JTAM (Jurnal Teori dan Aplikasi Matematika)
http://journal.ummat.ac.id/index.php/jtam
p-ISSN 2597-7512 | e-ISSN 2614-1175
Vol. 7, No. 3, July 2023, pp. 712-725

Prime Graph over Cartesian Product over Rings and Its Complement

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Article History:
Received : 05-05-2023
Revised : 09-06-2023
Accepted : 02-07-2023
Online : 18-07-2023

Keywords:
Prime graph;
Commutative ring;
Cartesian product.

ABSTRACT

Graph theory is a branch of algebra that is growing rapidly both in concept and application studies. This graph application can be used in chemistry, transportation, cryptographic problems, coding theory, design communication network, etc. There is currently a bridge between graphs and algebra, especially an algebraic structures, namely theory of graph algebra. One of researchs on graph algebra is a graph that formed by prime ring elements or called prime graph over ring R. The prime graph over commutative ring R $(PG(R))$ is a graph construction with set of vertices $V\big(PG(R)\big) = R$ and two vertices x and y are adjacent if satisfy $xRy = \{0\}$, for $x \neq y$. Girth is the shortest cycle length contains in $PG(R)$ or can be written $gr\big(PG(R)\big)$. Order in $PG(R)$ denoted by $\big|V\big(PG(R)\big)\big|$ and size in $PG(R)$ denoted by $\big|E\big(PG(R)\big)\big|$. In this paper, we discussed prime graph over cartesian product over rings $\mathbb{Z}_m \times \mathbb{Z}_n$ and its complement. We focused only for $m = p_1$, $n = p_2$ and $m = p_1$, $n = {p_2}^2$, where p_1 and p_2 are prime numbers. Then, we obtained some properties related to order and size, degree, and girth. We also observe some examples. Moreover, we found that a correction in the statement of (Pawar & Joshi, 2019) about the complement graph over prime graph over a ring and gave a counter example for that.

Crossref
https://doi.org/10.31764/jtam.v7i3.14987

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A. INTRODUCTION

One of branches of algebra is graph theory. The graph theory is growing rapidly both in concept and application studies. This graph application can be used in chemistry, transportation, cryptographic problems, coding theory, design communication network, thermodynamic characteristic of networks, modeling network intrusions, crystallography, etc. We can see application of graph in (Easttom, 2020; Gowda et al., 2021; Kumar & Kumar vats, 2020; N. Lakshmi Prasanna, 2014; Ye et al., 2015).

A graph G consist of a finite non empty set whose elements are called vertices $(V(G))$ and edges $\big(E(G)\big)$ such that the edges are identified by pairs of vertices (Axler & Ribet, 2008). A simple graph is a graph which containing no multiple edges and loop (Vasudev, 2006). A graph that containing no edge is trivial graph. A graph if exist at least one path is a connected graph, otherwise that graph is a disconnected graph. A vertex with zero degree called by isolated vertex (Vasudev, 2006). Girth of G $\big(gr(G)\big)$ is the shortest cycle length in G (Chartrand, 2016). The number of edge connecting with x is degree of a vertex x (Vasudev, 2006). Size of G is number of edge and order is number of vertex (Johnson, 2017). For result related to order, size, and girth of graph see (Chang & Lu, 2011; Erskine & Tuite, 2023; Gani & Begum, 2010; Jajcay et al., 2018; Jannesari, 2015; Potočnik & Vidali, 2019).

In this paper, order and size of graph G written by $|V(G)|$ and $|E(G)|$, respectively. In graph G, the vertices that are adjacent to a vertex x called by neighborhood of x (Chartrand, 2016). Prime graph over a ring ($PG(R)$) is a graph with the set of vertices constitutes of all elements of rings and two vertices form an edge if it multipled by that rings equal to zero set (Satyanarayana, 2010). Complement graph over a ring ($(PG(R)^c)$) is a graph which the set of vertices constitutes of elements of rings and two vertices are adjacent in $(PG(R)^c)$ if these vertices are not adjacent in $PG(R)$.

In this paper, we study a connection between algebra and graph theory. In 2010, (Satyanarayana, 2010) introduced a new concept about graph theory and algebra that is prime graph over a ring. (Satyanarayana, 2010) discussed $PG(\mathbb{Z}_p)$, for any prime p and some properties. (Rajendra et al., 2019) gives a corrected version of the theorem of (Satyanarayana, 2010) and gives some counter examples for that. Pawar & Joshi (2019) introduced a complement over prime graph over a ring $\mathbb{Z}_p$, for prime p and determined degree and girth of prime graph over a ring and the complement graph of that. In 2013, Kalita & Patra (2013) defining a prime graph of cartesian product of rings. (Satyanarayana et al., 2015) compared cartesian product of prime graph over ring and prime graph over cartesian product of rings. (Fatimah et al., 2023; Krisnawati et al., 2023) discussed some characteristics of prime graph of ring $\mathbb{Z}_m \times \mathbb{Z}_n$, for $m = p_1$, $n = p_2$, and $m = p_2$, $n = {p_2}^2$, where p_1 and p_2 are primes. Study of prime graph over a ring are expanded by (Joshi et al., 2018; Joshi & Pawar, 2020; Kalita et al., 2014; Kalita & Patra, 2021; Pawar & Joshi, 2017).

