpha(k), \mu_\beta(k) \right), \max(\nu_\alpha(k), \nu_\alpha(k)) \right) \mid k \in S \right\} \tag{3}$$

Definition 2.5. Let $(R, +, \cdot)$ is a ring. An IFS $\alpha = \{(k, \mu_\alpha(k), \nu_\alpha(k)) \mid k \in R\}$ is called intuitionistic fuzzy ring (IFR) of R if it admits the following axioms:
1. $\mu_\alpha(k_1 + k_2) \geq \min(\mu_\alpha(k_1), \mu_\alpha(k_2))$
2. $\mu_\alpha(k_1 k_2) \geq \min(\mu_\alpha(k_1), \mu_\alpha(k_2))$
3. $\mu_\alpha(-k_1) \geq \mu_\alpha(k_1)$
4. $\nu_\alpha(k_1 + k_2) \leq \max(\nu_\alpha(k_1), \nu_\alpha(k_2))$
5. $\nu_\alpha(k_1 k_2) \leq \max(\nu_\alpha(k_1), \nu_\alpha(k_2))$
6. $\nu_\alpha(-k_1) \leq \nu_\alpha(k_1)$
   for every $k_1, k_2 \in R$ (Yan, 2008).

Remark 2.6. On the next section, ring $(R, +, \cdot)$ will be denoted by R.

Theorem 2.7. Let α and β are IFR of R, then $\alpha \cap \beta$ is also an IFR.

Proof. Based on (3) in Definition 2.2., one have for any $k_1, k_2 \in R$:

1. $\mu_{\alpha\cap\beta}(k_1 + k_2) = \min\left(\mu_\alpha(k_1 + k_2), \mu_\beta(k_1 + k_2)\right)$

$$\geq \min\left(\min(\mu_\alpha(k_1), \mu_\alpha(k_2)), \min\left(\mu_\beta(k_1), \mu_\beta(k_2)\right)\right)$$

$$= \min\left(\min\left(\mu_\alpha(k_1), \mu_\beta(k_1)\right), \min\left(\mu_\alpha(k_2), \mu_\beta(k_2)\right)\right)$$

$$= \min\left(\mu_{\alpha\cap\beta}(k_1), \mu_{\alpha\cap\beta}(k_2)\right)$$

2. $\mu_{\alpha\cap\beta}(k_1 k_2) = \min\left(\mu_\alpha(k_1 k_2), \mu_\beta(k_1 k_2)\right)$

$$\geq \min\left(\min(\mu_\alpha(k_1), \mu_\alpha(k_2)), \min\left(\mu_\beta(k_1), \mu_\beta(k_2)\right)\right)$$

$$= \min\left(\min\left(\mu_\alpha(k_1), \mu_\beta(k_1)\right), \min\left(\mu_\alpha(k_2), \mu_\beta(k_2)\right)\right)$$

$$= \min\left(\mu_{\alpha\cap\beta}(k_1), \mu_{\alpha\cap\beta}(k_2)\right)$$

Similarly, one can prove that,

$$\nu_{\alpha\cap\beta}(k_1 + k_2) \leq \max\left(\nu_{\alpha\cap\beta}(k_1), \nu_{\alpha\cap\beta}(k_2)\right)$$

and

$$\nu_{\alpha\cap\beta}(k_1 k_2) \leq \max\left(\nu_{\alpha\cap\beta}(k_1), \nu_{\alpha\cap\beta}(k_2)\right).$$

Moreover,

3. $\mu_{\alpha\cap\beta}(-k_1) = \min\left(\mu_\alpha(-k_1), \mu_\beta(-k_1)\right)$

$$\geq \min\left(\mu_\alpha(k_1), \mu_\beta(k_1)\right)$$

$$= \mu_{\alpha\cap\beta}(k_1)$$

Similarly, one can prove that

$$\nu_{\alpha\cap\beta}(-k_1) \leq \nu_{\alpha\cap\beta}(k_1).$$

Then, $\alpha \cap \beta$ is also an IFR is proved.

Lemma 2.8. The union of IFR's is also an IFR.
Proof. Based on (2) in Definition 2.4., that will be used in the same method as the Theorem 2.7.
Next, the definition of intuitionistic fuzzy ideal is presented.
Definition 2.9. Let R is a ring. An IFS α over R is called intuitionistic fuzzy ideal (IFI) of R if it admits the following axioms:

1. $\mu_\alpha(k_1 - k_2) \geq \min(\mu_\alpha(k_1), \mu_\alpha(k_2))$
2. $\mu_\alpha(k_1 k_2) \geq \max(\mu_\alpha(k_1), \mu_\alpha(k_2))$
3. $\nu_\alpha(k_1 - k_2) \leq \max(\nu_\alpha(k_1), \nu_\alpha(k_2))$
4. $\nu_\alpha(k_1 k_2) \leq \min(\nu_\alpha(k_1), \nu_\alpha(k_2))$

   for every $k_1, k_2 \in R$ (Basnet, 2019).

Definition 2.10. Let α and β are IFI's over R. The sum of α and β is defined as

$$\alpha + \beta = \left\{\left(k, (\mu_{\alpha+\beta})(k), (\nu_{\alpha+\beta})(k)\right) \mid k \in R\right\} \tag{5}$$

where

$$(\mu_{\alpha+\beta})(k) = \sup_{k=m+n}\left(\min\left(\mu_\alpha(m), \mu_\beta(n)\right)\right), \quad \forall k \in R \tag{6}$$

and

$$(v_{\alpha+\beta})(k) = \inf_{k=m+n} \left(\max\left(v_\alpha(m), v_\beta(n) \right) \right), \quad \forall k \in R \tag{7}$$

(Basnet, 2019).

