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The Clique Number and The Chromatics Number Of The Coprime Graph for The Generalized Quarternion Group

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ABSTRACT

Graph theory can give a representation of abstract mathematical systems such as groups or rings. We have many graph representations for a group, in this study we use the coprime graph representation for a generalized quaternion group to find the numerical invariants of the graph, which are the clique number and the chromatic number. The main results obtained from this study are the clique number of the coprime graph representation for the generalized quaternion group is equal to the chromatic number of the coprime graph representation for the generalized quaternion group for each case of the order.

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A. INTRODUCTION

Graph theory is a useful tool to describe a real-world problem as a mathematics problem such as the in schedulling problem, in chemical graph topological indices, Jahandideh et al. (2015) or in atom bond connectivity index Hua et al. (2019), so the solution can be found easily. In recent years, a graph can give a representation of abstract mathematical systems such as groups or rings (Zavarnitsine, 2006). With graphs, we can give meaning to groups or rings, such as the visualization of groups or rings and we can define the distance between the elements of groups or rings. We have many graph representations for a group, such as the coprime graph Alimon et al. (2020), the non-coprime graph Mansoori et al. (2016), the intersection graph Akbari et al. (2015), and the power graph (Aşkin & Büyükköse, 2021).

There have been several studies regarding coprime graphs from finite groups such as coprime graphs and non-coprime graphs from the generalized quaternion group Nurhabibah et al. (2021), the dihedral group Syarifudin et al. (2021), the integer modulo group Series & Science (2021) and representation of non-coprime graphs from an integer modulo (Misuki et al., 2021) and non-coprime graph of the generalized quaternion group (Nurhabibah et al., 2022). Other popular studies in graph representation are the intersection graph for the dihedral group Ramdani et al. (2022), the prime graph Satyanarayana (2010) and the power graph for the dihedral group Asmarani et al. (2021) and integer modulo group (Syechah et al., 2022). Based on the study of the coprime graph on generalized quaternion group and the search for clique number and chromatic number (Husni et al., 2022) it can be analyzed further about the properties of the graph. In this study, the authors analyze the clique number and chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}).

B. METHODS

In this study, the authors searched various literature related to generalized quaternion groups, coprime graphs, as well as clique numbers, and chromatic numbers. Then analyze several examples so that a certain pattern is obtained which is then expressed as a conjecture. The conjecture is then proved to obtain its truth value. If the conjecture proves to be true, then it is stated as a theorem and if not, then the writer will construct a new conjecture until the correct conjecture is obtained.

C. RESULT AND DISCUSSION

In this research, the writer determines the clique number and chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}). The generalized quaternion group (Q_{4n}) is one of the finite groups with the following definition.

Definition 1 The generalized quaternion group (Q_{4n}) is a $4n$ order group composed of two elements (a, b) or can be written $Q_{4n} = \{a, b | b^2 = a^n, a^{2n} = e, bab^{-1} = a^{-1}\}$ where e is the identity element with $n \geq 2$. This group can be represented in several types of graphs, one of which is a coprime graph.

Suppose G is a finite group, the coprime graph of G is denoted by $\Gamma_{Q_{4n}}$ are vertices with elements of G and two different vertices x and y are adjacent if and only if $(|x|.|y|) = 1$ (Ma et al., 2014). Some results regarding the coprime graph of the generalized quaternion group (Q_{4n}) are given in the following 4 theorems (Nurhabibah et al., 2021).

Theorem 1 Suppose Q_{4n} is a generalized quaternion group, if $n = 2^k$ then the coprime graph of Q_{4n} is completely bipartite.

Theorem 2 Suppose Q_{4n} is a generalized quaternion group, if $n = p$ with p odd primes, then the coprime graph of Q_{4n} is tripartite (Nurhabibah et al., 2021).

Theorem 3 Suppose Q_{4n} is a generalized quaternion group, if $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$ then the coprime graph of Q_{4n} is $m + 1 - partite$ (Nurhabibah et al., 2021).