Based on some literature about prime graph over a ring, we know that the research on prime graph over ring is still relatively few. Moreover, the ring that studied is only focus on $\mathbb{Z}_n$. Therefore, in this paper, we study prime graph over ring where the ring is cartesian product of rings. We obtain some properties of prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement, for $m = p_1$, $n = p_2$, and $m = p_2$, $n = {p_2}^2$, where p_1 and p_2 are prime numbers. The properties of prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ such as order, size, degree, and girth. Also, some examples of prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement have been observed. This paper divided into four section. In section B, we explain the method used in this paper. In section C, we discuss the main result such as order, size, degree, and girth of prime graph over cartesian product over rings and its complement. In the last section, we conclude the main result in section C and we give some suggestions for the next research.

B. METHODS

This study uses the literature research method with reference to some related articles about prime graph over a ring. In particular, the analysis of finding the order and size, degree, and girth, it is tried first by examining the construction of small order and size graph, then expanded on the construction with a larger order and size. However, if we still have not found a suitable pattern, repeat the finding for the suitable pattern from the graph construction structure with small to large order and size, so that the pattern can be generalized and formed theorem. The research methodology of this research is shown in Figure 1.

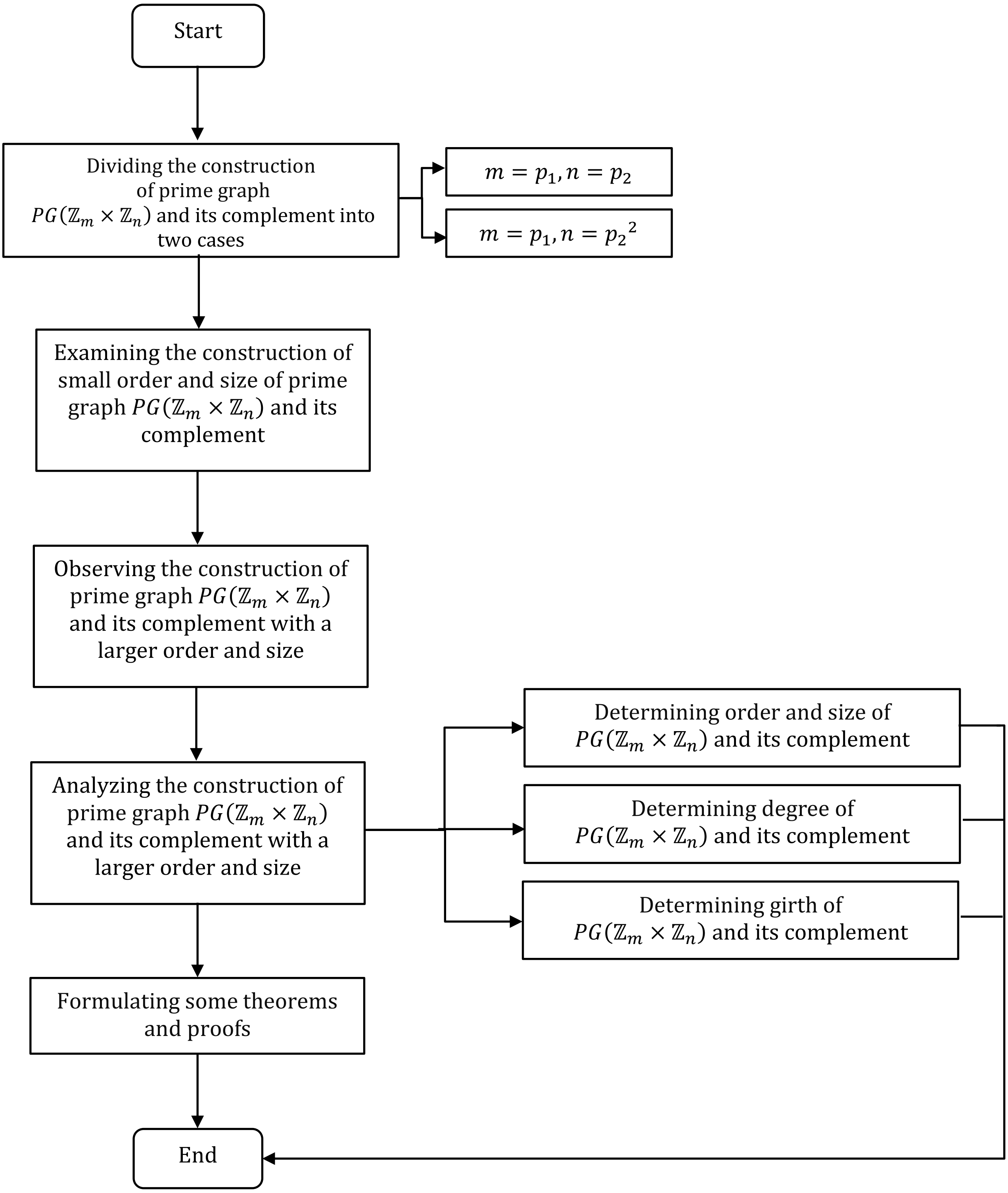

Figure 1. Flowchart of Research Methodology

Next, the definitions are used in this paper as follows:

Definition 2.1 (Gallian, 2017) : Let $a, b \in R$, $a, b \neq 0$. Element a is a zero divisor in R if $ab = 0$ or $ba = 0$. The set of all zero divisors in R is $Z(R)$.