Definition 2.11. Let α and β are IFI's over R. The product of α and β is defined as

$$\alpha\beta = \left\{ \left(k, (\mu_{\alpha\beta})(k), (v_{\alpha\beta})(k) \right) \mid k \in R \right\} \tag{8}$$

where

$$(\mu_{\alpha\beta})(k) = \sup_{k=\sum_{i<\infty} m_i n_i} \left(\min\left(\min\left(\mu_\alpha(m_i), \mu_\beta(n_i) \right) \right) \right) \tag{9}$$

and

$$(v_{\alpha\beta})(k) = \inf_{k=\sum_{i<\infty} m_i n_i} \left(\max\left(\max\left(\mu_\alpha(m_i), \mu_\beta(n_i) \right) \right) \right) \tag{10}$$

(Basnet, 2019).

    Then, one present the averaging operator to obtain the degree of membership and non membership of η-intuitionistic fuzzy set first as below.

Definition 2.12. Let $\alpha = \left\{ \left(k, \mu_\alpha(k), v_\alpha(k) \right) \mid k \in S \right\}$ and $\beta = \left\{ \left(k, \mu_\beta(k), v_\beta(k) \right) \mid k \in S \right\}$ are IFS's of S. The averaging operator of IFS's α and β in S, denoted by $\alpha\$\beta$, is defined as

$$\alpha\$\beta = \left\{ \left(k, \sqrt{\mu_\alpha(k) \cdot \mu_\beta(k)}, \sqrt{v_\alpha(k) \cdot v_\beta(k)} \right) \mid k \in S \right\} \tag{4}$$

which is $\sqrt{\mu_\alpha(k) \cdot \mu_\beta(k)}$ will be denoted by $\Phi\left(\mu_\alpha(k), \mu_\beta(k) \right)$ and $\sqrt{v_\alpha(k) \cdot v_\beta(k)}$ will be denoted by $\Phi'\left(v_\alpha(k), v_\beta(k) \right)$ on the next discussion (Shuaib et al., 2020).

Definition 2.13. Let S is an empty set. Let α is an IFS over S, and $\eta \in [0,1]$. An IFS $\alpha^\eta = \left\{ \left(k, \mu_{\alpha^\eta}(k), v_{\alpha^\eta}(k) \right) \mid k \in S \right\}$ which is
1. $\mu_{\alpha^\eta}(k) = \Phi(k, \eta)$
2. $v_{\alpha^\eta}(k) = \Phi'(k, \eta)$
is called η-intuitionistic fuzzy set (η-IFS) for every $k \in S$ (Shuaib et al., 2020).

Note: Defined mappings $\Phi: S \times [0,1] \rightarrow [0,1]$, where $\Phi(k, \eta) = \sqrt{\mu_A(k) \cdot \eta}$, and $\Phi': S \times [0,1] \rightarrow [0,1]$ where $\Phi'(k, \eta) = \sqrt{v_A(k) \cdot (1 - \eta)}$.

C.   RESULT AND DISCUSSION

    In this section will be discussed some definitions and its properties of η-intuitionistic fuzzy rings and η-intuitionistic fuzzy ideals. Based on Figure 1, one know that η-intuitionistic fuzzy ring and η-intuitionistic fuzzy ideal is combined of intuitionistic fuzzy set and any value $\eta \in [0,1]$. This is similar to research that built η-intuitionistic fuzzy subgroup from fuzzy subgroup, intuitionistic fuzzy set, and any value $\eta \in [0,1]$, that was conducted by Shuaib et.al (2020). For the first, one will discuss the η-intuitionistic fuzzy ring as below.

1. η-Intuitionistic Fuzzy Ring

Definition 3.1. Let R is a ring. Let α is an IFS over R, and $\eta \in [0,1]$. An $\alpha^\eta = \{(k, \mu_{\alpha^\eta}(k), \nu_{\alpha^\eta}(k)) \mid k \in R\}$ is called η-intuitionistic fuzzy ring (η-IFR) if it admits the following axioms:

   a. $\mu_{\alpha^\eta}(k_1 + k_2) \geq \min(\mu_{\alpha^\eta}(k_1), \mu_{\alpha^\eta}(k_2))$
   b. $\mu_{\alpha^\eta}(k_1 k_2) \geq \min(\mu_{\alpha^\eta}(k_1), \mu_{\alpha^\eta}(k_2))$
   c. $\mu_{\alpha^\eta}(-k_1) \geq \mu_{\alpha^\eta}(k_1)$
   d. $\nu_{\alpha^\eta}(k_1 + k_2) \leq \max(\nu_{\alpha^\eta}(k_1), \nu_{\alpha^\eta}(k_2))$
   e. $\nu_{\alpha^\eta}(k_1 k_2) \leq \max(\nu_{\alpha^\eta}(k_1), \nu_{\alpha^\eta}(k_2))$
   f. $\nu_{\alpha^\eta}(-k_1) \leq \nu_{\alpha^\eta}(k_2)$
      for every $k_1, k_2 \in R$.

Note: $\mu_{A^\eta}(k)$ is the degree of membership of η-intuitionistic fuzzy set and $\nu_{A^\eta}(k)$ is the degree of non membership of η-intuitionistic fuzzy set that were explained on Definition 2.13.

Example 3.2. Let $(\mathbb{Z}_6, +, \cdot)$ is a ring. Defined $\mu_\alpha : \mathbb{Z}_6 \to [0,1]$ and $\nu_\alpha : \mathbb{Z}_6 \to [0,1]$, which is

$$\mu_\alpha(k) = \begin{cases} 0,23 & , k = \bar{1}, \bar{2}, \bar{4}, \bar{5} \\ 0,40 & , k = \bar{0}, \bar{3} \end{cases} \quad \text{and} \quad \nu_\alpha(k) = \begin{cases} 0,77 & , k = \bar{1}, \bar{2}, \bar{4}, \bar{5} \\ 0,25 & , k = \bar{0}, \bar{3} \end{cases}$$