Theorem 4 Suppose Q_{4n} is a generalized quaternion group, if $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$ with $p_i \neq 2$, then the coprime graph of Q_{4n} is $m + 2 - partite$. The first discussion in this study is the clique number of the coprime graph of the generalized quaternion group (Q_{4n}) (Nurhabibah et al., 2022).

1. Clique Number

The clique number is one of the properties of a graph which is defined based on the complete subgraphs in the graph. A formal definition of the clique number is given in Definition 2 below.

Definition 2 The clique number $\omega(G)$ of graph G is the maximum order among complete subgraphs in G (Syarifudin et al., 2021).

The definition of a complete graph is contained in Definition 3 as follows

Definition 3 A complete graph is a simple graph in which every vertex has edges to all other vertices (Nurhabibah et al., 2022). A complete graph with n vertices is denoted by K_n.

The following is an example of the clique number of the coprime graph of the generalized quaternion group (Q_{4n}).

Example 1

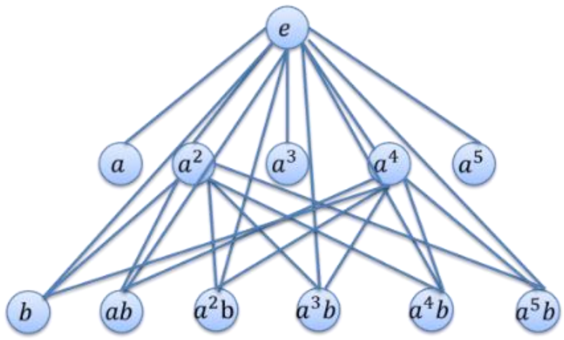

Figure 1. Examples Images a Coprime Graph for $Q_{4.3}$

The graph in figure 1 has several complete subgraphs and the maximum order of these complete subgraphs is 3, so based on Definition 2, $\omega(\Gamma_{Q_{4.3}}) = 3$. With an analysis similar to Example 1, the author derives several theorems about the clique numbers of the coprime graph of the group generalized quaternion.

Theorem 5 Let $\Gamma_{Q_{4n}}$ be a coprime graph of Q_{4n}. If $n = 2^k$ with $k \in \mathbb{N}$ then $\omega\ (\Gamma_{Q_{4n}}) = 2$.

Proof: Let $\Gamma_{Q_{4n}}$ be a coprime graph. Take $n = 2^k$ with $k \in \mathbb{N}$. We will show $\omega(\Gamma_{Q_{4n}}) = 2$. This means that it will be shown that there is K_2 which is a complete subgraph $\Gamma_{Q_{4n}}$ and there is no complete subgraph K_z with $z > 2$. The author obtains a complete subgraph K_2 from $\Gamma_{Q_{4n}}$ which is a graph with $V(K_2) = \{e, v_1\} \in Q_{4n}$ with $v_1 \in Q_{4n} \backslash \{e\}$. Suppose that there is a K_z complete subgraph of $\Gamma_{Q_{4n}}$ with $z > 2$. This means that $\Gamma_{Q_{4n}}$ must be a $k - partite$ graph with $k > 2$. This contradicts Theorem 1 which states that $\Gamma_{Q_{4n}}$ is a complete bipartite graph. So K_2 is a complete subgraph of $\Gamma_{Q_{4n}}$ of maximal order. It is proved that $\omega(\Gamma_{Q_{4n}}) = 2$. ■

Theorem 6 below is the clique number of the coprime graph of the group generalized quaternion (Q_{4n}) with $n = p$ where p is an odd prime number.

Theorem 6 Let $\Gamma_{Q_{4n}}$ be a coprime graph of Q_{4n}. If $n = p$ with p odd primes then $\omega(\Gamma_{Q_{4n}}) = 3$.