Definition 2.2 (Kalita & Patra, 2013): Let $R = R_1 \times \ldots \times R_n$ be a ring. Let $(x_1, x_2), (y_1, y_2) \in R$. Prime graph over a ring R ($PG(R)$) is a graph with set of vertices $V\big(PG(R)\big) = R$ and two vertices are adjacent if $(x_1, x_2)(R)(y_1, y_2) = \{(0,0)\}$.

Definition 2.3 (Pawar & Joshi, 2019): A complement of prime graph over a ring R ($\big(PG(R)\big)^c$) defined as a graph with $V\Big(\big(PG(R)\big)^c\Big) = R$ and the set of edges is $E = \{(x, y) | xRy \neq 0,\ x \neq y\}$.

Definition 2.4 (Pawar & Joshi, 2019): The shortest cycle length in G called by girth ($gr(G)$). If G containing no cycle, then $gr(G) = \infty$.

C. RESULT AND DISCUSSION

We obtain the result related to prime graph over cartesian product of rings and its complement with some examples, order and size, degree, and girth. First, we discuss some examples of prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement, for $m = p_1$, $n = p_2$ and $m = p_1$, $n = {p_2}^2$, where p_1 and p_2 are primes. Let us construct $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$**.** Based on Definition 2.2, we know that the set of vertices in $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is $V\big(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\big) = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{1}, \bar{0}), (\bar{1}, \bar{1})\}$. Neighborhood of vertex in $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$, we can see in Table 1.

Table 1. Neighborhood of vertex in $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$

$(\overline{x_1}, \overline{x_2}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$	$(\overline{y_1}, \overline{y_2}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$	$(\overline{x_1}, \overline{x_2})(\mathbb{Z}_2 \times \mathbb{Z}_2)(\overline{y_1}, \overline{y_2})$
$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{0})$
	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{0})$
	$(\bar{1}, \bar{1})$	$(\bar{0}, \bar{0})$
$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{0})$
	$(\bar{1}, \bar{1})$	$(\bar{0}, \bar{1})$
$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{0})$

Based on Definition 2.2 and Table 1, two vertices in prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$ are adjacent if satisfy $(\overline{x_1}, \overline{x_2})(\mathbb{Z}_2 \times \mathbb{Z}_2)(\overline{y_1}, \overline{y_2}) = \{(\bar{0}, \bar{0})\}$ (marked in blue). The set of edges in $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$ can be written

$$E\big(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\big) = \Big\{\big((\bar{0}, \bar{0}), (\bar{0}, \bar{1})\big), \big((\bar{0}, \bar{0}), (\bar{1}, \bar{0})\big), \big((\bar{0}, \bar{0}), (\bar{1}, \bar{1})\big), \big((\bar{0}, \bar{1}), (\bar{1}, \bar{0})\big)\Big\}$$

Therefore, $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is shown in Figure 2.

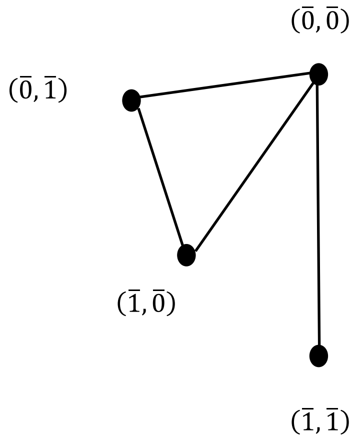

Figure 2. $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Next, we construct the complement prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Based on Definition 2.3, we can write the set of vertices $V\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the set of edges

$$E\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \{(\bar{x},\bar{y}) | (\overline{x_1},\overline{x_2})(\mathbb{Z}_2 \times \mathbb{Z}_2)(\overline{y_1},\overline{y_2}) \neq 0\}$$

It means that two vertices are adjacent in $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$ if those vertices are not adjacent in $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Therefore, $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$ is shown in Figure 3.

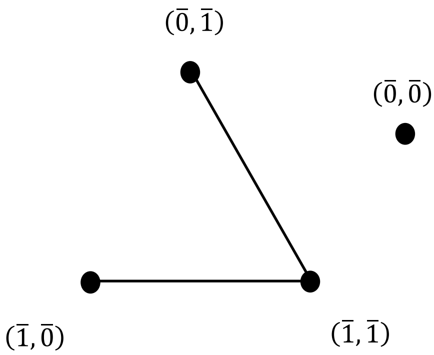

Figure 3. $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$

Based on Figure 3, the set of edges of $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$ is

$$E\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \left\{\left((\bar{0},\bar{1}),(\bar{1},\bar{1})\right),\left((\bar{1},\bar{0}),(\bar{1},\bar{1})\right)\right\}$$

A vertex $(\bar{0},\bar{0})$ forms a subgraph trivial. Since $(\bar{0},\bar{0})$ is adjacent to all other vertices in $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$, so $(\bar{0},\bar{0})$ is not adjacent to all other vertices in $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$. Similarly, we can construct $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$, $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$, $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$, and $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$ used Definition 2.2 and Definition 2.3. Those graphs are shown in Figure 4, Figure 5, Figure 6, and Figure 7.