Let $\eta = 0,65$, thus $1 - \eta = 0,35$. One refer to the averaging operation in Definition 2.3., we have $\mu_{\alpha^\eta}(k) = 0,39$ and $\nu_{\alpha^\eta}(k) = 0,52$ for $k = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}$, while $\mu_{\alpha^\eta}(k) = 0,51$ and $\nu_{\alpha^\eta}(k) = 0,30$ for $k = \{\bar{0}, \bar{3}\}$. It can be shown that η-IFR based on the following description:

   a. Let any $k_1 = \bar{1}$ and $k_2 = \bar{3}$, thus $k_1 + k_2 = \bar{4}$. As we know that $\mu_{\alpha^\eta}(\bar{1}) = 0,39$ and $\mu_{\alpha^\eta}(\bar{3}) = 0,51$. So, we have $\mu_{\alpha^\eta}(\bar{1} + \bar{3}) = \mu_{\alpha^\eta}(\bar{4}) = 0,39$.
   The value of $\min(\mu_{\alpha^\eta}(\bar{1}), \mu_{\alpha^\eta}(\bar{3})) = \min(0,39, 0,51) = 0,39$.
   Therefore, we have $\mu_{\alpha^\eta}(\bar{1} + \bar{3}) \geq \min(\mu_{\alpha^\eta}(\bar{1}), \mu_{\alpha^\eta}(\bar{3}))$.

   b. Let $k_1 = \bar{2}$ dan $k_2 = \bar{4}$, thus $k_1 + k_2 = \bar{0}$. As we know that $\nu_{\alpha^\eta}(\bar{2}) = 0,52$ dan $\nu_{\alpha^\eta}(\bar{4}) = 0,52$.
   So, we have that degree of non-membership, $\nu_{\alpha^\eta}(\bar{2} + \bar{4}) = \nu_{\alpha^\eta}(\bar{0}) = 0,30$.
   The value of $\max(\nu_{\alpha^\eta}(\bar{2}), \nu_{\alpha^\eta}(\bar{4})) = \max(0,52, 0,52) = 0,52$.
   Therefore, $\nu_{\alpha^\eta}(\bar{2} + \bar{4}) \leq \max(\nu_{\alpha^\eta}(\bar{2}), \nu_{\alpha^\eta}(\bar{4}))$.
   To show the other axioms can use the same method as above.
   Then, that can be proved that $\alpha^\eta = \{(k, \mu_{\alpha^\eta}(k), \nu_{\alpha^\eta}(k)) \mid k \in \mathbb{Z}_6\}$ is an η-IFR.

Theorem 3.3. Let α^η and β^η are η-IFR's of R, then $\alpha^\eta \cap \beta^\eta$ is also an η-IFR.

Proof. Based on (3) in Definition 2.4., one have for any $k_1, k_2 \in R$:

   a. $\mu_{\alpha^\eta \cap \beta^\eta}(k_1 + k_2) = \min\left(\mu_{\alpha^\eta}(k_1 + k_2), \mu_{\beta^\eta}(k_1 + k_2)\right)$

$$= \min\left(\Phi(\mu_\alpha(k_1 + k_2), \eta), \Phi(\mu_\beta(k_1 + k_2), \eta)\right)$$

$$\geq \min\left(\min\begin{pmatrix} \Phi(\mu_\alpha(k_1), \eta), \\ \Phi(\mu_\alpha(k_2), \eta) \end{pmatrix}, \min\begin{pmatrix} \Phi(\mu_\beta(k_1), \eta), \\ \Phi(\mu_\beta(k_2), \eta) \end{pmatrix}\right)$$

$$\geq \min\left(\min\begin{pmatrix}\Phi(\mu_\alpha(k_1),\eta),\\\Phi(\mu_\beta(k_1),\eta)\end{pmatrix}, \min\begin{pmatrix}\Phi(\mu_\alpha(k_2),\eta),\\\Phi(\mu_\beta(k_2),\eta)\end{pmatrix}\right)$$

$$= \min\left(\min\left(\mu_{\alpha^\eta}(k_1),\mu_{\beta^\eta}(k_1)\right), \min\left(\mu_{\alpha^\eta}(k_2),\mu_{\beta^\eta}(k_2)\right)\right)$$

$$= \min\left(\mu_{\alpha^\eta\cap\beta^\eta}(k_1),\mu_{\alpha^\eta\cap\beta^\eta}(k_2)\right)$$

Similarly, it can be proved that,

$$\mu_{\alpha^\eta\cap\beta^\eta}(k_1k_2) \geq \min\left(\mu_{\alpha^\eta\cap\beta^\eta}(k_1),\mu_{\alpha^\eta\cap\beta^\eta}(k_2)\right).$$

Further,

b. $v_{\alpha^\eta\cap\beta^\eta}(k_1 + k_2) = \max\left(v_{\alpha^\eta}(k_1 + k_2), v_{\beta^\eta}(k_1 + k_2)\right)$

$$= \max\left(\Phi'(v_\alpha(k_1 + k_2), 1 - \eta), \Phi'(v_\beta(k_1 + k_2), 1 - \eta)\right)$$

$$\leq \max\left(\max\begin{pmatrix}\Phi'(v_\alpha(k_1), 1 - \eta),\\\Phi'(v_\alpha(k_2), 1 - \eta)\end{pmatrix}, \max\begin{pmatrix}\Phi'(v_\beta(k_1), 1 - \eta),\\\Phi'(v_\beta(k_2), 1 - \eta)\end{pmatrix}\right)$$

$$\leq \max\left(\max\begin{pmatrix}\Phi'(v_\alpha(k_1), 1 - \eta),\\\Phi'(v_\beta(k_1), 1 - \eta)\end{pmatrix}, \max\begin{pmatrix}\Phi'(v_\alpha(k_2), 1 - \eta),\\\Phi'(v_\beta(k_2), 1 - \eta)\end{pmatrix}\right)$$

$$= \max\left(\max\left(v_{\alpha^\eta}(k_1),v_{\beta^\eta}(k_1)\right), \max\left(v_{\alpha^\eta}(k_2),v_{\beta^\eta}(k_2)\right)\right)$$

$$= \max\left(v_{\alpha^\eta\cap\beta^\eta}(k_1),v_{\alpha^\eta\cap\beta^\eta}(k_2)\right)$$

