

The Ideal Over Semiring of the Non-Negative Integer

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ABSTRACT

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Assumed that $(S, +, \cdot)$ is a semiring. Semiring is a algebra structure as a generalization of a ring. A set $I \subseteq S$ is called an ideal over semiring S if for any $\alpha, \beta \in I$, we have $\alpha - \beta \in I$ and $s\alpha = \alpha s \in I$ for every s in semiring S . Based on this definition, there is a special condition namely prime ideal P , when for any $\alpha\beta \in P$, then we could prove that α or β are elements of ideal P . Furthermore, an ideal I of S is irreducible if Ia is an intersection ideal from any ideal A and B on S , then $I = A$ or $I = B$. We also know the strongly notion of the irreducible concept. The ideal I of S is a strongly irreducible ideal when I is a subset of the intersection of A and B (ideal of S), then I is a subset of A , or I is a subset of B . In this paper, we discussed the characteristics of the semiring of the non-negative integer set. We showed that $p\mathbb{Z}^+$ is an ideal of semiring of the non-negative integer \mathbb{Z}^+ over addition and multiplication. We find a characteristic that $p\mathbb{Z}^+$ is a prime ideal and also a strongly irreducible ideal of the semiring \mathbb{Z}^+ with p is a prime number.



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A. INTRODUCTION

The algebraic structure group G is defined as a non-empty set G with binary operation $*$ that satisfies the associative law. Moreover, there exists an element e_G such that $e_G * \alpha = \alpha * e_G = \alpha$ for every α in G , and for each element of G , there exists $\alpha^{-1} \in G$ such that $\alpha * \alpha^{-1} = \alpha^{-1} * \alpha = e_G$ (Dummit & Foote, 2004; Herstein, 1975). Based on the group theory, a new structure can be constructed by taking any subset H of G that satisfies the group axioms with the same binary operations as group G . Furthermore, this structure is called a subgroup of group G (Herstein, 1995). There is a special condition on subgroup H namely the left coset of H is similar to the right coset, i.e., $\alpha H = H\alpha$ for any α element of group G . This structure is known as a normal subgroup (Ghaani Farashahi, 2023; Hao & Zhuxuan, 2013).

On the other hand, we have $R \neq \emptyset$ with two binary operations, namely addition and multiplication. A structure R over addition and multiplication is called a ring $(R, +, \cdot)$ if $(R, +)$ is a commutative group, (R, \cdot) is a semigroup, i.e., the multiplicative operation satisfies associative law, and the left and right distributive laws are applied (Wahyuni et al., 2016). In the structure ring, we have a subset of R with the same binary operation as the ring $(R, +, \cdot)$ such that it satisfies the axioms that apply to a ring called a subring. We have a special condition in the ring theory, that is for any $(R, +, \cdot)$ is a ring and $I \subseteq R$ with $I \neq \emptyset$, then the set I is called the left

(right) ideal of R if for any $r \in R$ and $x, y \in I$ follow that $x - y \in I, rx \in I (xr \in I)$ (Cohen, 1946; John B. Fraleigh, n.d.; Piciu, 2021). Furthermore, the explanation of rings is written in Chapter 4 of (Herstein, 1995) and Part II of (Dummit & Foote, 2004).

In 1934, HS. Vandiver founded the concept of semiring (Vandiver, 1934). A semiring is a generalization of a ring (Alarcon et al., 1994). A semiring is a ring that reduces the inverse element existence axiom in the additive operation. In the mentioned paper, given a set S with addition and multiplication, we have a semiring $(S, +, \cdot)$ with $(S, +)$ is a commutative monoid, (S, \cdot) is a semigroup, and there exists $0 \in S$ as the additive identity, and $s \cdot 0 = 0 \cdot s = 0$ for all $s \in S$, both are connected by ring-like distributive laws (Allen et al., 2006; Atani & Atani, 2008; Feng et al., 2008). A non-empty subset I of S is an ideal if for any $\alpha, \beta \in I$ and $s \in S$ implies $\alpha + \beta \in I$, and $s\alpha \in I$ (Atani & Atani, 2008).