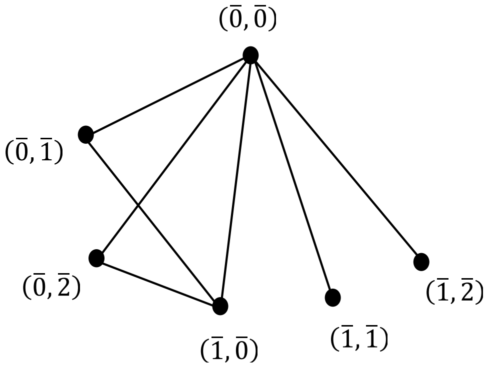

Figure 4. $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$

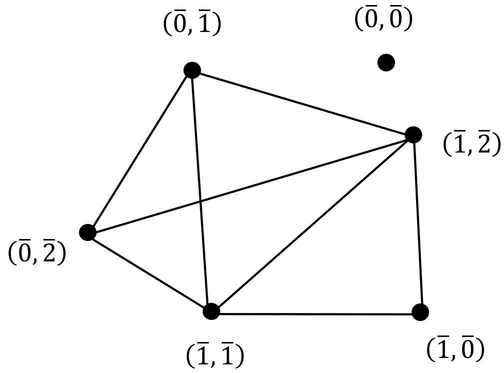

Figure 5. $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$

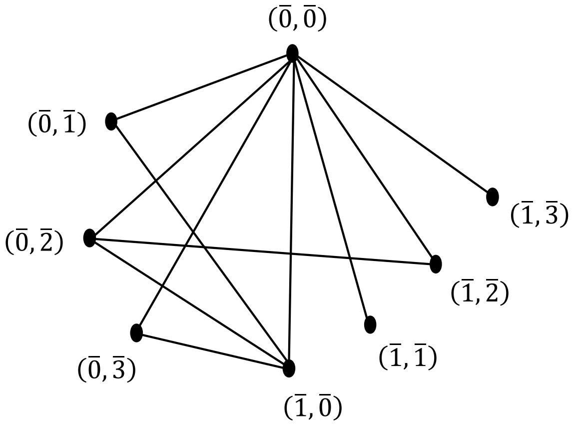

Figure 6. $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$

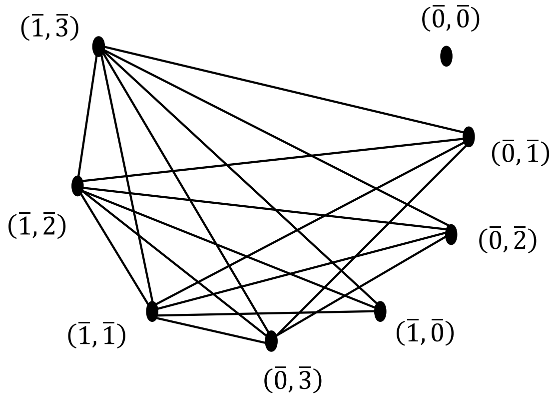

Figure 7. $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$

1. In Figure 4, $V\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right) = \{(\bar{0},\bar{0}),(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{1},\bar{2})\}$. Based on Definition 2.2, one of edges in prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is $\left((\bar{0},\bar{2}),(\bar{1},\bar{0})\right)$. Since $(\bar{0},\bar{2})(\mathbb{Z}_2 \times \mathbb{Z}_3)(\bar{1},\bar{0}) = \{(\bar{0},\bar{0})\}$. Similary, we know that the other edges in prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is $\left((\bar{0},\bar{0}),(\bar{0},\bar{1})\right)$, $\left((\bar{0},\bar{0}),(\bar{0},\bar{2})\right)$, $\left((\bar{0},\bar{0}),(\bar{1},\bar{0})\right)$, $\left((\bar{0},\bar{0}),(\bar{1},\bar{1})\right)$, $\left((\bar{0},\bar{0}),(\bar{1},\bar{2})\right)$, and $\left((\bar{0},\bar{1}),(\bar{1},\bar{0})\right)$.
2. In Figure 5, complement of prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$ can construct by observing the vertices in $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$ that are adjacent or not. Two vertices are adjacent in $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$, if those vertices not adjacent in $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$. Therefore, one of edge in $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$ is $\left((\bar{0},\bar{1}),(\bar{0},\bar{2})\right)$ due to $(\bar{0},\bar{1})(\mathbb{Z}_2 \times \mathbb{Z}_3)(\bar{0},\bar{2}) \neq \{(\bar{0},\bar{0})\}$. So, we get other edges in $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$ that is $\left((\bar{0},\bar{1}),(\bar{1},\bar{1})\right)$, $\left((\bar{0},\bar{1}),(\bar{1},\bar{2})\right)$, $\left((\bar{1},\bar{2}),(\bar{0},\bar{1})\right)$, $\left((\bar{1},\bar{2}),(\bar{1},\bar{1})\right)$, $\left((\bar{1},\bar{2}),(\bar{1},\bar{0})\right)$, $\left((\bar{1},\bar{0}),(\bar{1},\bar{1})\right)$, and $\left((\bar{0},\bar{1}),(\bar{1},\bar{1})\right)$.
3. In Figure 6, by Definition 2.2 the set of vertices in $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is $\mathbb{Z}_2 \times \mathbb{Z}_4$. A vertex $(\bar{0},\bar{0})$ is adjacent to all other vertices in prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$. Then, two other vertices are adjacent in $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$ that is $\left((\bar{0},\bar{1}),(\bar{1},\bar{0})\right)$, $\left((\bar{0},\bar{2}),(\bar{1},\bar{0})\right)$, $\left((\bar{0},\bar{3}),(\bar{1},\bar{0})\right)$, and $\left((\bar{0},\bar{2}),(\bar{1},\bar{2})\right)$.
4. In Figure 7, a vertex $(\bar{0},\bar{0})$ is not adjacent to all other vertices in complement prime graph $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$. Since a vertex $(\bar{0},\bar{0})$ is adjacent to all other vertices in $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$. As well as the other vertices in $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$, to determine the

neighborhood of vertex in complement prime graph $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$, observe the neighborhood of vertex in prime graph $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Based on Figure 3, Figure 5, and Figure 7, we can easily write this following lemma.

