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Development of ACERA Learning Model Based on Proof Construction Analysis

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ABSTRACT

Proof constructing is the process of justifying a claim using the methods and concepts of proof to produce mathematical proof. Proof constructing is also an aspect of proof, and is often the only way to assess student performance. However, proof construction is still a constant problem (difficulty) for every student. The cause of this difficulty is not only because of the content of proof in textbooks/sources, over-reliance on examples, understanding, underlying logic, and the ability to use proof writing strategies, but also due to the lack of proof discussion activities that train students to understand and answer proof practice questions, give proof reasoning against the proof that has been constructed, and validating own and other colleagues' answers. Thus, this study aims to develop a valid and practical ACERA (Activities, Classroom Discussion, Exercises, Reason, and Audience) learning model and has a potential effect on students' ability to proof construction. This study uses research design research development methods in three stages, namely the preliminary stage, the model development stage and the assessment stage. The research subjects were 23 students of the Mathematics Education study program at the University of Mataram. The development of the ACERA model offers an alternative solution to reduce the difficulty of proof construction, thus enabling this model to have characteristics that are valid, practical and have a potential effect in increasing the productivity of student proof construction.

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A. INTRODUCTION

Proof and proving are two important concepts that must be mastered by every undergraduate mathematics student because they can be the main means of developing and communicating new mathematical results or ideas in the mathematics community (de Villiers, 1990; Lakatos, 2015). Separately, proof refers to a mathematical argument that serves as the justification or refutation of a mathematical claim Stylianides (2007), while proving refers to the activity of producing proof (Shongwe, 2019). The ability to elaborate proof concepts such as axioms, definitions, lemmas, and theorems to justify a mathematical claim is identified as part of constructing a proof (Stylianides, 2007; Selden & Selden, 2009). Self-proof construction refers to the activity of constructing correct proofs at the level expected of university mathematics students (Chamberlain & Vidakovic, 2017). However, it turns out that proof construction to produce correct evidence at the university level is still a problem for every student (Weber, 2003). The causes of this difficulty include the content of proof in mathematics textbooks Stylianides (2009), sources for constructing proof Rabin & Quarfoot (2021), over-reliance on examples to justify the truth of a statement Knuth et al. (2019), understanding of the proof reading strategies and written Rabin & Quarfoot (2021); Lew et al. (2020); underlying mathematical logic Knipping (2008); Moore (1994); Stavrou (2014), and the ability to use proof writing approaches or strategies (Hanna, 2000; Hoyles, 1997; Moore, 1994; Weber, 2001, 2004).

Proof teaching and learning actions are needed that can be used as research-based interventions in mathematics classes Stylianides & Stylianides (2017), namely through the development of an activity-oriented learning model, classroom discussions, exercises, giving reasons, and audience validation; or abbreviated as ACERA learning (Activities, Classroom discussion, Exercises, Reason, and Audience). The development of this model is based on the ACE model developed by Dubinsky in 2001 (Dubinsky, 2001). The ACE learning approach is a pedagogical approach that facilitates collaborative activities in a computer environment and is considered efficacious in teaching and learning on various mathematics topics at the elementary, secondary, and tertiary levels (Vidakovic et al., 2018). The idea that a mathematical proof requires validation from mathematicians or the mathematical community (Reid, 2001). So learning to proof construction is not enough with activities, class discussions, and exercises. However, proof construction requires efforts to provide reasons and validation from experts (Selden & Selden, 2003). The addition of the letter R (reason) to the acronym ACERA, is based on the type of two-column proof on geometric proof which is intended to organize various thoughts related to the proof problems faced (Pair et al., 2021). While the letter A (audience) is based on proof written by considering the audience's considerations. Audience is also based on the definition of argumentation as a system consisting of three parts, namely a collection of premises, conclusions, and a chain of reasoning (Corcoran, 1989). This definition is considered appropriate to be developed into an argument as a product that reflects the audience's considerations (Ashton, 2021).

This last definition implies that a mathematical proof (argument) is developed taking into account the audience's correction. In this audience are mathematicians, researchers, or lecturers, so that the construction of proof reflects a standard of reasonableness that can be realized by the audience. This theoretical background shows the importance of developing a model to avoid making assumptions about certain mathematical views that can be replicated in pedagogical situations because the teaching of proof and proof is multifaceted, dynamic, and evolving (Lesseig & Hine, 2021). So it is important to develop a model based on the results of the analysis of students' difficulties in constructing of proof, namely the ACERA learning model. A learning model developed on the idea that constructing of proof is not enough through learning activities, class discussions, and exercises either in offline or online classes but also requires activities to give reasons for what has been constructed to convince readers or other proof to validate the construction results.