The application of semiring can be found in modelling of economies, queueing theory, social networks analysis, consensus theory, etc. The researches about the ideal over semiring have been done by any researcher in the world. (Bae Jun & Sik Kim, 1996) defined and studied normal L-fuzzy ideal in semiring at 1996. (Kim & Park, 1996) have studied k-fuzzy ideal over a semiring. The ideals over commutative semiring have been written in the paper (Atani & Atani, 2008). The concept of fuzzy k -ideal in semiring has studied by (Abdurrahman, 2022; Kar et al., 2015) At 2018, Nasehpour have been studied on minimalprime ideals in semiring (Nasehpour, 2018). The ideals of any semiring also have been written by Nasehpour at 2019 (Nasehpour, 2019). Ideal I is said to be a completely prime if for every $\alpha\beta \in I$ implies either $\alpha \in I$ or $\beta \in I$ (Nasehpour, 2019). A prime ideal has properties that $I \neq S$ is a prime ideal if and only if A, B are ideals in S such that $AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$, with $AB = \{\alpha\beta | \alpha \in A, \beta \in B\}$ (Atani & Atani, 2008).

On the other hand, the strongly irreducible ideal for a commutative ring was written by (Heinzer et al., 2002). Analogue with the concept ideal over a ring, an ideal I of semiring S is a strongly irreducible ideal if for any $A \cap B \subseteq I$, obtained either $A \subseteq I$ or $B \subseteq I$ (Atani & Atani, 2008). We could also said that an ideal I is a maximal ideal if there exists an ideal M of semiring S such that $I \subseteq M \subseteq S$, then $I = M$ or $M = S$ (Atani & Atani, 2008). From these definitions, we state that if I is a prime ideal of S , then I is a strongly irreducible ideal (Hasnani & Puspita, 2021). As the explanation above, we could construct the state of the art of the previous researches likely Figure 1.

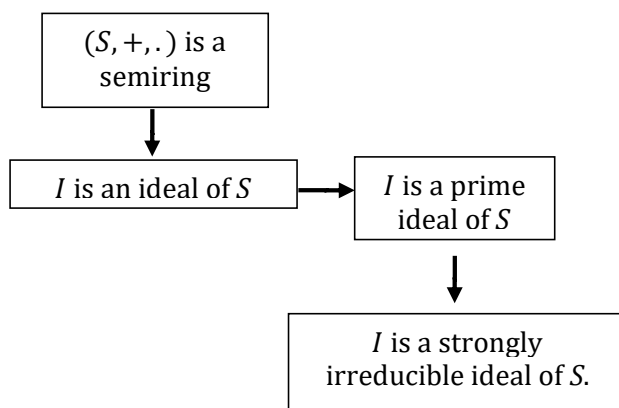


Figure 1. The state of the art of the previous researches.

Furthermore, the aims of this paper is to analyze characteristics of the ideal over semiring of the non-negative integer. Firstly, we take the non-negative integer set over addition and multiplication as a semiring. Then, we construct the ideals over semiring $(\mathbb{Z}^+, +, \cdot)$.

B. METHODS

This is a pure mathematical research where the results are a deepening of a semiring theory. The ideals over a semiring are divided into a prime ideal, a strongly irreducible ideal, and a reducible ideal. This research uses the concept of ideals that applies to semiring. The research method includes the analysis of ideals over the semiring $(\mathbb{Z}^+, +, \cdot)$. Ideals over the semiring $(\mathbb{Z}^+, +, \cdot)$ is constructed with defining the concept of semiring and ideals of a semiring in advance. At this stage, literature studies are carried out to determine the definition of semiring and ideals over semiring and the types of ideals that can be constructed over a semiring. After that, we construct ideals over the semiring of the non-negative integer. The definition of the ideal over semiring that explained by (Atani & Atani, 2008) is the foundation to construct ideals over the semiring $(\mathbb{Z}^+, +, \cdot)$. Furthermore, we analyze the ideal over semiring $(\mathbb{Z}^+, +, \cdot)$ and construct the prime ideal over semiring $(\mathbb{Z}^+, +, \cdot)$ by its definition. Then, based on definition of strongly irreducible ideal, we construct the strongly irreducible ideal over semiring $(\mathbb{Z}^+, +, \cdot)$. Moreover, we analyze the relationship between the prime ideal and the strongly irreducible ideal over the semiring $(\mathbb{Z}^+, +, \cdot)$. Based on the existing theorem that for each prime ideal is a strong irreducible ideal, we will analyze the applicability of that theorem to the prime ideal and the strongly irreducible ideal over the semiring $(\mathbb{Z}^+, +, \cdot)$. Furthermore, the state of the art of this research will show in Figure 2.