Similarly,

$$v_{\alpha^\eta\cap\beta^\eta}(k_1k_2) \leq \max\left(v_{\alpha^\eta\cap\beta^\eta}(k_1),v_{\alpha^\eta\cap\beta^\eta}(k_2)\right).$$

Moreover,

c. $\mu_{\alpha^\eta\cap\beta^\eta}(-k_1) = \min\left(\mu_{\alpha^\eta}(-k_1),\mu_{\beta^\eta}(-k_1)\right)$

$$\geq \min\left(\mu_{\alpha^\eta}(k_1),\mu_{\beta^\eta}(k_1)\right)$$

$$= \mu_{\alpha^\eta\cap\beta^\eta}(k_1)$$

Similarly, it can be proved that,

$$v_{\alpha^\eta\cap\beta^\eta}(-k_1) \leq v_{\alpha^\eta\cap\beta^\eta}(k_1).$$

Then, $\alpha^\eta \cap \beta^\eta$ is also an η-IFR is proved.

Lemma 3.4. The union of η-IFR's is also an η-IFR.

Proof. Based on (2) in Definition 2.4., one have for any $k_1, k_2 \in R$:

a. $\mu_{\alpha^\eta\cup\beta^\eta}(k_1 + k_2) = \max\left(\mu_{\alpha^\eta}(k_1 + k_2), \mu_{\beta^\eta}(k_1 + k_2)\right)$

$$= \max\left(\Phi(\mu_\alpha(k_1 + k_2), \eta), \Phi(\mu_\beta(k_1 + k_2), \eta)\right)$$

$$\geq \max\left(\min\begin{pmatrix}\Phi(\mu_\alpha(k_1),\eta),\\\Phi(\mu_\alpha(k_2),\eta)\end{pmatrix}, \min\begin{pmatrix}\Phi(\mu_\beta(k_1),\eta),\\\Phi(\mu_\beta(k_2),\eta)\end{pmatrix}\right)$$

$$\geq \min\left(\max\begin{pmatrix}\Phi(\mu_\alpha(k_1),\eta),\\\Phi(\mu_\beta(k_1),\eta)\end{pmatrix}, \max\begin{pmatrix}\Phi(\mu_\alpha(k_2),\eta),\\\Phi(\mu_\beta(k_2),\eta)\end{pmatrix}\right)$$

$$= \min\left(\max\left(\mu_{\alpha^\eta}(k_1),\mu_{\beta^\eta}(k_1)\right), \max\left(\mu_{\alpha^\eta}(k_2),\mu_{\beta^\eta}(k_2)\right)\right)$$

$$= \min\left(\mu_{\alpha^\eta\cup\beta^\eta}(k_1),\mu_{\alpha^\eta\cup\beta^\eta}(k_2)\right)$$

Similarly, it can be proved that,

$$\mu_{\alpha^\eta \cup \beta^\eta}(k_1 k_2) \geq \min\left(\mu_{\alpha^\eta \cup \beta^\eta}(k_1), \mu_{\alpha^\eta \cup \beta^\eta}(k_2)\right).$$

Further,

b. $\nu_{\alpha^\eta \cup \beta^\eta}(k_1 + k_2) = \min\left(\nu_{\alpha^\eta}(k_1 + k_2), \nu_{\beta^\eta}(k_1 + k_2)\right)$

$$= \min\left(\Phi'(\nu_\alpha(k_1 + k_2), 1 - \eta), \Phi'(\nu_\beta(k_1 + k_2), 1 - \eta)\right)$$

$$\leq \min\left(\max\begin{pmatrix}\Phi'(\nu_\alpha(k_1), 1 - \eta), \\ \Phi'(\nu_\alpha(k_2), 1 - \eta)\end{pmatrix}, \max\begin{pmatrix}\Phi'(\nu_\beta(k_1), 1 - \eta), \\ \Phi'(\nu_\beta(k_2), 1 - \eta)\end{pmatrix}\right)$$

$$\leq \max\left(\min\begin{pmatrix}\Phi'(\nu_\alpha(k_1), 1 - \eta), \\ \Phi'(\nu_\beta(k_1), 1 - \eta)\end{pmatrix}, \min\begin{pmatrix}\Phi'(\nu_\alpha(k_2), 1 - \eta), \\ \Phi'(\nu_\beta(k_2), 1 - \eta)\end{pmatrix}\right)$$

$$= \max\left(\min\left(\nu_{\alpha^\eta}(k_1), \nu_{\beta^\eta}(k_1)\right), \min\left(\nu_{\alpha^\eta}(k_2), \nu_{\beta^\eta}(k_2)\right)\right)$$

$$= \max\left(\nu_{\alpha^\eta \cup \beta^\eta}(k_1), \nu_{\alpha^\eta \cup \beta^\eta}(k_2)\right)$$

Similarly,

$$\nu_{\alpha^\eta \cup \beta^\eta}(k_1 k_2) \leq \max\left(\nu_{\alpha^\eta \cup \beta^\eta}(k_1), \nu_{\alpha^\eta \cup \beta^\eta}(k_2)\right).$$

Furthermore,

c. $\mu_{\alpha^\eta \cup \beta^\eta}(-k_1) = \max\left(\mu_{\alpha^\eta}(-k_1), \mu_{\beta^\eta}(-k_1)\right)$

$$\geq \max\left(\mu_{\alpha^\eta}(k_1), \mu_{\beta^\eta}(k_1)\right)$$

$$= \mu_{\alpha^\eta \cup \beta^\eta}(k_1)$$

Similarly, it can be proved that,

$$\nu_{\alpha^\eta \cup \beta^\eta}(-k_1) \leq \nu_{\alpha^\eta \cup \beta^\eta}(k_1).$$

Then, $\alpha^\eta \cup \beta^\eta$ is also an η-IFR is proved.