Lemma 3.1. Let $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ be a complement graph of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$. Construction prime graph of $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ is two components disconnected graph that is a trivial subgraph and a simple subgraph with $(mn-1)$ vertices.

Note. One component of complement prime graph $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ is a trivial subgraph with vertex $(\bar{0},\bar{0})$. Hence a vertex $(\bar{0},\bar{0})$ is always adjacent to all other vertices in prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$. Therefore, a vertex $(\bar{0},\bar{0})$ is not adjacent to all vertices in $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$. A vertex $(\bar{0},\bar{0})$ refers to isolated vertex. Next, we give a correction statement of (Pawar & Joshi, 2019). We begin by giving some counter examples.

Example 3.2. Let construct $PG(\mathbb{Z}_p)$ and $\left(PG(\mathbb{Z}_p)\right)^c$, where $p = 2$ and $p = 4$. The set of vertices in prime graph $PG(\mathbb{Z}_2)$ is $V(PG(\mathbb{Z}_2)) = \{\bar{0},\bar{1}\}$. By the Definition 2.2, it is clear that vertex $\bar{0}$ and $\bar{1}$ are adjacent. Since those two vertices are adjacent, then vertex $\bar{0}$ and $\bar{1}$ are not adjacent in complement prime graph $\left(PG(\mathbb{Z}_2)\right)^c$. Then, $V(PG(\mathbb{Z}_4)) = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$. A vertex $\bar{0}$ is adjacent to all other vertices in $PG(\mathbb{Z}_4)$ and there are no vertices are adjacent. In complement prime graph of $\left(PG(\mathbb{Z}_4)\right)^c$, a vertex $\bar{1}$ and $\bar{2}$ are adjacent, by Definition 2.3. Similary, we can get other vertices in $\left(PG(\mathbb{Z}_4)\right)^c$ are adjacent. Prime graph $PG(\mathbb{Z}_2)$, $PG(\mathbb{Z}_4)$ and complement prime graph $\left(PG(\mathbb{Z}_2)\right)^c$ and $\left(PG(\mathbb{Z}_4)\right)^c$ are shown in Figure 8, Figure 9, Figure 10, and Figure 11.

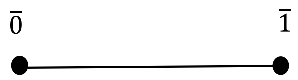

Figure 8. $PG(\mathbb{Z}_2)$

Figure 9. $\left(PG(\mathbb{Z}_2)\right)^c$

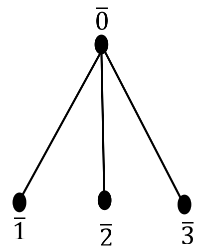

Figure 10. $PG(\mathbb{Z}_4)$

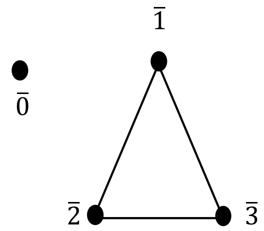

Figure 11. $\left(PG(\mathbb{Z}_4)\right)^c$

We see that $\left(PG(\mathbb{Z}_2)\right)^c$ not containing complete graph but both are trivial graph. Then, $\left(PG(\mathbb{Z}_4)\right)^c$ is a disconnected graph consisting of a trivial subgraph and $(p-1)$ vertices that form a complete graph. The statement of the Observation 3.4 (1) in (Pawar & Joshi, 2019) is as follows:

Observation 3.4 (1) (Pawar & Joshi, 2019)**:**
Let $\mathbb{Z}_p$ be a ring and $\left(PG(\mathbb{Z}_p)\right)^c$ be the complement of prime graph. For any prime p, then $\left(PG(\mathbb{Z}_p)\right)^c$ is a disconnected graph with two components in which one component is a trivial graph containing one vertex and other component is always a complete graph with $(p-1)$ vertices. By our Example 3.2, it follows that the Observation 3.4 (1) is not true for $p=2$ and alsotrue for $p=4$. The corrected version of that observation shown in lemma as follows:

Lemma 3.3
Let $\mathbb{Z}_p$ be a ring and $\left(PG(\mathbb{Z}_p)\right)^c$ be the complement of prime graph of a ring. For prime $p>2$ and $p=4$, then $\left(PG(\mathbb{Z}_p)\right)^c$ is a disconnected graph with two components in which one component is a trivial subgraph containing one vertex and other component is always a complete subgraph with $(p-1)$ vertices.

Now, we obtain the result about the order and size of complement prime graph $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$, for $m=p_1, n=p_2$ and $m=p_1, n={p_2}^2$, where p_1 and p_2 are primes. Firstly, we review the theorem about order and size of prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$.