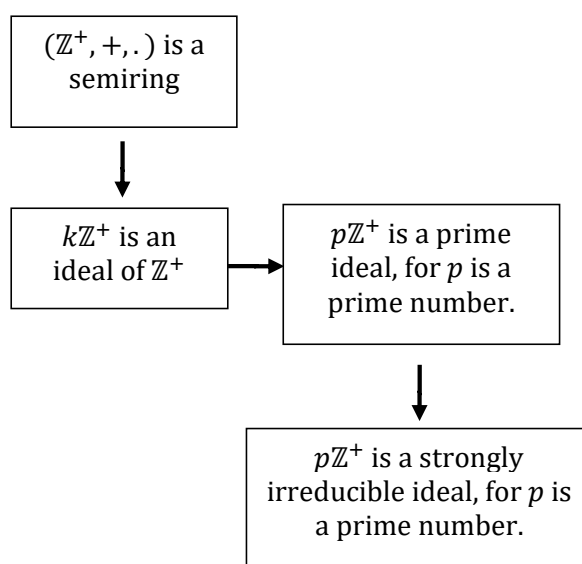


Figure 2. The state of the art of this research.

C. RESULTS AND DISCUSSION

Let us recall that by definition, a set S over addition and multiplication is said to be semiring $(S, +, \cdot)$ if the set S over addition is a commutative monoid, the set S over multiplication is semigroup, there exists zero element in S , and also satisfies the distributive laws. In this paper, we use the non-empty set \mathbb{Z}^+ over addition and multiplication. The set \mathbb{Z}^+ be a semiring $(\mathbb{Z}^+, +, \cdot)$ since satisfies the following conditions:

1. The set \mathbb{Z}^+ over addition $(\mathbb{Z}^+, +)$ is a commutative monoid
2. The set \mathbb{Z}^+ over multiplication (\mathbb{Z}^+, \cdot) is a semigroup
3. There exists $0 \in \mathbb{Z}^+$ such that $0 \cdot \alpha = \alpha \cdot 0 = 0$, for all $\alpha \in \mathbb{Z}^+$
4. The tuple $(\mathbb{Z}^+, +, \cdot)$ satisfies the distributive laws, i.e.
 - a. For any $\alpha, \beta, \gamma \in \mathbb{Z}^+$, we have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ (left distribution law)
 - b. For any $\alpha, \beta, \gamma \in \mathbb{Z}^+$, we have $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ (right distribution law)

Proof. Given the non-empty set \mathbb{Z}^+ over addition and multiplication. We will show that $(\mathbb{Z}^+, +, \cdot)$ is a semiring.

1. The tuple $(\mathbb{Z}^+, +)$ is a commutative monoid since for any $\alpha, \beta, \gamma \in \mathbb{Z}^+$ over addition operation over the integer, obtained $\alpha + \beta = \beta + \alpha \in \mathbb{Z}^+$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. There is an element 0 such that $\alpha + 0 = 0 + \alpha = \alpha$.
2. The tuple (\mathbb{Z}^+, \cdot) is a semigroup since for any $\alpha, \beta, \gamma \in \mathbb{Z}^+$ over multiplication operation over the integer, obtained $\alpha \cdot \beta \in \mathbb{Z}^+$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
3. There exists $0 \in \mathbb{Z}^+$, over multiplication operation over the integer, obtained $\alpha \cdot 0 = 0 \cdot \alpha = 0$, for all $\alpha \in \mathbb{Z}^+$.
4. By the distribution concept over integers, we have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ satisfies a left distribution law and $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ satisfies a right distribution law for any $\alpha, \beta, \gamma \in \mathbb{Z}^+$. ■

For any semiring S , we could construct the ideal over the semiring S . Analog with the definition of ideal over a ring, a non-empty subset I of S is an ideal if for any $\alpha, \beta \in I$ and $s \in S$ implies $\alpha + \beta \in I$, and $s\alpha \in I$ (Piciu, 2021). The set $\{0\} \subseteq S$ and $S \subseteq S$ are ideals over semiring S , and both of them is called trivial ideals. From the semiring $(\mathbb{Z}^+, +, \cdot)$, we could construct the ideal over the semiring \mathbb{Z}^+ as follows.