Proof: Let $\Gamma_{Q_{4n}}$ be a coprime graph. Take $n = p$ with p odd primes. It will show $\omega(\Gamma_{Q_{4n}}) = 3$. This means that it will be shown that there is K_3 which is a complete subgraph $\Gamma_{Q_{4n}}$ and there is no complete subgraph K_z with $z > 3$. The author obtains a complete subgraph K_3 from $\Gamma_{Q_{4n}}$ which is a graph with $V(K_3) = \{ e, v_1, v_2 \}$ ⍰ Q_{4n} with $v_1 \in P_1$ where P_1 is the set of all vertices in $\Gamma_{Q_{4n}}$ of order p and $v_2 \in P_2$ where P_2 is the set of all vertices in $\Gamma_{Q_{4n}}$with even order. Suppose that there is a K_z complete subgraph of $\Gamma_{Q_{4n}}$ with $z > 3$. This means that $\Gamma_{Q_{4n}}$ must be a $k - partite$ graph with $k > 3$. This contradicts Theorem 2 which states $\Gamma_{Q_{4n}}$ is a tripartite graph. So K_3 is a complete subgraph of Γ_{Q4n} with maximum order, it is proved that $\omega(\Gamma_{Q4n}) = 3$.

Theorem 7 Let Γ_{Q4n} be a coprime graph of Q_{4n}. If $n = p_1^{k1}p_2^{k2}p_3^{k3} \dots \dots p_m^{km}$, then $\omega(\Gamma_{Q4n}) = m + 1$.

Proof: Let $\Gamma_{Q_{4n}}$ be a coprime graph. Take $n = p_1^{k1}p_2^{k2}p_3^{k3} \dots \dots p_m^{km}$. It will show $\omega(\Gamma_{Q_{4n}}) = m + 1$. This means that it will be shown that there exists K_{m+1} which is a complete subgraph of $\Gamma_{Q_{4n}}$ and there is no complete subgraph K_z with $z > m + 1$. The writer obtains a complete subgraph K_{m+1} from $\Gamma_{Q_{4n}}$ is a graph with $V(K_{m+1}) = \{e, v_1, v_2, \dots., v_m\}$ ⍰ Q_{4n} with $v_i \in P_i$ for $i = 1, \dots \dots .., m$. Suppose that there are K_z complete subgraphs of $\Gamma_{Q_{4n}}$ with $z > m + 1$. This means that $\Gamma_{Q_{4n}}$ must be a $k - partite$ graph with $k > m + 1$. This contradicts Theorem 3 which states that $\Gamma_{Q_{4n}}$ is a $m + 1 - partite$ graph. So K_{m+1} is a complete subgraph of $\Gamma_{Q_{4n}}$ with maximal order, it is proved that $\omega(\Gamma_{Q_{4n}}) = m + 1$. ■

Theorem 8 below is the clique number of the coprime graph of the group generalized quaternion (Q_{4n}) with $n = p_1^{k1}p_2^{k2}p_3^{k3} \dots \dots p_m^{km}$ with $p \neq 2$

Theorem 8 Let Γ_{Q4n} be a coprime graph of Q_{4n}. If $n = p_1^{k1}p_2^{k2}p_3^{k3} \dots \dots p_m^{km}$ with $p \neq 2$, then $\omega(\Gamma_{Q_{4n}}) = m + 2$.

Proof: Let Γ_{Q4n} be a coprime graph. Take $n = p_1^{k1}p_2^{k2}p_3^{k3} \dots \dots p_m^{km}$ with $p_i \neq 2$. Will show $\omega(\Gamma_{Q_{4n}}) = m + 2$. This means that it will be shown that there exists K_{m+2} which is a complete subgraph of $\Gamma_{Q_{4n}}$ and there is no complete subgraph K_z with $z > m + 2$. The writer obtains a complete subgraph K_{m+2} from $\Gamma_{Q_{4n}}$ is a graph with $V(K_{m+2}) = \{e, u, v_1, v_2, \dots., v_m\} \in Q_{4n}$ with $u \in R$ where R is the set of all vertices in $\Gamma_{Q_{4n}}$ with even order and $v_i \in P_i$. Suppose that there are K_z complete subgraphs of $\Gamma_{Q_{4n}}$ with $z > m + 2$. This means that $\Gamma_{Q_{4n}}$ must be a $k - partite$ graph with $k > m + 2$. This contradicts Theorem 4 which states $\Gamma_{Q_{4n}}$ $m + 2 - partite$ graph.