B. METHODS

The research mechanism is carried out through four stages. The *first stage* is to analyze students' difficulties in proof construction. The difficulty analysis stage is preceded by the assumption that there will be difficulties faced by students who are involved in the activity of proof construction. The instrument used to see students' difficulties in proof construction consisted of a proof test instrument and the researcher himself as the interviewer. The proof test instrument contains statements that must be proven by the indirect proof method, namely by using proof by contradiction.

Prove that $\sqrt{3}$ is an irrational number!

The results of the analysis of the difficulties in constructing propositions on the above evidentiary instrument are taken into consideration when developing the ACERA model framework. Next is the *second stage*, namely the ACERA model development stage. The development of the ACERA model uses a design research type of development studies, which consists of two stages, namely: the preliminary stage, and the development or prototyping phase (Van den Akker et al., 2013). At the development stage, the researcher conducted a formative study of the materials used and made from the results of a concept map based on the ACERA development model by conducting a self-evaluation, expert, one-to-one (one-on-one review), small group, and field test. The subjects in this study were 23 mathematics education undergraduate students at the University of Mataram who were programming calculus courses in semester 2 of years 2021.

C. RESULT AND DISCUSSION

1. Analysts have difficulty in proof construction

Based on the results of the analysis of student difficulties in proof construction against the propositions proposed in the problem of proof (first stage) to 23 research subjects who are willing to be involved, the results of the analysis of difficulties are summarized in Table 1.

Table 1. Types of difficulties in proof construction using the indirect method

No	Type of difficulty	Frequency	Description of the difficulty of proof construction
1	Difficulty understanding formal proof or proof by contradiction deduction steps,	7	Difficulty understanding the step of contradiction deduction refers to the ability to show sufficient proof with definitions or proofs with pictures, not with deductive constructing properties or theorems.
2	Difficulty making suppositions	16	Difficulty making a supposition, refers to the incompleteness of the elements in the supposition, for example, if $\sqrt{3}$ is rational, then $\sqrt{3} = \frac{a}{b}$, and $a, b \in \mathbb{Z}$
3	Difficulty connecting ideas logically	7	Difficulty connecting ideas logically refers to the difficulty in making assumptions such as "if p^2 is divisible by 3, then p is also divisible by 3, for every integer p" as a basis for generating a result expression to obtain a contradiction.
4	Difficulty in refuting supposition	2	Difficulty in refuting suppositions refers to the inability to generate suppositions, due to incompleteness in constructing suppositions.

Table 1 above, shows that 7 subjects have difficulty understanding the steps of proof by contradiction, 16 subjects have difficulty in making complete assumptions, 7 subjects have difficulty connecting ideas logically, and 2 people have difficulty in refuting suppositions. It should be emphasized that this type of difficulty is based on the general structure of proof of contradiction, such as making assumptions, making assumptions, connecting ideas logically, and generating contradictions. Furthermore, from the 23 samples of proof answers, the following will describe the results of the analysis of difficulties in proof construction with proof by contradiction, and in the following description, only 3 samples of proof answers will be shown, in which the three proof samples have difficulties in proof construction.

a. The subject of proof I (DWH)

Figure 1 shows the results of constructing the proof by DWH subjects in constructing the proposition "$\sqrt{3}$ is an irrational number" are still incomplete. The subject does not make a definition of the assumptions create so it is not enough to be the basis for the assumption that a contradiction has been found. It can be seen that the subject still does not understand the steps to construct proof by contradiction, as shown in Figure 1.