Proposition 1. Given a semiring $(\mathbb{Z}^+, +, \cdot)$. For $k \in \mathbb{Z}^+$, we have $k\mathbb{Z}^+ \subseteq \mathbb{Z}^+$ that can be written as $k\mathbb{Z}^+ = 0\mathbb{Z}^+, 1\mathbb{Z}^+, 2\mathbb{Z}^+, \dots$ is an ideal of semiring $(\mathbb{Z}^+, +, \cdot)$.

Proof. Given a semiring $(\mathbb{Z}^+, +, \cdot)$ and let $k\mathbb{Z}^+ \subseteq \mathbb{Z}^+$, where $k \in \mathbb{Z}^+$. It will be proved that $k\mathbb{Z}^+$ is an ideal of semiring $(\mathbb{Z}^+, +, \cdot)$. For any $km, kn \in k\mathbb{Z}^+$ and $\gamma \in \mathbb{Z}^+$, we have

1. $km + kn = k(m + n) \in k\mathbb{Z}^+$
2. $km \cdot \gamma = k \cdot m \cdot \gamma = k \cdot m\gamma \in k\mathbb{Z}^+$ and
3. $\gamma \cdot km = \gamma \cdot k \cdot m = k \cdot \gamma \cdot m = k \cdot \gamma m \in k\mathbb{Z}^+$

Based on (i) and (ii), it is proven that $k\mathbb{Z}^+$ is an ideal of \mathbb{Z}^+ . ■

Example 2.

The set $5\mathbb{Z}^+ \subseteq \mathbb{Z}^+ = \{5n | n \in \mathbb{Z}^+\}$ is the ideal over semiring \mathbb{Z}^+ since for any $5n, 5m \in 5\mathbb{Z}^+$ and $\gamma \in \mathbb{Z}^+$, we have

1. $5m + 5n = 5(m + n)$. Since $(m + n)$ is element of \mathbb{Z}^+ , so $5(m + n) \in 5\mathbb{Z}^+$.
2. $5m \cdot \gamma = 5 \cdot m \cdot \gamma = 5 \cdot (m\gamma)$. Since $m\gamma$ is element of \mathbb{Z}^+ , so that $5 \cdot (m\gamma) \in 5\mathbb{Z}^+$.
3. $\gamma \cdot 5m = \gamma \cdot 5 \cdot m = 5 \cdot \gamma \cdot m = 5(\gamma m) \in 5\mathbb{Z}^+$.

From (1) – (3), we prove that $5\mathbb{Z}^+$ is the ideal over semiring \mathbb{Z}^+ .

From the ideal over a semiring, we have a special condition namely a prime ideal P , when for any $\alpha\beta \in P$, then we could prove that α or β are elements of ideal P (Nasehpour, 2019). We will show the prime ideal of semiring \mathbb{Z}^+ with the following proposition.

Proposition 3. Given a semiring $(\mathbb{Z}^+, +, \cdot)$ and let $p\mathbb{Z}^+ \subset \mathbb{Z}^+$ with a prime number p . The set $p\mathbb{Z}^+$ is a prime ideal of semiring $(\mathbb{Z}^+, +, \cdot)$ if for any $\alpha, \beta \in \mathbb{Z}^+$ and $\alpha \cdot \beta \in p\mathbb{Z}^+$, then $\alpha \in p\mathbb{Z}^+$ or $\beta \in p\mathbb{Z}^+$.

Proof. Let a semiring $(\mathbb{Z}^+, +, \cdot)$ and the set $p\mathbb{Z}^+ \subset \mathbb{Z}^+$. We will prove that $p\mathbb{Z}^+$ is a prime ideal. It will be proved that for any $\alpha, \beta \in \mathbb{Z}^+$ if $\alpha \cdot \beta \in p\mathbb{Z}^+$, then $\alpha \in p\mathbb{Z}^+$ or $\beta \in p\mathbb{Z}^+$. We will prove with the concept of contraposition. Let $\alpha \notin p\mathbb{Z}^+$ and $\beta \notin p\mathbb{Z}^+$, for a prime number p , we mean $\alpha \in k\mathbb{Z}^+$ and $\beta \in l\mathbb{Z}^+$, for k is a composite number. A prime number is a number that only has two factors, namely one and itself. Based on the fundamental theorem of arithmetic, a composite number is a product of two or more prime numbers, i.e., $k = p_1 \cdot p_2 \cdot \dots \cdot p_n$, with p is a prime number and $n = 1, 2, 3, \dots$