So K_{m+2} is a complete subgraph of $\Gamma_{Q_{4n}}$ with maximal order, it is shown that $\omega(\Gamma_{Q_{4n}}) = m + 2$. ■

The second discussion in this study is the chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}).

2. Chromatic Number

Besides the clique number, the chromatic number is also one of the properties of a graph which is defined based on the coloring of the graph vertices. A formal definition of a chromatic number is given in Definition 4 below.

Definition 4 The chromatic number of a graph G is the minimum number of colors needed to color all the vertices of G such that every two neighboring vertices get a different color (Syarifudin, et al., 2021). The chromatic number of a graph G, denoted by $\chi(G)$.

The following is an example of the chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}).

Example 2: For example, for $n = 3$, the coprime graph of the generalized quaternion group (Q_{4n}) is as shown in Figure 2.

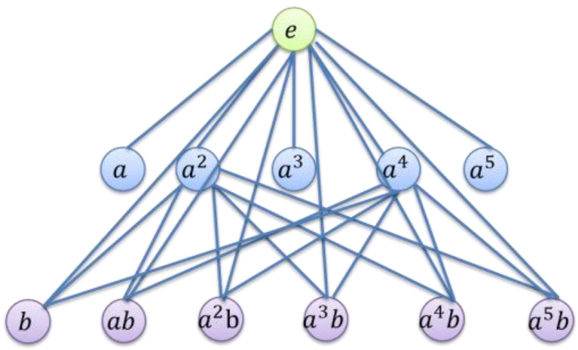

Figure 2. Examples of Images of Coloring Copprime Graph for $Q_{4.3}$

Based on the image, the minimum color needed to color the graph in figure 2 is 3, so $\chi(\Gamma_{Q_{12}}) = 3$. With more or less the same steps, the authors derive several theorems about the chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}).

Theorem 9 Let $\Gamma_{Q_{4n}}$ be a coprime graph of Q_{4n} . If $n = 2^k$ with $k \in N$ then $\chi(\Gamma_{Q_{4n}}) = 2$.

Proof: Let $\Gamma_{Q_{4n}}$ be a coprime graph. Take $n = 2^k$ with $k \in N$. It will be shown that $\chi(\Gamma_{Q_{4n}}) = 2$. This means that it will be shown that the minimum number of colors needed to color the vertices of $\Gamma_{Q_{4n}}$ so that every two neighboring vertices with different colors is 2. According to Theorem 1, $\Gamma_{Q_{4n}}$ is a complete bipartite graph, namely a star graph. The partition consists of 2 subsets, namely $P_1 = \{e\}, P_2 = \{Q_{4n}\} \setminus \{e\}$. Notice that node e is next to every node in P_2. So that every vertex in P_2 cannot have the same color as $\{e\}$. Also, note that none of the vertices in

P_2 are adjacent to each other so that the color of each vertex in P_2 is the same. So, the minimum number of colors required is 2. It is proved that $\chi(\Gamma_{Q_{4n}}) = 2$. ■

Theorem 10 Let $\Gamma_{Q_{4n}}$ be a coprime graph of Q_{4n}. If $n = p$ with p odd prime numbers. Then $\chi(\Gamma_{Q_{4n}}) = 3$.

Proof: Let $\Gamma_{Q_{4n}}$ be a coprime graph. Take $n = p$ with p odd primes. It will show $\chi(\Gamma_{Q_{4n}}) = 3$. This means that it will be shown that the minimum number of colors needed to color the vertices of $\Gamma_{Q_{4n}}$ so that every two neighboring vertices with different colors is 3. According to Theorem 2, $\Gamma_{Q_{4n}}$ is a tripartite graph. The partition consists of 3 subsets, namely $\{e, v_1, v_2\}$ with $v_1 \in P_1$ where P_1 is the set of all vertices in $\Gamma_{Q_{4n}}$ with order p and $v_2 \in P_2$ where P_2 is the set of all vertices in $\Gamma_{Q_{4n}}$ with even order. Note that every vertex in P_1 and P_2 will be next door to $\{e\}$, so every vertex in P_1 and P_2 cannot be the same color as $\{e\}$. Also, note that there are nodes in P_1 that are neighbors to P_2 so the nodes in P_1 and P_2 must have different colors. So, the minimum color needed to color the graph is 3. ■

Theorem 11 below is the chromatic number of the coprime graph of the group generalized quaternion (Q_{4n}) with $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$.

