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Partial Fourier Transform Method for Solution Formula of Stokes Equation with Robin Boundary Condition in Half-space

Sri Maryani[1], Dede Bagus Suhada[2], Bambang Hendriya Guswanto[3]
[1,2,3]Departement of Mathematics, Jenderal Soedirman University, Indonesia
sri.maryani@unsoed.ac.id[1], dedebagus274@gmail.com[2], Bambang.guswanto@unsoed.ac.id[3]

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ABSTRACT

The area of applied science known as fluid dynamics studied how gases and liquids moved. The motion of the fluid in the liquid and vapour phases is described by a special system of partial differential equations. The research purpose of this article investigated the solution formula of incompressible Stokes equation with the Robin boundary condition in half-space case. The solution formula for Stokes equation was calculated using the partial Fourier transform. This calculation was carried out over the Weis's multipliers theorem. Our calculation showed that the solution formula of Stokes equation with Robin boundary condition in half-space for velocity and pressure were contained multipliers as due to work Shibata & Shimada. Due to our consideration of the half-space situation, the partial Fourier transform approach is the most appropriate one to use to get the velocity and pressure for the Stokes equation with Robin boundary condition. Furthermore, research methods in this article, in the first stage, we use the resolvent problem of the model. Secondly, we apply the partial Fourier transform to the model problem and finally, we use inverse partial Fourier transform to get the solution formula of the incompressible type of Stokes equation for velocity and pressure. This result indicates that Weis' multiplier theorem also allows us to find the local well-posedness of the model problem in addition to the maximal Lp-Lq regularity class (Gerard-Varet et al., 2020).

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A. INTRODUCTION

Fluid dynamics is the branch of applied science that is concerned with the movement of liquids and gases. In other word, fluid dynamic is the subject which is study of fluid and how forces affect them. The Convective heat transfer and species mass transfer are described by scalar transport equations, but fluid dynamics imply fluid movement and accompanying forces defined by vector equations. Even fluid mechanics is broken down into various areas. Hydrodynamics is the term used to describe the study of the motion of fluids that are essentially incompressible (such as liquids, particularly water, and gases at low speeds) (Kleinstreuer, 2018).

Because of the strong cohesive interactions between the molecules in a liquid, molecular pieces can move around in relation to one another, but the volume stays largely fixed. As a result, a liquid assumes the shape of the container it is in and, in the presence of gravity, produces a free surface in a bigger container. In contrast, a gas enlarges until it touches the container's walls and fills the full volume. Due to their large spacing and weak cohesive forces, gas molecules behave in this way. Gases cannot create a free surface, in contrast to liquids (John & Yunus, 2014). Recently, there are many researchers considering reated to fluid, not only fluid mechanics but also fluid dynamics.

Vapor and gas are frequently used interchangeably. When a substance is above the critical temperature, its vapor phase is commonly referred to as a gas. Vapor often denotes a gas that is close to the condensation stage. Both in daily activities and in the design of contemporary technical systems, ranging from vacuum cleaners to supersonic aircraft, fluid mechanics is extensively used. Therefore, it's critical to gain a solid understanding of fluid mechanics' fundamental concepts.

To start, fluid mechanics is important to the human body. The lungs are the locations of airflow in alternating directions, and the heart is constantly pumping blood to all areas of the body through the arteries and veins. Naturally, fluid dynamics is used in the design of all artificial hearts, breathing apparatuses, and dialysis devices. Think about how a fluid might go through a pipe that isn't moving or through a solid surface that isn't porous. A fluid in motion stops completely at the surface and assumes a zero velocity relative to the surface, according to all experimental observations. As a result of viscous effects, a fluid in direct contact with a solid "sticks" to the surface; slippage is therefore not possible. The no-slip condition is what is meant by this (John & Yunus, 2014).

On the other hand, in the middle of the 20th century may be regarded as the heyday of fluid mechanics applications. The fluid characteristics and qualities established by existing theories were sufficient for the tasks at hand. These enabled the enormous growth of the industrial, water resources, chemical, and aeronautical sectors, each of which advanced fluid mechanics in novel ways (John & Yunus, 2014). The emergence of the digital computer in the late 20th century dominated fluid mechanics study and work. Some researchers have benefited from the ability to address difficult issues of a scale that the fluid mechanics pioneers of the eighteenth century could not have foreseen, such as global climate modelling or optimizing the design of a turbine blade. In this article, we consider fluid dynamics which is explaining motion of the flow.

In some cases, fluid motion are studying about friction of the layer. A friction force forms between two fluid layers that are moving in relation to one another, and the slower layer seeks to slow the faster layer down. The fluid property viscosity, which is a gauge of the fluid's internal stickiness, quantifies this internal resistance to flow. Cohesive interactions between molecules in liquids and molecular collisions in gases are what generate viscosity. Since there is no fluid with zero viscosity, viscous effects are present in all fluid flows to some extent. Viscous flows are those in which frictional effects are prominent. The Robin boundary condition can be used in fluid dynamics to express no-slip boundary conditions in terms of vorticity in n -dimensional case ($n \geq 2$). Specifically, the Robin-type boundary condition is used to enforce moment relations under Stokes flow, which ensures that the affine invarienat manifold is invariant and the no-slip law at the boundary is satisfied (Gorshkov, 2021).