For any $km, ln \in k\mathbb{Z}^+$, as the product of two prime numbers, i.e.,

$$\begin{aligned} km \cdot ln &= k(mkn) \in k\mathbb{Z}^+ \\ km \cdot ln &= k(mkn) \notin p\mathbb{Z}^+ \end{aligned}$$

Since $k = p_1 \cdot p_2 \cdot \dots \cdot p_n$, k has more than two factors, it is not a prime number. Thus, it is proven that for any $\alpha, \beta \in \mathbb{Z}^+$, if $\alpha \cdot \beta \in p\mathbb{Z}^+$, then $\alpha \in p\mathbb{Z}^+$ or $\beta \in p\mathbb{Z}^+$. ■

Example 4. Let a semiring $(\mathbb{Z}^+, +, \cdot)$ and the set $11\mathbb{Z}^+ \subset \mathbb{Z}^+$. We will prove that $11\mathbb{Z}^+$ is a prime ideal. By the contraposition concept, let $a = 11n + x \notin 11\mathbb{Z}^+$ and $b = 11n + y \notin 11\mathbb{Z}^+$ with $n, x, y \in \mathbb{Z}^+$ and $x, y \neq 0$. Thus, we have

$$\begin{aligned} (11n + x)(11n + y) &= 11^2n^2 + 11ny + 11nx + xy \\ &= 11(11n^2 + ny + nx) + xy \end{aligned}$$

Since $n, x, y \in \mathbb{Z}^+$ and $x, y \neq 0$, thus $11(11n^2 + ny + nx) + xy \in 11\mathbb{Z}^+ + \mathbb{N}$ or it means that $11(11n^2 + ny + nx) + xy \notin 11\mathbb{Z}^+$.

Thus, we could prove that $11\mathbb{Z}^+$ is a prime ideal.

A prime ideal has properties that $I \neq S$ is a prime ideal if and only if A, B are ideals in S such that $AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$, with $AB = \{\alpha\beta | \alpha \in A, \beta \in B\}$ (Atani & Atani, 2008). Moreover, from Proposition 3, we have a proposition that applies to ideal $p\mathbb{Z}^+$ as follows.

Proposition 5. Let a semiring $(\mathbb{Z}^+, +, \cdot)$ and A, B are ideals over semiring $(\mathbb{Z}^+, +, \cdot)$.

1. Ideal $p\mathbb{Z}^+$ is a prime ideal if and only if for any A, B are ideals in \mathbb{Z}^+ with $AB \subseteq p\mathbb{Z}^+$, then either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$, where $AB = \{\alpha\beta | \alpha \in A \text{ and } \beta \in B.\}$
2. $AB \subseteq A \cap B$.

Proof. Let a semiring $(\mathbb{Z}^+, +, \cdot)$ and A, B are ideals over semiring $(\mathbb{Z}^+, +, \cdot)$.

1. (\Rightarrow) Given the set $p\mathbb{Z}^+$ is a prime ideal over semiring $(\mathbb{Z}^+, +, \cdot)$. We will prove that for any A, B are ideals in \mathbb{Z}^+ with $AB \subseteq p\mathbb{Z}^+$, then either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. For any $\sum_{i=1}^n \alpha_i \beta_i \in AB$ with $\alpha_i \in A$ and $\beta_i \in B$, $\alpha_i \beta_i \in AB \subseteq p\mathbb{Z}^+$ for all $i = 1, \dots, n$. Since $p\mathbb{Z}^+$ is a prime ideal, hence $\alpha_i \in p\mathbb{Z}^+$ or $\beta_i \in p\mathbb{Z}^+$ for any $i = 1, \dots, n$. Thus, it is proven that either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$.