Theorem 11 Let $\Gamma_{Q_{4n}}$ be a coprime graph of Q_{4n}. If $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$ then $\chi(\Gamma_{Q_{4n}}) = m + 1$.

Proof: Let $\Gamma_{Q_{4n}}$ be a coprime graph. Take $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$. We will show that $\chi(\Gamma_{Q_{4n}}) = m + 1$. According to Nurhabibah (Nurhabibah et al., 2021), $\Gamma_{Q_{4n}}$ is an $m + 1 - partite$ graph. That is, $V(\Gamma_{Q_{4n}})$ can be partitioned into $m + 1$ node sets. Suppose the partitions are $P_0, P_1, P_2, \dots \dots .., P_m$ with $P_0 = \{e\}$ and $P_i = \{v \in V(\Gamma_{Q_{4n}}) \, |p_i||v| \text{ and } p \nmid |v| \text{for } p < p_i \}$. Notice that node $\{e\}$ is next to every other node. Note also that in every set P_i for $i = 1,2, \dots \dots, m$ there is always $x \in P_i$ with $|x| = p_i$. As a result, it can be ascertained that for every $i \neq j \; \exists \, y \in P_j$ so that $(|x|, |y|) = 1$. Thus, $\deg(x) \geq m + 1$. This shows that the coloring of the vertices of this graph requires at least $m + 1$ colors. It is proved that $\chi(\Gamma_{Q_{4n}}) = m + 1$. ■

Theorem 12 below is the chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}) with $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$ with $p_i \neq 2$.

Theorem 12 Let $\Gamma_{Q_{4n}}$ be a coprime graph of Q_{4n}. If $n = p_1^{k1} p_2^{k2} p_3^{k3} \dots \dots p_m^{km}$ with $p \neq 2$, then $\chi(\Gamma_{Q4n}) = m + 2$. ■

Proof: Direct result of Theorem 11

D. CONCLUSION AND SUGGESTIONS

The clique number and chromatic number of the coprime graph of the generalized quaternion group (Q_{4n}) are $2,3, m + 1$, and $m + 2$. There are many open problem that can be brought up after this result, such as the topological indices of the coprime graph of the generalized quaternion group.

REFERENCES

Akbari, S., Heydari, F., & Maghasedi, M. (2015). The intersection graph of a group. *Journal of Algebra and Its Applications*, *14*(5). https://doi.org/10.1142/S0219498815500656

Alimon, N. I., Sarmin, N. H., & Erfanian, A. (2020). The Szeged and Wiener indices for coprime graph of dihedral groups. *AIP Conference Proceedings*, *2266*. https://doi.org/10.1063/5.0018270

Aşkin, V., & Büyükköse, Ş. (2021). The Wiener Index of an Undirected Power Graph. *Advances in Linear Algebra & Matrix Theory*, *11*(01), 21–29. https://doi.org/10.4236/alamt.2021.111003

Asmarani, E. Y., Syarifudin, A. G., Adhitya, G., Wardhana, W., & Switrayni, W. (2021). Eigen Mathematics Journal The Power Graph of a Dihedral Group. *Eigen Mathematics Journal*, *4*(2), 80–85. https://doi.org/10.29303/emj.v4i2.117

Hua, H., Das, K. C., & Wang, H. (2019). On atom-bond connectivity index of graphs. *Journal of Mathematical Analysis and Applications*, *479*(1), 1099–1114. https://doi.org/10.1016/j.jmaa.2019.06.069