Theorem 3.4 (Krisnawati et al., 2023): Let $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ be a prime graph of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, where p_1 and p_2 are prime numbers. Then, $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ has order and size

$$\left|V\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)\right| = p_1p_2 \text{ , and}$$
$$\left|E\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)\right| = 2p_1p_2 - p_1 - p_2$$

Theorem 3.5 (Fatimah et al., 2023): Let $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})$ be a prime graph $\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}$, where p_1 and p_2 are prime numbers. Then, $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})$ has order and size

$$\left|V\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})\right)\right| = p_1{p_2}^2 \text{ , and}$$
$$\left|E\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})\right)\right| = \frac{1}{2}\left(p_2(6p_1p_2 - 4p_1 - 3p_2 + 1)\right)$$

The order and size of complement prime graph $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ for $m=p_1, n=p_2$ and $m=p_1, n={p_2}^2$, where p_1 and p_2 are primes in the next two following theorems.

Theorem 3.6. Let $\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)^c$ be a complement of prime graph of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, where p_1 and p_2 are prime numbers. Then, $\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)^c$ has order and size

$$\left|V\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c\right)\right| = p_1p_2 \text{ , and}$$
$$\left|E\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c\right)\right| = \frac{1}{2}({p_1}^2{p_2}^2 - 5p_1p_2 + 2p_1 + 2p_2)$$

Proof. Based on Definition 2.3, $V\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c\right) = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, so it is clear that $\left|V\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c\right)\right| = p_1p_2$.

Suppose the size in complete graph with p_1p_2 vertices is $\frac{1}{2}\left(p_1p_2(p_1p_2 - 1)\right)$.
By Theorem 3.4, the number of edges in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ is $2p_1p_2 - p_1 - p_2$.
We know that two distinct vertices are adjacent in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c$ if these vertices are not adjacent in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$. Therefore, the number of edges in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c$ can be written

$$\begin{aligned} &\frac{1}{2}\left(p_1p_2(p_1p_2 - 1)\right) - (2p_1p_2 - p_1 - p_2) \\ &= \frac{1}{2}({p_1}^2{p_2}^2 - 5p_1p_2 + 2p_1 + 2p_2) \end{aligned}$$

Thus, we obtain $\left|E\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c\right)\right| = \frac{1}{2}({p_1}^2{p_2}^2 - 5p_1p_2 + 2p_1 + 2p_2)$. ■

Theorem 3.7. Let $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c$ be the complement of prime graph of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}$, where p_1 and p_2 are prime numbers. Then, $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c$ has order and size

$$\left|V\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c\right)\right| = p_1{p_2}^2 \text{ , and}$$
$$\left|E\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c\right)\right| = \frac{1}{2}\left(p_2({p_2}^3{p_1}^2 + (3 - 7p_1)p_2 + 4p_1 - 1)\right)$$

Proof. It follows from Definition 2.3, that $\left|V\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c\right)\right| = p_1{p_2}^2$.
Suppose the size n complete graph with $p_1{p_2}^2$ vertices is $\frac{1}{2}\left(p_1{p_2}^2(p_1{p_2}^2 - 1)\right)$.
By Theorem 3.5, the number of edges in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)$ is $\frac{1}{2}\left(p_2(6p_1p_2 - 4p_1 - 3p_2 + 1)\right)$.
Therefore, the number of edges in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c$ can be written

$$\begin{aligned} &\frac{1}{2}\left(p_1{p_2}^2(p_1{p_2}^2 - 1)\right) - \left(\frac{1}{2}\left(p_2(6p_1p_2 - 4p_1 - 3p_2 + 1)\right)\right) \\ &= \frac{1}{2}\left(p_2({p_2}^3{p_1}^2 + (3 - 7p_1)p_2 + 4p_1 - 1)\right) \end{aligned}$$

Thus, we obtain $\left|E\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2}\right)\right)^c\right)\right| = \frac{1}{2}\left(p_2({p_2}^3{p_1}^2 + (3 - 7p_1)p_2 + 4p_1 - 1)\right)$. ■

For each vertex in prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement has a different degree. So, we give theorem about degree of a vertex in $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement, for $m = p_1, n = p_2$ and $m = p_1, n = {p_2}^2$, where p_1 and p_2 are primes.

Theorem 3.8
Suppose $(\bar{x}, \bar{y})$ be a vertex in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, then a vertex $(\bar{x}, \bar{y})$ has degree

$$\deg\big((\bar{x},\bar{y})\big) = \begin{cases} p_1p_2 - 1 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} = 0 \\ p_1 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} = 0 \\ p_2 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} \neq 0 \\ 1 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} \neq 0 \end{cases}$$

Proof. Degree of of vertex $(\bar{x}, \bar{y}) \in V\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)$ divided into four cases as follows:

1. For $\bar{y} = 0$ and $\bar{x} = 0$.
   Since a vertex $(\bar{0}, \bar{0})$ is adjacent to all vertices of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, then $(\bar{0}, \bar{0})$ has degree $p_1p_2 - 1$.
2. For $\bar{y} \neq 0$ and $\bar{x} = 0$.
   Since a vertex $(\bar{0}, \bar{y})$ is adjacent to $(\bar{x}, \bar{0})$ and $(\bar{0}, \bar{0})$, then $(\bar{0}, \bar{y})$ has degree $(p_1 - 1) + 1 = p_1$.
3. For $\bar{y} = 0$ and $\bar{x} \neq 0$.
   Since a vertex $(\bar{x}, \bar{0})$ is adjacent to $(\bar{0}, \bar{y})$ and $(\bar{0}, \bar{0})$, then $(\bar{x}, \bar{0})$ has degree $(p_2 - 1) + 1 = p_2$.
4. For $\bar{y} \neq 0$ and $\bar{x} \neq 0$.
   Since a vertex $(\bar{x}, \bar{y})$ is not adjacent to all other vertices in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ except $(\bar{0}, \bar{0})$, then $(\bar{x}, \bar{y})$ has degree 1. ■

Then, the following theorem describe the degree of each vertex in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})$.