(\Leftarrow) Assume that if $AB \subseteq p\mathbb{Z}^+$, then either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. We need to prove that $p\mathbb{Z}^+$ is a prime ideal. Based on the assumption, for any $i = 1, \dots, n$, $\alpha_i \beta_i \in AB \subseteq p\mathbb{Z}^+$ obtained $\alpha_i \in A \subseteq p\mathbb{Z}^+$ or $\beta_i \in B \subseteq p\mathbb{Z}^+$. Since it applies for any $i = 1, \dots, n$, it follows that $\sum_{i=1}^n \alpha_i \beta_i \in p\mathbb{Z}^+$, then $\alpha_i \in p\mathbb{Z}^+$ or $\beta_i \in p\mathbb{Z}^+$. Thus, it is proven that $p\mathbb{Z}^+$ is a prime ideal.

Based on (\Rightarrow) and (\Leftarrow), it is proven that $p\mathbb{Z}^+$ is a prime ideal if only if for any A, B are ideals in \mathbb{Z}^+ such that $AB \subseteq p\mathbb{Z}^+$, then either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. ■

2. We need to prove that $AB \subseteq A \cap B$. For any $x \in AB$, we will show that $x \in A \cap B$, i.e., $x \in A$ and $x \in B$. Let $x = \sum_{i=1}^n \alpha_i \beta_i$, $\alpha_i \in A$, and $\beta_i \in B$.
 - a. Considering that $\beta_i \in \mathbb{Z}^+$, $\alpha_i \in A$, since A is an ideal over semiring $(\mathbb{Z}^+, +, \cdot)$, then $\alpha_i \beta_i \in A$ for every $i = 1, \dots, n$. Thus, $\sum_{i=1}^n \alpha_i \beta_i \in A$.
 - b. Considering that $\alpha_i \in \mathbb{Z}^+$, $\beta_i \in B$, since B is an ideal over semiring $(\mathbb{Z}^+, +, \cdot)$, then $\alpha_i \beta_i \in B$ for every $i = 1, \dots, n$. Thus, $\sum_{i=1}^n \alpha_i \beta_i \in B$.

From a and b, it is shown that for any $x \in AB$ implies $x \in A \cap B$. Thus, it is proven that $AB \subseteq A \cap B$. ■

Moreover, the following example is to show one of the prime ideal over semiring $(\mathbb{Z}^+, +, \cdot)$.

Example 6. Let semiring $(\mathbb{Z}^+, +, \cdot)$ and the set A and B are ideals over semiring \mathbb{Z}^+ . We have an ideal $7\mathbb{Z}^+$ over the semiring \mathbb{Z}^+ , the set $7\mathbb{Z}^+$ is a prime ideal if for any ideal A, B in \mathbb{Z}^+ with $AB \subseteq 7\mathbb{Z}^+$, then $A \subseteq 7\mathbb{Z}^+$ or $B \subseteq 7\mathbb{Z}^+$. We prove that for any A, B are ideals in \mathbb{Z}^+ with $AB \subseteq 7\mathbb{Z}^+$, then either $A \subseteq 7\mathbb{Z}^+$ or $B \subseteq 7\mathbb{Z}^+$. For any $\sum_{i=1}^n \alpha_i \beta_i \in AB$ with $\alpha_i \in A$ and $\beta_i \in B$, $\alpha_i \beta_i \in AB \subseteq 7\mathbb{Z}^+$ for all $i = 1, \dots, n$. Since $7\mathbb{Z}^+$ is a prime ideal, hence $\alpha_i \in 7\mathbb{Z}^+$ or $\beta_i \in 7\mathbb{Z}^+$ for any $i = 1, \dots, n$. Thus, it is proven that either $A \subseteq 7\mathbb{Z}^+$ or $B \subseteq 7\mathbb{Z}^+$.

Analog with the concept of ideal over a ring, for arbitrary semiring we could find an ideal namely a strongly irreducible ideal if for any $A \cap B \subseteq I$, obtained either $A \subseteq I$ or $B \subseteq I$. The following proposition show the strongly irreducible ideal over the semiring $(\mathbb{Z}^+, +, \cdot)$.