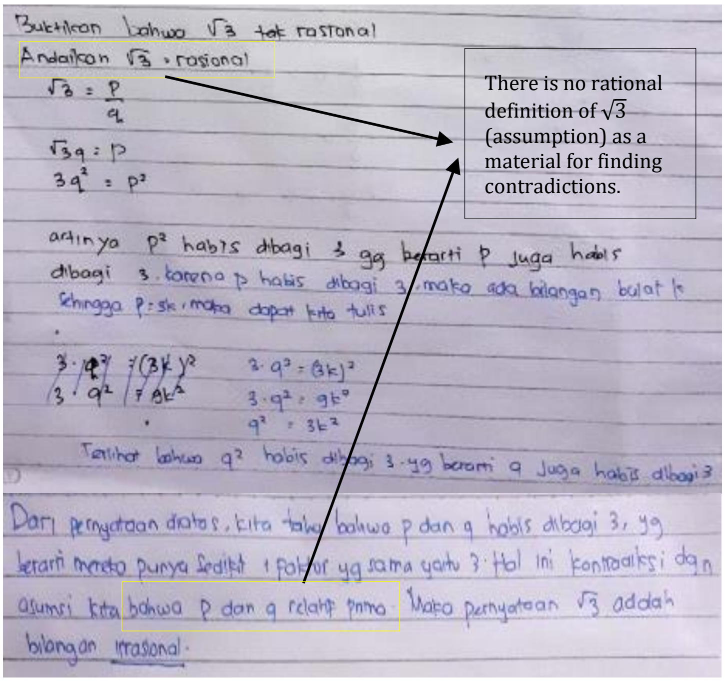

Figure 1. Proof construction from DWH subject

To clarify the results of the construction in Figure 1 above. In the following, the results of the construction by the DWH subject will be rewritten using English.

Prove that $\sqrt{3}$ *irrational*

Proof.

Suppose that $\sqrt{3}$ *rational*

$$\sqrt{3} = \frac{p}{q}$$

$$\sqrt{3}q = p$$

$$3q^2 = p^2$$

This means that p^2 *is divisible by* 3, *which means that* p *is also divisible by* 3. *Since* p *is divisible by* 3, *then there are integers* k *so that* $p = 3k$, *so we can write.*

$$3q^2 = (3k)^2$$
$$3q^2 = 9k^2$$
$$q^2 = 3k^2$$

It can be seen that q^2 *is divisible by* 3 *which means that* q *is also divisible by* 3.
From the statement above, we know that p *and* q *are divisible by* 3, *which means they have at least one factor in common, which is* 3. *This contradicts our assumption that* p *and* q *are relatively prime. Then the statement that* $\sqrt{3}$ *is an irrational number.*

The results of the DWH subject's proof construction in Figure 1 above, show that there are two deficiencies in the proof construction, which results in the subject having no basis in denying the assumption to conclude that the claim is true. The first drawback, the subject does not provide complete information on the assumption that $\sqrt{3}$ is a rational number, for example by providing explanations such as: suppose $\sqrt{3}$ is a rational number, then there are integers p and q; and $q \neq 0$, so $\sqrt{3} = \frac{p}{q}$. The rational form $\frac{p}{q}$ is the smallest form, so p and q are relatively prime. Then from the result expression $p^2 = 3q^2$ we get $p = 3k$ with k integer elements (by assuming that "p^2 is divisible by 3, then p is also divisible by 3"), so $q^2 = 3k^2$. The result of $q^2 = 3k^2$ the subject again assumes that "q^2 is divisible by 3, then q is also divisible by 3" but does not determine the value of q?. From these two assumptions, the subject concludes that p and q have at least 1 prime factor, namely 3. However, there is no basis for giving rise to a contradiction because it has not assumed that p and q are relatively prime to justify $\sqrt{3}$ is a rational number. Then from the description of this analysis, it can be concluded that DWH subjects experienced three types of difficulties, namely difficulties in 1) making complete assumptions, 2) connecting ideas logically (if $q^2 = 3k^2$, then $q = \cdots ?$), and 3) difficulty in refuting suppositions (having no basis for suppositions and being unable to distinguish between suppositions and assumptions).

b. The subject of Proof II (SDR)
Figure 2 shows the results of the construction of the proof for the proposition $\sqrt{3}$ is an irrational number by the SDR subject using the method of proof by contradiction. The results of the construction show that the subject does not make a supposition about the proposed proposition has difficulty defining the supposition, is unable to construct a supposition, and has difficulty finding contradictions as material to refute supposition**,** as shown in Figure 2.