2. η-Intuitionistic Fuzzy Ideal

In this section, one will discuss the η-intuitionistic fuzzy ideal concept and its operations that is admitted.

Definition 3.5. Let R is a ring and $\eta \in [0,1]$. α^η over R is called η-intuitionistic fuzzy ideal (η-IFI) of R if it admits the following conditions:

a. $\mu_{\alpha^\eta}(k_1 - k_2) \geq \min(\mu_{\alpha^\eta}(k_1), \mu_{\alpha^\eta}(k_1))$
b. $\mu_{\alpha^\eta}(k_1 k_2) \geq \max(\mu_{\alpha^\eta}(k_1), \mu_{\alpha^\eta}(k_1))$
c. $\nu_{\alpha^\eta}(k_1 - k_2) \leq \max(\nu_{\alpha^\eta}(k_1), \nu_{\alpha^\eta}(k_2))$
d. $\nu_{\alpha^\eta}(k_1 k_2) \leq \min(\nu_{\alpha^\eta}(k_1), \nu_{\alpha^\eta}(k_2))$
   for every $k_1, k_2 \in R$.

Example 3.6. Let $(\mathbb{Z}_3, +, \cdot)$ is a ring. Defined $\mu_\alpha : \mathbb{Z}_3 \to [0,1]$ and $\nu_\alpha : \mathbb{Z}_3 \to [0,1]$, which is

$$\mu_\alpha(k) = \begin{cases} 0{,}48, & k = \bar{0} \\ 0{,}13, & k = \bar{1}, \bar{2} \end{cases} \quad \text{and} \quad \nu_\alpha(k) = \begin{cases} 0{,}35, & k = \bar{0} \\ 0{,}51, & k = \bar{1}, \bar{2} \end{cases}$$

Let $\eta = 0{,}65$, thus $1 - \eta = 0{,}35$. As the same method as Example 3.2., we have $\mu_{\alpha^\eta}(k) = 0{,}29$ and $\nu_{\alpha^\eta}(k) = 0{,}42$ for $k = \{\bar{1}, \bar{2}\}$, while $\mu_{\alpha^\eta}(k) = 0{,}56$ and $\nu_{\alpha^\eta}(k) = 0{,}35$ for $k = \{\bar{0}\}$. It can be shown that η-IFI based on the following description:

a. Let $k_1 = \bar{1}$ and $k_2 = \bar{2}$, thus $-k_2 = \bar{1}$. So, we have $k_1 + (-k_2) = \bar{2}$. As we know that $\mu_{\alpha^\eta}(\bar{1}) = 0{,}29$ and $\mu_{\alpha^\eta}(\bar{2}) = 0{,}29$. Thus, we have $\mu_{\alpha^\eta}(\bar{1} + \bar{1}) = \mu_{\alpha^\eta}(\bar{2}) = 0{,}29$.

   The value of $\min(\mu_{\alpha^\eta}(\bar{1}), \mu_{\alpha^\eta}(\bar{1})) = \min(0{,}29, 0{,}29) = 0{,}29$.

   Therefore, $\mu_{\alpha^\eta}(\bar{1} + \bar{1}) \geq \min(\mu_{\alpha^\eta}(\bar{1}), \mu_{\alpha^\eta}(\bar{2}))$.

b. Let $k_1 = \bar{1}$ and $k_2 = \bar{2}$, thus $-k_2 = \bar{1}$. As we know that $\nu_{\alpha^\eta}(\bar{1}) = 0{,}42$ and $\nu_{\alpha^\eta}(\bar{2}) = 0{,}42$.

   Thus, we have $\nu_{\alpha^\eta}(\bar{1} + \bar{1}) = \nu_{\alpha^\eta}(\bar{2}) = 0{,}42$.

   The value of $\max(\nu_{\alpha^\eta}(\bar{1}), \nu_{\alpha^\eta}(\bar{1})) = \max(0{,}42, 0{,}42) = 0{,}42$.

   Therefore, $\nu_{\alpha^\eta}(\bar{1} + \bar{1}) \leq \max(\nu_{\alpha^\eta}(\bar{1}), \nu_{\alpha^\eta}(\bar{2}))$.

To show the other axioms can use the same method as above.

Then, it can be proved that $\alpha^\eta = \{(k, \mu_{\alpha^\eta}(k), \nu_{\alpha^\eta}(k)) \mid k \in \mathbb{Z}_6\}$ is an η-IFI.

Theorem 3.7. Let α^η and β^η are η-IFI's of R, then $\alpha^\eta \cap \beta^\eta$ is also an η-IFI.

Proof. Based on (3) in Definition 2.4., one have for every $k_1, k_2 \in R$:

a.
$$\mu_{\alpha^\eta \cap \beta^\eta}(k_1 - k_2) = \min\left(\mu_{\alpha^\eta}(k_1 - k_2), \mu_{\beta^\eta}(k_1 - k_2)\right)$$
$$= \min\left(\Phi(\mu_\alpha(k_1 - k_2), \eta), \Phi(\mu_\beta(k_1 - k_2), \eta)\right)$$
$$\geq \min\left(\min\begin{pmatrix}\Phi(\mu_\alpha(k_1), \eta), \\ \Phi(\mu_\alpha(k_2), \eta)\end{pmatrix}, \min\begin{pmatrix}\Phi(\mu_\beta(k_1), \eta), \\ \Phi(\mu_\beta(k_2), \eta)\end{pmatrix}\right)$$
$$= \min\left(\min\begin{pmatrix}\Phi(\mu_\alpha(k_1), \eta), \\ \Phi(\mu_\beta(k_1), \eta)\end{pmatrix}, \min\begin{pmatrix}\Phi(\mu_\alpha(k_2), \eta), \\ \Phi(\mu_\beta(k_2), \eta)\end{pmatrix}\right)$$
$$= \min\left(\min\left(\mu_{\alpha^\eta}(k_1), \mu_{\beta^\eta}(k_1)\right), \min\left(\mu_{\alpha^\eta}(k_2), \mu_{\beta^\eta}(k_2)\right)\right)$$
$$= \min\left(\mu_{A^\eta \cap B^\eta}(x), \mu_{A^\eta \cap B^\eta}(y)\right)$$