Husni, M. N., Syafitri, H., Siboro, A. M., Syarifudin, A. G., Aini, Q., & Wardhana, I. G. A. W. (2022). The Harmonic Index And The Gutman Index Of Coprime Graph Of Integer Group Modulo With Order Of Prime Power. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *16*(3), 961–966. https://doi.org/10.30598/barekengvol16iss3pp961-966

Jahandideh, M., Sarmin, N. H., & Omer, S. M. S. (2015). The topological indices of non-commuting graph of a finite group. *International Journal of Pure and Applied Mathematics*, *105*(1), 27–38. https://doi.org/10.12732/ijpam.v105i1.4

Ma, X., Wei, H., & Yang, L. (2014). The Coprime graph of a group. *International Journal of Group Theory*, *3*(3), 13–23. https://doi.org/10.22108/ijgt.2014.4363

Mansoori, F., Erfanian, A., & Tolue, B. (2016). Non-coprime graph of a finite group. *AIP Conference Proceedings*, *1750*(June 2016). https://doi.org/10.1063/1.4954605

Misuki, W. U., Wardhana, G. A. W., & Switrayni, N. W. (2021). Some Characteristics of Prime Cyclic Ideal On Gaussian Integer Ring Modulo. *IOP Conference Series: Materials Science and Engineering*, *1115*(1), 012084. https://doi.org/10.1088/1757-899X/1115/1/012084

Nurhabibah, Malik, D. P., Syafitri, H., & Wardhana, I. G. A. W. (2022). Some results of the non-coprime graph of a generalized quaternion group for some n. *AIP Conference Proceedings*, *2641*(December 2022), 020001. https://doi.org/10.1063/5.0114975

Nurhabibah, N., Syarifudin, A. G., & Wardhana, I. G. A. W. (2021). Some Results of The Coprime Graph of a Generalized Quaternion Group Q_4n. *InPrime: Indonesian Journal of Pure and Applied Mathematics*, *3*(1), 29–33. https://doi.org/10.15408/inprime.v3i1.19670

Ramdani, D. S., Wardhana, I. G. A. W., & Awanis, Z. Y. (2022). The Intersection Graph Representation Of A Dihedral Group With Prime Order And Its Numerical Invariants. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *16*(3), 1013–1020. https://doi.org/10.30598/barekengvol16iss3pp1013-1020

Syarifudin, A. G., Adhitya, I. G., Wardhana, W., & Switrayni, N. W. (n.d.). *The Degree , Radius , and Diameter of Coprime Graph of Dihedral Group*.

Satyanarayana, B. (2010). *Prime Graph of a Ring dimension theory of associative rings view project problems for competitive EXAMS View project*. https://www.researchgate.net/publication/259007924

Series, I. O. P. C., & Science, M. (2021). *Some Characteristics of Prime Cyclic Ideal On Gaussian Integer Ring Some Characteristics of Prime Cyclic Ideal On Gaussian Integer Ring Modulo*. https://doi.org/10.1088/1757-899X/1115/1/012084

Syarifudin, A. G., Nurhabibah, Malik, D. P., & dan Wardhana, I. G. A. W. (2021). Some characterizatsion of coprime graph of dihedral group D2n. *Journal of Physics: Conference Series*, *1722*(1). https://doi.org/10.1088/1742-6596/1722/1/012051

Syarifudin, A. G., Wardhana, I. G. A. W., Switrayni, N. W., & Aini, Q. (2021). The Clique Numbers and Chromatic Numbers of The Coprime Graph of a Dihedral Group. *IOP Conference Series: Materials Science and Engineering*, *1115*(1), 012083. https://doi.org/10.1088/1757-899x/1115/1/012083

Syechah, B. N., Asmarani, E. Y., Syarifudin, A. G., Anggraeni, D. P., & Wardhana, I. G. A. W. W. (2022). Representasi Graf Pangkat Pada Grup Bilangan Bulat Modulo Berorde BilanganPrima. *Evolusi: Journal of Mathematics and Sciences*, *6*(2), 99–104.

Zavarnitsine, A. V. (2006). Recognition of finite groups by the prime graph. *Algebra and Logic*, *45*(4), 220–231. https://doi.org/10.1007/s10469-006-0020-9