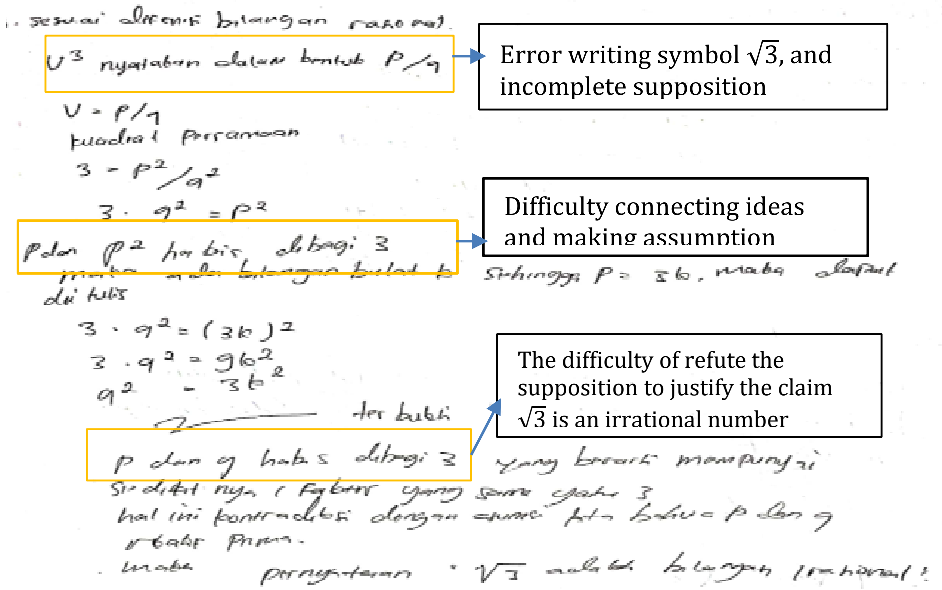

Figure 2. Proof construction from SDR subject

Clarifying the results of the SDR subject construction in Figure 2 above. In the following, the results of the construction will be rewritten in English.

Problem: Prove that $\sqrt{3}$ *irrational*

Proof.

According to the definition of a rational number

$\sqrt{3}$ *expressed in the form* $\frac{p}{q}$

$$\sqrt{3} = \frac{p}{q}$$

Square the equation

$$3 = \frac{p^2}{q^2}$$

$$3q^2 = p^2$$

p and p^2 *divisible by* 3

then there are integer k such that $p = 3k$, *then it can be written*

$$3q^2 = (3k)^2$$

$$3q^2 = 9k^2$$

$$q^2 = 3k^2$$

p and q are divisible by 3 which means they have at least 1 factor in common, namely 3. This contradicts our assumption that p and q are relatively prime. then statement $\sqrt{3}$ *is an irrational number.*

The results of the construction of the proof above, it is assumed that the subject has difficulty in making a complete supposition that $\sqrt{3}$ is a rational number, followed by not defining of $\sqrt{3} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, and $GCD(p, q) = 1$. The command "square of the equation" $\sqrt{3} = \frac{p}{q}$ to produce $3q^2 = p^2$ is an incorrect command. The emergence of the assumption that p and p^2 is divisible by 3 based on the expression $3q^2 = p^2$ does not have a logical basis for a deduction, however, based on this assumption, $p = 3k$ with $k \in \mathbb{Z}$ is obtained so that we get $q^2 = 3k^2$. With the result expression $q^2 = 3k^2$, assume that the claim has been proven. But again, convincingly assuming that p and q are divisible by 3 means they have at least 1 factor in common, namely 3, and states that this result contradicts the assumption that p and q are relatively prime. However, the subject has no basis for refuting the supposition to conclude that 3 is irrational. From a series of written arguments, it shows that the subject has 3 types of difficulties, namely: difficulty in (1) making complete suppositions and correct assumptions; (2) connecting logically connected ideas; and (3) refuting suppositions.

c. The subject of Proof III (ABS)

The result of the construction of the proposition $\sqrt{3}$ is that the irrational number in Figure 3 is given by the ABS subject. The result of the construction is proof using examples, or in other words, the subject does not understand the step of proof by contradiction, as shown in Figure 3.

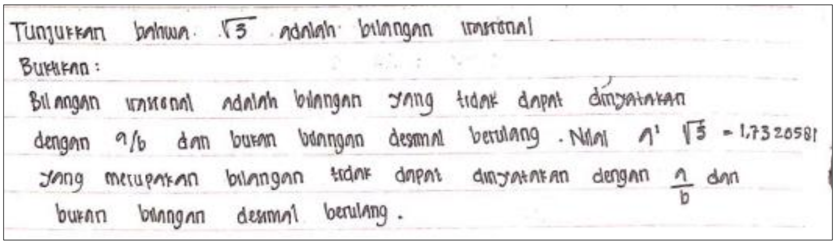

Figure 3. Answer constructions from ABS subject

Clarifying the results of the ABS subject construction in Figure 3 above. In the following, the results of the construction will be rewritten in English.