   Similarly, it can be proved that follows Definition 3.5,
$$\mu_{\alpha^\eta \cap \beta^\eta}(k_1 k_2) \geq \max\left(\mu_{A^\eta \cap B^\eta}(x), \mu_{A^\eta \cap B^\eta}(y)\right).$$

   Further,

b.
$$\nu_{\alpha^\eta \cap \beta^\eta}(k_1 - k_2) = \max\left(\nu_{\alpha^\eta}(k_1 - k_2), \nu_{\beta^\eta}(k_1 - k_2)\right)$$
$$= \max\left(\Phi'(\nu_\alpha(k_1 - k_2), 1 - \eta), \Phi'(\nu_\beta(k_1 - k_2), 1 - \eta)\right)$$
$$\leq \max\left(\max\begin{pmatrix}\Phi'(\nu_\alpha(k_1), 1 - \eta), \\ \Phi'(\nu_\alpha(k_2), 1 - \eta)\end{pmatrix}, \max\begin{pmatrix}\Phi'(\nu_\beta(k_1), 1 - \eta), \\ \Phi'(\nu_\beta(k_2), 1 - \eta)\end{pmatrix}\right)$$
$$= \max\left(\max\begin{pmatrix}\Phi'(\nu_\alpha(k_1), 1 - \eta), \\ \Phi'(\nu_\beta(k_1), 1 - \eta)\end{pmatrix}, \max\begin{pmatrix}\Phi'(\nu_\alpha(k_2), 1 - \eta), \\ \Phi'(\nu_\beta(k_2), 1 - \eta)\end{pmatrix}\right)$$
$$= \max\left(\max\left(\nu_{\alpha^\eta}(k_1), \nu_{\beta^\eta}(k_1)\right), \max\left(\nu_{\alpha^\eta}(k_2), \nu_{\beta^\eta}(k_2)\right)\right)$$
$$= \max\left(\nu_{\alpha^\eta \cap \beta^\eta}(k_1), \nu_{\alpha^\eta \cap \beta^\eta}(k_2)\right)$$

   Similarly,
$$\nu_{\alpha^\eta \cap \beta^\eta}(k_1 k_2) \leq \min\left(\nu_{\alpha^\eta \cap \beta^\eta}(k_1), \nu_{\alpha^\eta \cap \beta^\eta}(k_2)\right).$$

Then, $\alpha^\eta \cap \beta^\eta$ is also an η-IFI is proved.

Lemma 3.8. The union of η-IFI's is also an η-IFI.

Proof. That will be used in the same method as the Theorem 3.7 and Lemma 3.4.

The following are definitions of sum and product between η-IFI's and its properties.

Definition 3.9. Let α^η dan β^η are η-IFI's over R. The sum of α^η and β^η is defined as,

$$\alpha^\eta + \beta^\eta = \left\{ \left(k, (\mu_{\alpha^\eta+\beta^\eta})(k), (\nu_{\alpha^\eta+\beta^\eta})(k) \right) \mid k \in R \right\} \tag{11}$$

where

$$(\mu_{\alpha^\eta+\beta^\eta})(k) = \sup_{k=m+n} \left(\min \left(\mu_{\alpha^\eta}(m), \mu_{\beta^\eta}(n) \right) \right), \quad \forall k \in R \tag{12}$$

and

$$(\nu_{\alpha^\eta+\beta^\eta})(k) = \inf_{k=m+n} \left(\max \left(\nu_{\alpha^\eta}(m), \nu_{\beta^\eta}(n) \right) \right), \quad \forall k \in R. \tag{13}$$

Theorem 3.10. Let α^η and β^η are η-IFI's of R, then $\alpha^\eta + \beta^\eta$ is also an η-IFI.

Proof. Given $k_1, k_2 \in R$. Let $k_1 = m+n$ and $k_2 = p+1$, for any $m, n, p, q \in R$. Based on Definition 4.5, (12), and (13), one have:

a. $\displaystyle \mu_{\alpha^\eta+\beta^\eta}(k_1 - k_2) = \sup_{\substack{k_1=m+n \\ k_2=p+q}} \left(\min \left(\mu_{\alpha^\eta}(m-p), \mu_{\beta^\eta}(n-q) \right) \right)$

$$= \sup_{\substack{k_1=m+n \\ k_2=p+q}} \left(\min \left(\Phi(\mu_\alpha(m-p), \eta), \Phi(\mu_\beta(n-q), \eta) \right) \right)$$

$$\geq \sup_{\substack{k_1=m+n \\ k_2=p+q}} \left(\min \left(\min \begin{pmatrix} \Phi(\mu_\alpha(m), \eta), \\ \Phi(\mu_\alpha(p), \eta) \end{pmatrix}, \min \begin{pmatrix} \Phi(\mu_\beta(n), \eta), \\ \Phi(\mu_\beta(q), \eta) \end{pmatrix} \right) \right)$$

$$= \min \left(\sup_{\substack{k_1=m+n \\ k_2=p+q}} \left(\min \begin{pmatrix} \Phi(\mu_\alpha(m), \eta), \\ \Phi(\mu_\beta(n), \eta) \end{pmatrix} \right), \sup_{\substack{k_1=m+n \\ k_2=p+q}} \left(\min \begin{pmatrix} \Phi(\mu_\alpha(p), \eta), \\ \Phi(\mu_\beta(q), \eta) \end{pmatrix} \right) \right)$$

$$= \min \left(\sup_{k_1=m+n} \left(\min \left(\mu_{\alpha^\eta}(m), \mu_{\beta^\eta}(n) \right) \right), \sup_{k_2=p+q} \left(\min(\mu_{\alpha^\eta}(p), \mu_{\alpha^\eta}(q)) \right) \right)$$

$$= \min \left(\mu_{\alpha^\eta+\beta^\eta}(k_1), \mu_{\alpha^\eta+\beta^\eta}(k_2) \right)$$