Theorem 3.9
Suppose $(\bar{x}, \bar{y})$ be a vertex in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})$, then a vertex $(\bar{x}, \bar{y})$ has degree

$$\deg\big((\bar{x},\bar{y})\big) = \begin{cases} p_1{p_2}^2 - 1 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} = 0 \\ {p_2}^2 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} \neq 0 \\ p_1p_2 - 1 & \text{, if } \bar{y} \neq 0, \bar{x} = 0, \text{ and } \bar{y} \in Z(\mathbb{Z}_{{p_2}^2}) \\ p_2 & \text{, if } \bar{y} \neq 0, \bar{x} \neq 0, \text{ and } \bar{y} \in Z(\mathbb{Z}_{{p_2}^2}) \\ 1 & \text{, if } \bar{y} \neq 0, \bar{x} \neq 0, \text{ and } \bar{y} \notin Z(\mathbb{Z}_{{p_2}^2}) \end{cases}$$

Proof. Based on Definition 2.1, $Z(\mathbb{Z}_{{p_2}^2}) = \{p, 2p, 3p, \dots, (p-1)p\}$. It means that the number of zero divisor in $\mathbb{Z}_{{p_2}^2}$ is $(p_2 - 1)$. Degree of vertex $(\bar{x}, \bar{y}) \in V\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{{p_2}^2})\right)$ divided into five cases as follows:

1. For $\bar{y} = 0$ and $\bar{x} = 0$.

Since a vertex $(\bar{0},\bar{0})$ is adjacent to other vertices in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2})$, then $(\bar{0},\bar{0})$ has degree $p_1 p_2{}^2 - 1$.

2. For $\bar{y} = 0$ and $\bar{x} \neq 0$.
   Since a vertex $(\bar{x},\bar{0})$ is adjacent to $(\bar{0},\bar{y})$ and $(\bar{0},\bar{0})$, then $(\bar{0},\bar{x})$ has degree $(p_2{}^2 - 1) + 1 = p_2{}^2$.
3. For $\bar{y} \neq 0$, $\bar{x} = 0$, and $\bar{y} \in Z(\mathbb{Z}_{p_2{}^2})$.
   In this case, we consider three cases as follows:
   a. $(\bar{0},\bar{y})$ is adjacent to $(\bar{x},\bar{0})$ and $(\bar{0},\bar{0})$, then the degree of $(\bar{0},\bar{y})$ is $(p_1 - 1) + 1 = p_1$.
   b. $(\bar{0},\bar{y})$ is adjacent to $(\bar{x},\bar{y})$, then the degree of $(\bar{0},\bar{y})$ is $(p_2 - 1)(p_1 - 1)$.
   c. $(\bar{0},\bar{y})$ is adjacent to $(\bar{0},\bar{y})$, except it self, then the degree of $(\bar{0},\bar{y})$ is $(p_2 - 2)$.

   By (a), (b), and (c), the degree of $(\bar{0},\bar{y})$ is $p_1 + (p_2 - 1)(p_1 - 1) + (p_2 - 2) = p_1 p_2 - 1$.
4. For $\bar{x} \neq 0$, $\bar{y} \neq 0$, and $\bar{y} \in Z(\mathbb{Z}_{p_2{}^2})$
   Since a vertex $(\bar{x},\bar{y})$ is adjacent with $(\bar{0},\bar{y})$ and $(\bar{0},\bar{0})$, then the degree of $(\bar{x},\bar{y})$ is $(p_2 - 1) + 1 = p_2$.
5. For $\bar{x} \neq 0$, $\bar{y} \neq 0$ and $\bar{y} \notin Z(\mathbb{Z}_{p_2{}^2})$
   Since a vertex $(\bar{x},\bar{y})$ is only adjacent to $(\bar{0},\bar{0})$, then the degree of $(\bar{x},\bar{y})$ is 1. ■

Let $deg((\bar{x},\bar{y}))$ be a degree of a vertex $(\bar{x},\bar{y}) \in PG(\mathbb{Z}_m \times \mathbb{Z}_n)$. Based on the degree of prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$, we can obtain degree of complement prime graph $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ by $\left|V\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)\right| - 1 - deg(\bar{x},\bar{y})$, so that degree of $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ can be written on this following lemma:

Lemma 3.10

If $(\bar{x},\bar{y})$ be a vertex in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c$, then a vertex $(\bar{x},\bar{y})$ has degree

$$\deg((\bar{x},\bar{y})) = \begin{cases} p_1 p_2 - 2 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} \neq 0 \\ p_1 p_2 - p_2 - 1 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} \neq 0 \\ p_1 p_2 - p_1 - 1 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} = 0 \\ 0 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} = 0 \end{cases}$$

Proof. Based on Theorem 3.8, we can obtain the degree of vertex in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)^c$ as follows: (i) For $\bar{y} \neq 0$ and $\bar{x} \neq 0$: $p_1 p_2 - 1 - 1 = p_1 p_2 - 2$; (ii) For $\bar{y} = 0$ and $\bar{x} \neq 0$: $p_1 p_2 - p_2 - 1$; (iii) For $\bar{y} \neq 0$ and $\bar{x} = 0$: $p_1 p_2 - p_1 - 1$: (iv) For $\bar{y} = 0$ and $\bar{x} = 0$: $p_1 p_2 - 1 - (p_1 p_2 - 1) = 0$.