Problem: Show that $\sqrt{3}$ *irrational number*

Proof.

An irrational number is a number that cannot be represented by $\frac{a}{b}$ and is a repeating decimal. Value $\sqrt{3} = 1{,}7320581$ which is a number that cannot be expressed by $\frac{a}{b}$ and is not a repeating decimal number.

The proof given by the ABS subject cannot be said to be formal proof, because it only uses definitions and is accompanied by examples to show the irrationality of $\sqrt{3}$. So, it can be said that ABS subjects have difficulty using formal proof or contradiction deduction steps.

2. Development of ACERA model learning

Proof is an important topic of interest in the discipline of mathematics and its study in the last few decades (Stylianides & Stylianides, 2017; Miller et al., 2018; Davies et al., 2020; Davies & Jones, 2022). However, proof teaching and learning is an instructional area that is quite difficult to teach and learn at all levels of education (Barnard, 2000; Weber, 2003; Shaker & Berger, 2016; Güler, 2016; Miyazaki et al., 2017; Ioannou, 2017; Antonini & Mariotti, 2006; Quarfoot & Rabin, 2021; Rabin & Quarfoot, 2021). Based on the results of proof construction which states that there are difficulties experienced by students who are involved in the construction of proof with proof by contradiction, it can be a basis for developing models that have a potential effect on reducing and increasing students' abilities in constructing proof with a proof approach such as indirect proof, either contradiction or contradiction. The model offered here is the ACERA learning model, which is a model that is activated in activities, class discussions, exercises, giving reasons, and validating the audience.

This learning model develops ACE which was previously developed by Dubinsky (Dubinsky, 2001). The idea of developing this model is based on the statement that mathematical proof requires validation from mathematicians or the mathematical community (Reid, 2001). Learning to proof construction is not enough with independent study activities, class discussions, and exercises. However, learning to proof construction requires efforts to provide reasons and validation from experts (Selden & Selden, 2003). With this idea, the idea arose to develop the ACERA learning model, where the R stands for the ACERA acronym, referring to the two-column proof that is commonly found in proofs of geometric concepts, and is intended to organize various thoughts related to the proof problems encountered (Pair et al., 2021). Then the letter A, refers to the audience's consideration Ashton (2021) of the construction results of using proof. The audience comes from mathematicians or lecturers or researchers or colleagues, which in this case is the audience. Each orientation of this model is a learning step that has coherent work indicators as described below.

a. Activities (A)
   The activities step refers to concept map activities, for example preparing a discussion of the proof material to be delivered. So, in this step, the teacher's activities are:
   1) Develop a lesson plan,
   2) Create teaching materials that contain proof approach content, such as direct proof, indirect proof (both contraposition and contradiction), and other approaches,
   3) Providing presentation material that requires students to be able to recognize and understand several approaches to proof, for example: (a) With direct proof. Prove that if n is an even number, then n^2 is an even number; (b) With proof by contraposition. Prove that if k is an integer and k^2 is divisible by 3, then k is also divisible by 3; and (4) With proof by contradiction. Prove that 2 is an irrational number.
   4) Develop assessment instruments for each proof approach.

b. Classroom Discussion (C)
   Steps for classroom discussion. Refers to the social context where students can work in groups to solve the proving questions posed by the facilitator/researcher, as well as facilitate discussion of the problems that arise in the activities stage. So, in this step, the teacher's activities are:
   1) Establish discussion groups, taking into account the types of difficulties experienced by each student.
   2) Students work in groups to solve the following proving problems:
      a) With direct proof. (1) Prove that if p is an integer and p is not divisible by 2, then p^2 is also not divisible by 2; and (2) Prove that if n is an integer and n^2 is an even

number, then n is an even number.

b) With proof by contraposition. (1) Prove that if n is an integer and n^2 is an odd number, then n is an odd number; and (2) Prove that if k is an integer and $9k^2 + 6k + 1$ is not a multiple of 3, then $3k + 1$ is also not a multiple of 3.

c) With proof by contradiction. (1) Prove that 3 is an irrational number; and (2) Prove that if k is an integer and k^2 is not divisible by 3, then k is also not divisible by 3.