Similarly, it can be proved that follows Definition 4.5,

$$\mu_{\alpha^\eta+\beta^\eta}(k_1 k_2) \geq \max \left(\mu_{\alpha^\eta+\beta^\eta}(k_1), \mu_{\alpha^\eta+\beta^\eta}(k_2) \right).$$

Further,

b. $\displaystyle \nu_{\alpha^\eta+\beta^\eta}(k_1 - k_2) = \inf_{\substack{k_1=m+n \\ k_2=p+q}} \left(\max \left(\nu_{\alpha^\eta}(m-p), \nu_{\beta^\eta}(n-q) \right) \right)$

$$= \inf_{\substack{k_1=m+n \\ k_2=p+q}} \left(\max \left(\Phi'(\nu_\alpha(m-p), 1-\eta), \Phi'(\nu_\beta(n-q), 1-\eta) \right) \right)$$

$$\leq \inf_{\substack{k_1=m+n \\ k_2=p+q}} \left(\max \left(\max \begin{pmatrix} \Phi'(\nu_\alpha(m), 1-\eta), \\ \Phi'(\nu_\alpha(p), 1-\eta) \end{pmatrix}, \max \begin{pmatrix} \Phi'(\nu_\beta(n), 1-\eta), \\ \Phi'(\nu_\beta(q), 1-\eta) \end{pmatrix} \right) \right)$$

$$= \max \left(\inf_{\substack{k_1=m+n \\ k_2=p+q}} \left(\max \begin{pmatrix} \Phi'(\nu_\alpha(m), 1-\eta), \\ \Phi'(\nu_\beta(n), 1-\eta) \end{pmatrix} \right), \inf_{\substack{k_1=m+n \\ k_2=p+q}} \left(\max \begin{pmatrix} \Phi'(\nu_\alpha(p), 1-\eta), \\ \Phi'(\nu_\beta(q), 1-\eta) \end{pmatrix} \right) \right)$$

$$= \max\left(\inf_{k_1=m+n}\left(\max\left(\nu_{\alpha^\eta}(m), \nu_{\beta^\eta}(n)\right)\right), \inf_{k_2=p+q}\left(\max\left(\nu_{\alpha^\eta}(p), \nu_{\beta^\eta}(q)\right)\right)\right)$$

$$= \max\left(\nu_{\alpha^\eta+\beta^\eta}(k_1), \nu_{\alpha^\eta+\beta^\eta}(k_2)\right)$$

Similarly,

$$\nu_{\alpha^\eta+\beta^\eta}(k_1 k_2) \le \min\left(\nu_{\alpha^\eta+\beta^\eta}(k_1), \nu_{\alpha^\eta+\beta^\eta}(k_2)\right).$$

Then, $\alpha^\eta + \beta^\eta$ is also an η-IFI is proved.

Definition 3.11. Let α^η dan β^η are η-IFI's over R. The product of α^η and β^η is defined as,

$$\alpha^\eta\beta^\eta = \left\{\left(k, (\mu_{\alpha^\eta\beta^\eta})(k), (\nu_{\alpha^\eta\beta^\eta})(k)\right) \mid k \in R\right\} \tag{14}$$

where

$$(\mu_{\alpha^\eta\beta^\eta})(k) = \sup_{k=\sum_{i<\infty}m_i n_i}\left(\min\left(\min\left(\mu_{\alpha^\eta}(m_i), \mu_{\beta^\eta}(n_i)\right)\right)\right) \tag{15}$$

and

$$(\nu_{\alpha^\eta\beta^\eta})(k) = \inf_{k=\sum_{i<\infty}m_i n_i}\left(\max\left(\max\left(\mu_{\alpha^\eta}(m_i), \mu_{\beta^\eta}(n_i)\right)\right)\right). \tag{16}$$

Example 3.12. Consider that Example 3.6. First, the degree of membership an non membership of $k = \sum_{i<\infty}m_i n_i$ will be determined for any $m_i, n_i \in \mathbb{Z}_3$. Hence, for all $k \in \mathbb{Z}_3$, there are many possible forms $\sum_{i<\infty}m_i n_i$. Then, we determine the degree of membership and non membership of all $k \in \mathbb{Z}_3$, as shown in Table 1.

Table 1. The Degree of Membership and Non-membership $\alpha^\eta\beta^\eta$ for all $k \in \mathbb{Z}_3$

k	$(\mu_{\alpha^\eta\beta^\eta})(k)$	$(\nu_{\alpha^\eta\beta^\eta})(k)$
$\bar{0}$	0,49	0,35
$\bar{1}$	0,29	0,42
$\bar{2}$	0,29	0,42

Theorem 3.13. Let α^η and β^η are η-IFI's of R, then $\alpha^\eta\beta^\eta$ is also an η-IFI.

Proof. Given $k_1, k_2 \in R$. Let $k_1 = \sum_{i<\infty}m_i n_i$ and $k_2 = \sum_{i<\infty}p_i q_i$, for any $m_i, n_i, p_i, q_i \in R$ with $i < \infty$. Based on Definition 4.5, (15), and (16), one have:

a. $\mu_{\alpha^\eta\beta^\eta}(k_1 - k_2) = \sup_{\substack{k_1=\sum_{i<\infty}m_i n_i \\ k_2=\sum_{i<\infty}p_i q_i}}\left(\min\left(\min\left(\mu_{\alpha^\eta}(m_i - p_i), \mu_{\beta^\eta}(n_i - q_i)\right)\right)\right)$

$$= \sup_{\substack{k_1=\sum_{i<\infty}m_i n_i \\ k_2=\sum_{i<\infty}p_i q_i}}\left(\min\left(\min\left(\Phi(\mu_\alpha(m_i - p_i), \eta), \Phi(\mu_\beta(n_i - q_i), \eta)\right)\right)\right)$$