Lemma 3.11

If $(\bar{x},\bar{y})$ be a vertex in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2}\right)\right)^c$, then a vertex $(\bar{x},\bar{y})$ has degree

$$\deg\big((\bar{x},\bar{y})\big)=\begin{cases}0 & \text{, if } \bar{y}=0 \text{ and } \bar{x}=0\\ p_1{p_2}^2-{p_2}^2-1 & \text{, if } \bar{y}=0 \text{ and } \bar{x}\neq 0\\ p_1{p_2}^2-p_1p_2 & \text{, if } \bar{y}\neq 0, \bar{x}=0, \text{ and } \bar{y}\in Z\big(\mathbb{Z}_{{p_2}^2}\big)\\ p_1{p_2}^2-p_2-1 & \text{, if } \bar{y}\neq 0, \bar{x}\neq 0, \text{ and } \bar{y}\in Z\big(\mathbb{Z}_{{p_2}^2}\big)\\ p_1{p_2}^2-2 & \text{, if } \bar{y}\neq 0, \bar{x}\neq 0, \text{ and } \bar{y}\notin Z\big(\mathbb{Z}_{{p_2}^2}\big)\end{cases}$$

Proof. Based on Theorem 3.9, we can obtain the degree of vertex in $\big(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\big)^c$ as follows:

1. For $\bar{y}=0$ and $\bar{x}=0$: $p_1{p_2}^2-1-(p_1{p_2}^2-1)=0$;
2. For $\bar{y}=0$ and $\bar{x}\neq 0$: $p_1{p_2}^2-{p_2}^2-1$;
3. For $\bar{y}\neq 0, \bar{x}=0$, and $\bar{y}\in Z\big(\mathbb{Z}_{{p_2}^2}\big)$: $p_1{p_2}^2-1-(p_1p_2-1)=p_1{p_2}^2-p_1p_2$;
4. For $\bar{y}\neq 0, \bar{x}\neq 0$, and $\bar{y}\in Z\big(\mathbb{Z}_{{p_2}^2}\big)$: $p_1{p_2}^2-p_2-1$;
5. For $\bar{y}\neq 0, \bar{x}\neq 0$, and $\bar{y}\notin Z\big(\mathbb{Z}_{{p_2}^2}\big)$: $p_1{p_2}^2-1-1=p_1{p_2}^2-2$. ■

Next, we give theorem about girth of prime graph $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$ and its complement, where m and n are integers.

Theorem 3.12

Let $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$ be a prime graph over cartesian product of rings and $\big(PG(\mathbb{Z}_m\times\mathbb{Z}_n)\big)^c$ be a complement prime graph of $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$.

1. If $m,n\in\mathbb{Z}^+$ and $m,n\geq 2$, then $gr\big(PG(\mathbb{Z}_m\times\mathbb{Z}_n)\big)=3$.
2. $gr\Big(\big(PG(\mathbb{Z}_2\times\mathbb{Z}_2)\big)^c\Big)=\infty$.

Proof. (i) Based on Definition 2.2, a vertex $(\bar{0},\bar{0})$ is adjacent to other vertices in $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$ and a vertex $(\bar{x},\bar{0})$ is adjacent with $(\bar{0},\bar{y})$. Moreover, a pairwise vertices are adjacent if at least one of the vertex containing zero divisors. Hence, it is clear that in $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$ is always exist a subgraph which forms a triangle with one of the vertex in that triangle is $(\bar{0},\bar{0})$ and the probability of two other vertices is $(\bar{x},\bar{0})$ and $(\bar{0},\bar{y})$ or vertices are containing zero divisors. Therefore, it forms a shortest cycle of length 3. In other words, $gr\big(PG(\mathbb{Z}_m\times\mathbb{Z}_n)\big)=3$. If $m,n<2$, no triangles are forms. (ii) Based on example on section 1, $\big(PG(\mathbb{Z}_2\times\mathbb{Z}_2)\big)^c$ does not form the shortest cycle. Therefore, by Definition 2.4, $gr\Big(\big(PG(\mathbb{Z}_2\times\mathbb{Z}_2)\big)^c\Big)=\infty$. ■

D. CONCLUSION AND SUGGESTIONS

In this paper, we study a connection between algebra and graph theory. Especially an algebraic structures, namely theory of graph algebra. One of research on graph algebra is a graph that formed by prime ring elements or called prime graph over ring R. Based on section C, we obtain some properties of prime graph $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$ and complement prime graph $\big(PG(\mathbb{Z}_m\times\mathbb{Z}_n)\big)^c$ such as order and size, degree, and girth. The analysis of finding the order and size, degree, and girth, it is tried first by examining the construction of small order and size graph, then expanded on the construction with a larger order and size. On the next research, we can develop to observe prime graph $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$, where m, n are not prime numbers. Moreover, we can analyze some properties such as energy, Wiener index, line graph, coloring, labeling, etc.

ACKNOWLEDGEMENT

The authors would really like to thankful to the Department of Mathematics of University of Brawijaya have supported this research.

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