3) Provide facilitation/assistance to groups who have difficulty in answering questions of proof,
4) Conduct a structured evaluation of the results at the class discussion stage and the learning activity stage. In discussions in proving a proposition with several proof approaches, students can know when an approach can be used to justify a certain mathematical claim because in class discussions students will connect and communicate ideas to build new ideas.

c. Exercises (E)

Exercises steps refer to the activity of doing exercises or learning new proof concepts in proof based on group discussions. Exercise in the form of group assignments outside the classroom. So that teachers need to arrange practice questions by paying attention to understanding the steps of activity and group discussions and demanding that students be able to elaborate on their understanding of the proof approach. The instrument that can be used as practice questions is to prove the following proposition.

1) If n is an odd number, then n^2 is an odd number,
2) If a is an integer and a^2 is a multiple of 3, then a is also a multiple of 3,
3) $\sqrt{6}$ is an irrational number,
4) $\sqrt{n}$ is an irrational number, where n is not a square number.

It should be noted that the *exercises* in this paper should not be seen as *drilling*, as they refer to different problem-based or experience-based tasks, which are spread over different class periods, and each lead to the same basic ideas. Meanwhile, *drill* refers to repetitive, non-question-based exercises designed to improve acquired skills or procedures.

d. Reason (R)

Refers to giving individual reasons for one of the arguments given in practice questions using a two-column proof format (a two-column proof). Giving reasons (reasons) with two-column evidence is intended to organize various thoughts related to the problem of proof faced. A two-column proof is an arrangement that provides one column for known things (arguments) or a list of mathematical statements, and the next column for supporting evidence (reasons stating that the statement is true). So that each subjects answer will be reconfirmed (think aloud) at the reason step using a two-column proof format, as in the following table example, as shown in Table 2.

Table 2. A two-column proof from $\sqrt{6}$ is an irrational number

Statements/arguments	Reasons
$\sqrt{6}$ is an irrational number	Given
Suppose $\sqrt{6}$ is an irrational number, then there is $p, q \in \mathbb{Z}$ with $q \neq 0$, such that $\frac{p}{q} = \sqrt{6}$	Definition of rational number: for every $p, q \in \mathbb{Z}$ and $q \neq 0$, then $\frac{p}{q}$ is a rational number, and is usually denoted by the symbols $\mathbb{Q} = \left\{\frac{p}{q} \middle\| p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
The rational form $\frac{p}{q}$ is the smallest form, so p and q are relatively prime	The definition of the greatest common divisor of two integers p and q, both of which are non-zero, is the largest positive integer d, such that d is divisible by p and q. The greatest common factors p and q are generally denoted by $GCD(p, q)$. So that the smallest rational form can be symbolized by $\mathbb{Q} = \left\{\frac{p}{q} \middle\| p, q \in \mathbb{Z} \text{ dan } q \neq 0; GCD(p, q) = 1\right\}$
$\frac{p}{q} = \sqrt{6} \Leftrightarrow p^2 = 6q^2$	Squaring both sides (two sides)
p^2 is a multiple of 6	The properties divisibility of the number 6: If p^2 is a multiple of 6 then there is an integer q, such that $p^2 = 6q^2$
If p^2 is a multiple of 6, then p is a multiple of 6	Assumptions that can be justified by proof by contrapositive
Since p is a multiple of 6, it means that there is an integer k such that $p = 6k$	The consequence of p is a multiple of 6
$(6k)^2 = 6q^2 \Leftrightarrow q^2 = 6k^2$ So, q^2 is a multiple of 6	Properties divisibility of the number 6
If q^2 is a multiple of 6, then q is a multiple of 6	Assumptions that can be justified by proof by contrapositive
Since q is a multiple of 6, it means that there is an integer l such that $q = 6l$	Consequently, q is a multiple of 6
Since $p = 6k$ and $q = 6l$, it means that p and q are not relatively prime. It contradicts that p and q are relatively prime.	$GCD(6k, 6l) \neq 1$
So, the supposition $\sqrt{6}$ is a rational number is wrong. Q.E.D	Refute supposition

e. Audience (A)

Refers to the validation/correction of mathematical arguments that have been constructed based on the two-column proof format in the reason step (giving reasons). Audiences come from peer students in groups or outside groups, as well as from lecturers/researchers themselves. In this step, the audience will validate/correct the argument and will provide an evaluation of the evidence argument that is not yet valid. In addition, in validating the proof of the argument, the audience will read and reflect on each argument to determine the truth of the claims given. In other words, the audience will determine whether the sequence of arguments has the correct characteristics of proof.