Proposition 7. *The set $p\mathbb{Z}^+$ is a strongly irreducible ideal if for any ideals A, B in \mathbb{Z}^+ and $A \cap B \subseteq p\mathbb{Z}^+$ implies either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$.*

Proof. Given a semiring $(\mathbb{Z}^+, +, \cdot)$ and the set $p\mathbb{Z}^+ \subset \mathbb{Z}^+$, where p is a prime number. We need to prove that $p\mathbb{Z}^+$ is a strongly irreducible ideal. Then, for any A, B are ideals over semiring $(\mathbb{Z}^+, +, \cdot)$ with $A \cap B \subseteq p\mathbb{Z}^+$, we show that either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. Since \mathbb{Z}^+ and $p\mathbb{Z}^+$ are ideals in \mathbb{Z}^+ and the set $p\mathbb{Z}^+$ is a maximal ideal, it follows that $A \cap B \subseteq p\mathbb{Z}^+$ means not only $A \subseteq p\mathbb{Z}^+$ and $B \subseteq p\mathbb{Z}^+$ but also $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. Thus, it is proven that $p\mathbb{Z}^+$ belongs to a strongly irreducible ideal of semiring $(\mathbb{Z}^+, +, \cdot)$. ■

The research that has been done by (Hasnani & Puspita, 2021) states that for any prime ideals is a strongly irreducible ideal. Thus, from Proposition 2 at (Atani & Atani, 2008), we could construct a proposition that applies at the semiring \mathbb{Z}^+ as follows.

Proposition 8. *If $p\mathbb{Z}^+$ is a prime ideal of \mathbb{Z}^+ , then $p\mathbb{Z}^+$ is a strongly irreducible ideal.*

Proof. Given that $p\mathbb{Z}^+$ is a prime ideal, it will be proved that $p\mathbb{Z}^+$ is a strongly irreducible ideal. For any A, B are ideals of the semiring $(\mathbb{Z}^+, +, \cdot)$ such that $A \cap B \subseteq p\mathbb{Z}^+$, we need to prove either $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. Let $p\mathbb{Z}^+$ be a prime ideal and A, B are ideals in \mathbb{Z}^+ with $A \cap B \subseteq p\mathbb{Z}^+$. From Proposition 4, to show that $p\mathbb{Z}^+$ is a strongly irreducible ideal, it is shown that $A \subseteq p\mathbb{Z}^+$ or $B \subseteq p\mathbb{Z}^+$. By using contradiction, we assumed that $A \not\subseteq p\mathbb{Z}^+$ and $B \not\subseteq p\mathbb{Z}^+$. Since $p\mathbb{Z}^+$ is a prime ideal, where $A \not\subseteq p\mathbb{Z}^+$ or $B \not\subseteq p\mathbb{Z}^+$, then $AB \not\subseteq p\mathbb{Z}^+$. Based on Proposition 3, it is known that $AB \subseteq A \cap B \subseteq p\mathbb{Z}^+$, contradicts the fact that $AB \not\subseteq p\mathbb{Z}^+$. Thus, it is proven that if $p\mathbb{Z}^+$ is a prime ideal, then $p\mathbb{Z}^+$ is a strongly irreducible ideal. ■

D. CONCLUSIONS AND SUGGESTIONS

A ring can be generalized to be a semiring. A semiring is constructed from the non-empty set S over addition and multiplication operations. The set $(S, +)$ is a commutative monoid, (S, \cdot) is a semigroup, a neutral element exists, and the left and right distributive laws apply as in a ring. An ideal of a semiring can be constructed analogue with the definition of an ideal of a ring, i.e., the set $I \subseteq S$ is called an ideal of the semiring $(S, +, \cdot)$ if for any $\alpha, \beta \in I$ and $s \in S$, then $\alpha + \beta \in I$ and $\alpha \cdot s = s \cdot \alpha \in I$.

The object of this research is the set of non-negative integer. Previously, it has been shown that the set of non-negative integers over addition and multiplication operations, denoted by $(\mathbb{Z}^+, +, \cdot)$ is a semiring. One of the ideals of the semiring $(\mathbb{Z}^+, +, \cdot)$ is the set of numbers $k\mathbb{Z}^+ = 0\mathbb{Z}^+, 1\mathbb{Z}^+, 2\mathbb{Z}^+, \dots$. In addition, a prime ideal of the semiring $(\mathbb{Z}^+, +, \cdot)$ can be found, namely the set $p\mathbb{Z}^+$, with a prime number p . Moreover, we construct that $p\mathbb{Z}^+$ is also a strongly irreducible ideal of the semiring $(\mathbb{Z}^+, +, \cdot)$.