$$\ge \sup_{\substack{k_1=\sum_{i<\infty}m_i n_i \\ k_2=\sum_{i<\infty}p_i q_i}}\left(\min\left(\min\left(\min\binom{\Phi(\mu_\alpha(m_i), \eta),}{\Phi(\mu_\alpha(p_i), \eta)}\right), \min\binom{\Phi(\mu_\beta(n_i), \eta),}{\Phi(\mu_\beta(q_i), \eta)}\right)\right)\right)$$

$$= \min\left(\begin{array}{c}\sup_{\substack{k_1=\sum_{i<\infty}m_i n_i \\ k_2=\sum_{i<\infty}p_i q_i}}\left(\min\left(\min\binom{\Phi(\mu_\alpha(m_i), \eta),}{\Phi(\mu_\beta(n_i), \eta)}\right)\right), \\ \sup_{\substack{k_1=\sum_{i<\infty}m_i n_i \\ k_2=\sum_{i<\infty}p_i q_i}}\left(\min\left(\min\binom{\Phi(\mu_\alpha(p_i), \eta),}{\Phi(\mu_\beta(q_i), \eta)}\right)\right)\end{array}\right)$$

$$= \min \left(\begin{array}{c} \sup_{k_1 = \sum_{i < \infty} m_i n_i} \left(\min \left(\min \left(\mu_{\alpha^\eta}(m_i), \mu_{\beta^\eta}(n_i) \right) \right) \right), \\ \sup_{k_2 = \sum_{i < \infty} p_i q_i} \left(\min \left(\min \left(\mu_{\alpha^\eta}(p_i), \mu_{\beta^\eta}(q_i) \right) \right) \right) \end{array} \right)$$

$$= \min \left(\mu_{\alpha^\eta \beta^\eta}(k_1), \mu_{\alpha^\eta \beta^\eta}(k_2) \right)$$

Similarly, it can be proved that,

$$\mu_{\alpha^\eta \beta^\eta}(k_1 k_2) \geq \max \left(\mu_{\alpha^\eta \beta^\eta}(k_1), \mu_{\alpha^\eta \beta^\eta}(k_2) \right).$$

Furthermore,

b. $$\nu_{\alpha^\eta \beta^\eta}(k_1 - k_2) = \inf_{\substack{k_1 = \sum_{i < \infty} m_i n_i \\ k_2 = \sum_{i < \infty} p_i q_i}} \left(\max \left(\max \left(\nu_{\alpha^\eta}(m_i - p_i), \nu_{\beta^\eta}(n_i - q_i) \right) \right) \right)$$

$$= \inf_{\substack{k_1 = \sum_{i < \infty} m_i n_i \\ k_2 = \sum_{i < \infty} p_i q_i}} \left(\max \left(\max \left(\Phi'(\nu_\alpha(m_i - p_i), 1 - \eta), \Phi'(\nu_\beta(n_i - q_i), 1 - \eta) \right) \right) \right)$$

$$\geq \inf_{\substack{k_1 = \sum_{i < \infty} m_i n_i \\ k_2 = \sum_{i < \infty} p_i q_i}} \left(\max \left(\max \left(\max \begin{pmatrix} \Phi'(\nu_\alpha(m_i), 1 - \eta), \\ \Phi'(\nu_\alpha(p_i), 1 - \eta) \end{pmatrix}, \max \begin{pmatrix} \Phi'(\nu_\beta(n_i), 1 - \eta), \\ \Phi'(\nu_\beta(q_i), 1 - \eta) \end{pmatrix} \right) \right) \right)$$

$$= \max \left(\begin{array}{c} \inf_{\substack{k_1 = \sum_{i < \infty} m_i n_i \\ k_2 = \sum_{i < \infty} p_i q_i}} \left(\max \left(\max \begin{pmatrix} \Phi'(\nu_\alpha(m_i), 1 - \eta), \\ \Phi'(\nu_\beta(n_i), 1 - \eta) \end{pmatrix} \right) \right), \\ \inf_{\substack{k_1 = \sum_{i < \infty} m_i n_i \\ k_2 = \sum_{i < \infty} p_i q_i}} \left(\max \left(\max \begin{pmatrix} \Phi'(\nu_\alpha(p_i), 1 - \eta), \\ \Phi'(\nu_\beta(q_i), 1 - \eta) \end{pmatrix} \right) \right) \end{array} \right)$$

$$= \max \left(\begin{array}{c} \inf_{k_1 = \sum_{i < \infty} m_i n_i} \left(\max \left(\max \left(\nu_{\alpha^\eta}(m_i), \nu_{\beta^\eta}(n_i) \right) \right) \right), \\ \inf_{k_2 = \sum_{i < \infty} p_i q_i} \left(\max \left(\max \left(\nu_{\alpha^\eta}(p_i), \nu_{\beta^\eta}(q_i) \right) \right) \right) \end{array} \right)$$

$$= \max \left(\nu_{\alpha^\eta \beta^\eta}(k_1), \nu_{\alpha^\eta \beta^\eta}(k_2) \right)$$

Similarly,

$$\nu_{\alpha^\eta \beta^\eta}(k_1 k_2) \geq \min \left(\nu_{\alpha^\eta \beta^\eta}(k_1), \nu_{\alpha^\eta \beta^\eta}(k_2) \right).$$

Then, $A^\eta B^\eta$ is also an η-IFI is proved.

D. CONCLUSION AND SUGGESTIONS

Based on the result and discussion, a new structure of η-intuitionistic fuzzy ring and η-intuitionistic fuzzy ideal is obtained. Further, in η-intuitionistic fuzzy ideal, one obtained the sum and product operations between η-intuitionistic fuzzy ideals and its properties. On the next research, it is suggested to build a new structure of η-intuitionistic fuzzy quotient ring by using η-intuitionistic fuzzy ideal concept and homomorphism mapping and its properties in η-intuitionistic fuzzy ring and η-intuitionistic fuzzy ideal.

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