Based on the five steps of the ACERA learning model (activities, class discussions, exercises, giving reasons, and audience validation), class discussions (classroom discussions, exercises (exercise), giving reasons (reasons), and audience correction are a series of activities. These five steps can be described by the following cycle in Figure 4.

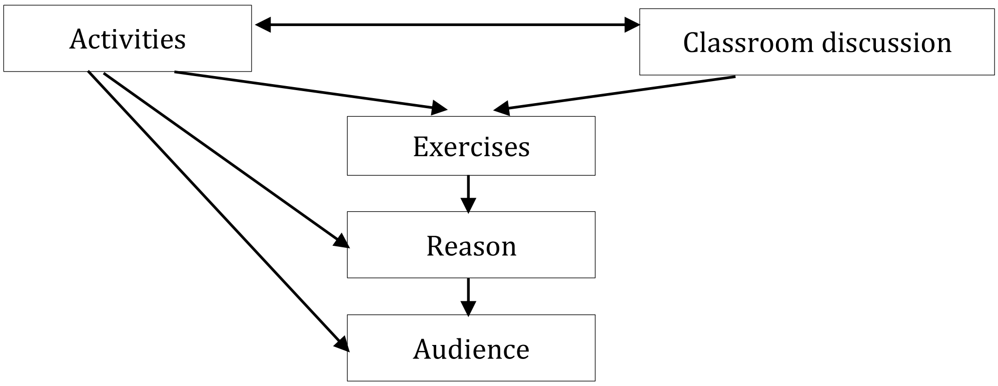

Figure 4. Cycle of ACERA Learning Model

Concerning the *activities* step, students are required to understand several types of proof approaches, especially when that approach can be used to show a certain claim. This *first* activity is in the form of understanding the presentation from the facilitator regarding the introductory proof material, then the *second* is discussing and working in groups to build ideas or cover disparities/gaps that have not been fully understood in the first activity, then the *third* work to solve the problem of proof in practice questions, the *fourth* chose one of the practice questions to be done individually, then worked on and finally the *fifth* corrected the answers of colleagues. Or in other words, the activity refers to the concept map/cycle activity of the ACERA learning model.

The second part of this cycle is class discussion, in which students work in groups to solve proof problems. This class discussion takes place in several meetings, so the questions that are worked on in each class discussion are different from the questions in the next discussion. The questions worked on in this discussion require students to work collaboratively to build ideas or cover the disparity of ideas that each subject has in the class group so that it is hoped that each group member has an understanding that is considered the same in terms of proof construction ideas. And this class discussion also covers understanding problems that arise at the activity stage. The purpose of this class discussion is to develop (1) an understanding of when a proof approach can be used to justify a particular mathematical claim; (2) collaborative activities in connecting and communicating ideas to build new ideas; (3) the habit of giving an argument against a particular claim; (4) assess whether and why our answers or peers are logical; and (5) opportunities to reflect on their work, such as providing definitions, offering explanations, and/or presenting an overview to bring together what colleagues have thought and done (Borji et al., 2018).

Group exercise outside the classroom is the third part of the cycle, which consists of practice questions in the form of propositions that must be proven using the appropriate approach. The appropriate evidence approach is intended because not all evidence approaches can be used, if they can be used, students will have difficulty demonstrating it. This is under the think Brown (2018), who said that "when students are given two proofs of the same theorem, students prefer the direct approximation to some theorems and the indirect approximation to other theorems. This illustrates that habit is a criterion that students bring to produce proof, when before choosing to consider the type of proof that is most convincing. This part of the cycle consists of 4 propositions (possibly more) that must be proven by an appropriate proof approach and are designed to strengthen class activities and discussions. Exercises help to support the continued development of mental constructs, to apply what has been understood, and to consider related mathematical ideas (Dubinsky et al., 2013). So in compiling practice questions it is necessary to pay attention to the ability to elaborate on understanding several proof approaches. Exercise should not be seen as drilling, because the exercise refers to tasks that demand different experiences, each of which leads to the same basic ideas. While Drill refers to repeated exercises, demanding the same experience because it is designed to improve procedural abilities (Van de Walle et al., 2020).