Numerous conclusions have already been made, all up to this point in Stokes equations. Maekawa et al. (2020), they proved the Stokes and Navier-Stokes equation in half-space case for initial data in class of locally uniform Lebesgue integrable functions. The local energy weak solutions for the Navier-Stokes equation in the same space have been demonstrated by the same authors (Maekawa et al., 2019). Ten years ago, de Almeida & Ferreira (2013) also investigated the Navier-Stokes in half-space case, with initial and boundary rough data in Morrey space.

Concerning the time analyticity for inhomogeneous parabolic equations and the Navier-Stokes equations in half space, Dong & Pan (2020) studied weak solutions for inhomogeneous parabolic equations with measurable coefficients in the half-space with either the Dirichlet boundary condition or the Dirichlet boundary condition under the supposition that the solution and the source term have an exponential increase of order 2 with the space variables. Our argument in this paper is completely different from the argument due to (Saal, 2006) but rather closed to that due (Shibata & Shimada, 2007). Saal (2006) investigated not only resolvent estimate but also H^∞ calculus.

The inviscid limit for general analytic data without having to construct Prandtl's boundary layer corrector studied by (Nguyen & Nguyen, 2018). In that article, they used forticity formulation on the boundary and Theorem of Cauchy-Kovalevskaya on the layer function spaces. Recently, the localized smoothing and concentration for the Navier-Stokes equations in half-space for the incompressible case can bee seen in (Albritton et al., 2023). In that article they consent to the non-local effects of the pressure in the half-space in 3-dimensional Euclidean space.

On the other hand, the main result for forward self-similar solution of the Navier-Stokes equations in the half-space is due to (Korobkov & Tsai, 2016). The boundedness and stabilization of the Navier-Stokes equation system with competitive kinetics in 2-dimensional case is studied by (Hirata et al., 2017). They showed not only the boundedness and stabilization of the solution to the two species chemotaxis model but also the global existence. As known that the partial differential equation can be described the natural phenomena. Moreover, there are some observation which reveal dynamics concerning pattern and spontaneous emergence of turbulence in population of aerobic bacteria suspended in sessile drops of water (Tuval et al., 2005). In 2017, Wang investigated the zero-viscosity limit of the Navier-Stokes in analytic setting (Wang et al., 2017). There are many type of fluid motion then this different of model fluid flow which be interesting point of view for many researchers not only in compressible case but also in incompressible case. Inna et al. (2020) investigated half-space problem for compressible fluid motion of Korteweg type with slip boundary condition. Furthermore, Maryani et al. (2022) prove the R-boundedness of the solution operator families of Navier-Lame equation problem by using partial Fourier transform. Ogawa & Shimizu (2020) considered the L^1-regularity class for the initial boundary value problem of parabolic equation in half-space. Other researchers who consider the korteweg model are (Bresch et al., 2019). They investigated the Euler Korteweg and Navier-Stokes-Korteweg in quantum application (Bresch et al., 2019). Furthermore, Farwig et. al studied the fundamental solution of linearized non-stationary Navier-Stokes equations of motion around a rotating and translating body (Farwig et al., 2014). They considered the motion of the viscous fluid around a rotating body in which the axis of the rotation of the body is not parallel to the velocity of the fluid at infinity.

The central issue in this paper is determining the solution formula of velocity and pressure in equation (1) in half-space case, with Robin boundary condition by using partial Fourier transform. We use the techniques in Shibata & Shimada (2007) together with some recent result due to (Dewi et al., 2022). However, little information of the technical manner use the partial Fourier transform for half-space case. Therefore, in this article, we show the detail getting solution formula of the model problem using partial Fourier transform. In spirit of Shibata & Shimada (2007), we determine the solution formula of velocity for the Stokes equation $\mathbf{u}$ and pressure p using inverse Fourier transform.

Notation $\mathbb{N}$ denotes the sets of natural numbers and we set $\mathbb{N}_\mathbf{0} = \mathbb{N} \cup \{\mathbf{0}\}$. $\mathbb{C}$ and $\mathbb{R}$ denote the sets of complex numbers and real numbers, respectively. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_\mathbf{0}^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial_x^\kappa = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$ with $x = (x_1, \dots, x_N)$. For scalar function f and N-vector of function $\mathbf{g}$, we get

$$\nabla f = (\partial_1 f, \dots, \partial_N f), \nabla \mathbf{g} = \{\, \partial_i g_j \mid i, j = 1, \dots, N\},$$
$$\nabla^2 f = \{\partial_i \partial_j f \mid i, j = 1, \dots, N\},\ \nabla^2 \mathbf{g} = \{\partial_i \partial_j g_k \mid i, j, k = 1, \dots, N\},$$
$$W_q^{m,\ell}(\Omega) \coloneqq \left\{ (\mathbf{f}, \mathbf{g}) \mid \mathbf{f} \in W_q^m(\Omega),\ \mathbf{g} \in W_q^\ell(\Omega) \right\}.$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and Fourier inverse transform, respectively, which are defined by

$$\mathcal{F}_x[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx,\ \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\, d\xi. \tag{1}$$