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REFERENCES

- Abdurrahman, S. (2022). Ideal Fuzzy Semiring Atas Level Subset. *Jurnal Fourier*, 11(1), 1–6. <https://doi.org/10.14421/fourier.2022.111.1-6>
- Alarcon, F., Anderson, D., & Anderson, D. D. (1994). Commutative Semirings and Their Lattices of Ideals. *Houston Journal of Mathematics*, 20(4), 571–590. <https://www.researchgate.net/publication/266952555>
- Allen, P. J., Neggers, J., & Kim, H. S. (2006). Ideal Theory in Commutative A-semirings. *KYUNGPOOK Math. J*, 46, 261–271.
- Atani, R. E., & Atani, S. E. (2008). Ideal Theory in Commutative Semirings. *MATEMATICA Number*, 2(57), 14–23.
- Bae Jun, Y., & Sik Kim, H. (1996). Normal L-fuzzy Ideals in Semirings. *Fuzzy Sets and Systems*, 82, 383–386.
- Cohen, I. S. (1946). On the Structure and Ideal Theory of Complete Local Rings. *Transactions of the American Mathematical Society*, 59(1), 54–106.
- Dummit, D. S., & Foote, R. M. (2004). *Abstract Algebra Third Edition* (J. Battista, A. Scanlan-Rohrer, J. Z. Lindstrom, & K. Santor, Eds.; 3rd ed.).
- Feng, F., Jun, Y. B., & Zhao, X. (2008). Soft Semirings. *Computers and Mathematics with Applications*, 56(10), 2621–2628. <https://doi.org/10.1016/j.camwa.2008.05.011>
- Ghaani Farashahi, A. (2023). Banach Convolution Modules of Group Algebras on Covariant Functions of Characters of Normal Subgroups. *Bulletin Des Sciences Mathématiques*, 103259. <https://doi.org/10.1016/j.bulsci.2023.103259>
- Hao, C., & Zhuxuan, J. (2013). Finite p-Groups Which Contain A Self-Centralizing Cyclic Normal Subgroup. *Acta Mathematica Scientia*, 33(1), 131–138. <http://actams.wipm.ac.cn>
- Hasnani, F., & Puspita, N. P. (2021). Ideal Tak Tereduksi Kuat Atas Semiring Komutatif. *Prosiding Konferensi Nasional Matematika XX*, 21–26. <https://doi.org/10.30598/PattimuraSci.2021.KNMXX>
- Heinzer, W. J., Ratliff Jr., L. J., & Rush, D. E. (2002). Strongly Irreducible Ideals of a Commutative Ring. *Journal of Pure and Applied Algebra*, 267–275.
- Herstein, I. N. (1975). *Topics in Algebra* (2nd ed.). John Wiley & Sons.
- Herstein, I. N. (1995). *Abstract Algebra Third Edition* (G. Lobell, J. Grant, & E. Wetterau, Eds.; 3rd ed.). A Viacom Company.
- John B. Fraleigh. (n.d.). *A First Course in Abstract Algebra* (5th ed.). Addison Wesley.
- Kar, S., Purkait, S., & davvas, B. (2015). Fuzzy k-Primary Decomposition of Fuzzy k-Ideal in a Semiring. *Fuzzy Information and Engineering*, 405–422.
- Kim, C. B., & Park, M.-A. (1996). k-Fuzzy Ideals in Semirings. *Fuzzy Sets and Systems*, 81, 281–286.
- Nasehpour, P. (2018). Pseudocomplementation and Minimal Prime Ideals in Semirings. *Algebra Universalis*, 79(1). <https://doi.org/10.1007/s00012-018-0496-x>
- Nasehpour, P. (2019). Some Remarks on Semirings and Their Ideals. *Asian-European Journal of Mathematics*, 12(7). <https://doi.org/10.1142/S1793557120500023>
- Piciu, D. (2021). Prime, Minimal Prime and Maximal Ideals Spaces in Residuated Lattices. *Fuzzy Sets and Systems*, 405, 47–64. <https://doi.org/10.1016/j.fss.2020.04.009>
- Vandiver, H. S. (1934). Note on a Simple Type of Algebra in Which The Cancellation Law of Addition does not Hold. *Vorlesungen Iiber Zahlen Und FunkHonenlehre*, 1(1), 4–30.
- Wahyuni, S., Wijayanti, I. E., Yuwaningsih, D. A., & Hartanto, A. D. (2016). *Teori Ring dan Modul* (1st ed.). Gadjah Mada University Press.