Giving reasons is the fourth cycle part of this model. Reason refers to giving reasons for each argument that gives to be able to attract a claim using the format of a two-column proof. The format of these two columns is good for organizing the variety of thoughts related to proof. It also answers the function of what evidence and why our answers are logical NCTM (2000) because the two columns of the two columns proof consist of a list of statements (left column) and the reasons why we know the statement is true (right column). In addition, a two-column proof can be an explanation of the claim structure/command that underlies all the proof and can be a useful tool for highlighting the underlying structure of proof (Pair et al., 2021). So that way, giving reasons in the format of a two-column proof can practice communication skills, as well as training in introducing/ expressing ideas, especially in propositions that require the ability to proof construction with the contradiction method that has been alleged more difficult than direct proof (Antonini & Mariotti, 2008).

Finally, the audience step is the fifth or the last cycle of this model. This last section contains the validation or correction of the argument/statement as well as the reasons underlying the argument based on the format of a two-column proof by an audience. In this case, the audience can come from peer students, lecturers/researchers, or mathematics and have access/authority to determine whether the argument sequence has the true characteristics of proof. Validation of this audience is based on mathematical proof in the form of arguments aimed at justifying or refuting a mathematical claim Stylianides (2007) and mathematicians develop proof by considering their universal audience (Ashton, 2021). Validation of proof by this audience (proof validation) refers to the reading and reflection of the proof effort to determine the truth of a claim (Selden & Selden, 2017). So that the standard of reasonableness of a proud that reflects the understanding of a method/approach used is realized by the validation of the universal audience (colleagues, lecturers, researchers, or mathematicians). Further validation or correction of the audience to the proud will help explain how the proud standards interact, and determine what methods of proof can be received. In other words, a comprehensive understanding of mathematical proof and accuracy requires further investigation from the audience that is intended because validation is a task that demands cognitive (Selden & Selden, 2003).

This model was developed by identifying students' difficulties in constructing of proof, then analysing the causes of difficulties such as understanding the problem of proof, methods of proof, steps of proof, learning activities of proof and proving, class discussions that involve activities of constructing of proof, experience practicing solving proof problems with various methods of proof, specific proof by contradictions, how to present construction results, and peer assessment of construction results.

The identification of constructing difficulties showed that of the 23 students involved in constructing of proof, there were 7 students who had difficulty understanding the step of proof by contradiction, 16 students had difficulty making supposition, 7 students had difficulty connecting ideas logically, and 2 students had difficulty refuting supposition. Then the analysis of the causes of difficulties in constructing of proof shows the importance of emphasizing transitional learning activities towards proof, class discussions in constructing of proof, experience in practicing constructing of proof, how to present construction results, and assessing peers to convince the mathematical community of the construction results. The results help with the analysis of the causes of the difficulty in constructing this proof, initiating a model that can accommodate learning activities, class discussions, experiences solving proof problems, how to present construction results, and peer-review to convince the mathematical community that construction results can be difficult proofs.

The model that was initiated was the ACERA model, a model consisting of a combination of learning activities (A), class discussions (C), exercises (E), reasoning (R), and peer assessment (A). This model has not been tested, but it is believed that it can offer alternative solutions to reduce difficulties in constructing of proof, thus enabling this model to have valid, practical characteristics and have potential effects in increasing the productivity of student proof construction. It is believed that each letter composing the ACERA acronym is an activity that can support student success in constructing of proof. For example Activity (A), learning activities towards proof require understanding, and when constructing student understanding it will increase, then with class discussions (C) can build ideas, ideas that are initially difficult to construct will be easy to build with collaborative efforts, coupled with solving experiences proving problems (E) will enrich the proof scheme in students' cognitive, not enough with learning activities (A), class discussions (C), and exercises solving proof problems (E), it turns out how to present the results of construction or give reasons (R) for constructs/proof is still needed, because proof is a logical argument that functions to justify or convince the mathematical community, in this case the mathematical community is the participant or audience (A).

D. CONCLUSION AND SUGGESTIONS

Based on the results of the identification of difficulties in constructing of proof, and an analysis of the causes of difficulties in constructing of proof, it can become the basis for developing the ACERA model. The theoretical series of each part of this model cycle, provides positive hope for reducing difficulties in proof construction, so that it is possible to have criteria that are valid, practical, and have a potential effect in increasing the productivity of proof construction, providing a constructive view that constructing of proof is not something difficult. to be studied and constructed, and provide a theoretical basis for the construction results in the mental structure of students.

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