In other hand, the partial Fourier transform with respect to $x' = (x_1, \dots, x_{N-1})$ and its inverse transform are defined as

$$\mathcal{F}_{x'}[u(x', x_N)](\xi) = \hat{u}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} u(x', x_N)\, dx', \tag{2}$$
$$\mathcal{F}_{\xi'}^{-1}[u(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} u(\xi', x_N) d\xi', \tag{3}$$

where $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$.

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform, respectively, which are defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\, dt,\quad \mathcal{L}^{-1}[g](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\tau)\, d\tau, \tag{4}$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. For $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$, we set $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^N x_j y_j$. For scalar functions f, g and N-vectors of function $\mathbf{k}, \mathbf{g}$, we get $(\mathbf{k}, \mathbf{g})_D = \int_D \mathbf{k} \cdot \mathbf{g}\, dx$, $(k, g)_\Gamma = \int_\Gamma \mathrm{kg}\, d\sigma$, $(\mathbf{k}, \mathbf{g})_\Gamma = \int_\Gamma \mathbf{k} \cdot \mathbf{g}\, d\sigma$, where σ is the surface element of Γ. For $N \times N$ matrices of function $\mathbf{F} = (F_{ij})$ and $\mathbf{G} = (G_{ij})$, we get $(\mathbf{F}, \mathbf{G})_D = \int_D \mathbf{F} : \mathbf{G}\, dx$, $(\mathbf{F}, \mathbf{G})_\Gamma = \int_\Gamma \mathbf{F} : \mathbf{G}\, d\sigma$, where $\mathbf{F} : \mathbf{G} \equiv \sum_{i,j=1}^N F_{ij} G_{ij}$. The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on $a, b, \dots$. The values of constants C and $C_{a,b,\dots}$ denote a positive constant which maybe different even in a single chain of inequalities. We use small boldface letter, e.g. $\mathbf{u}$ to denote vector-valued functions and capital boldface letters, e.g. **H** to denote matrix-valued functions, respectively. But, we also use the Greek letters, e.g $\sigma, \rho, \theta, \tau, \omega$ such as mass densities.

B. METHODS

The research strategy for this work draws on an examination of related academic literature, particularly the article by Shibata & Shimada, (2007). We defined the solution of velocity differently in this article because to their development of the velocity formula. In the subsequent phases, we will utilize resolvent problem, which is described in (1) and for the Robin boundary condition is described in (2). We also have the answer formula for equation (1) and (2) using the partial Fourier transform and inverse partial Fourier transform of the equation system. As a result, we arrive at the solution formula for pressure and velocity in the half-space situation. Therefore, the partial Fourier transform is the first and most important step in understanding the solution formula of model problem in half-space. In this study, the partial Fourier transform method is used to solve the Stokes equation for incompressible fluid flow in half-space under the Robin boundary conditions. The procedure of the research described, as shown in Figure 1.

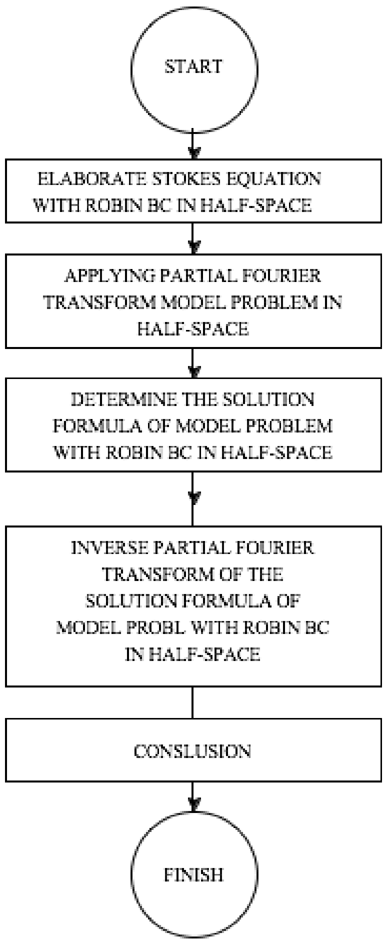

Figure 1. Procedure of the research

C. RESULT AND DISCUSSION

1. Stokes with Robin boundary condition

Stokes equation is the linearized of the Navier-Stokes equation (NSE). The equation system of the NSE describe the motion of viscous fluids, such as air and water. These materials are widely used in fluid dynamics as a phenomena including turbulence, flow around objects, and the behaviour of fluid in different environtments. The equations are set of the partial differential equations that govern the conservation of mass and momentum in a fluid. The compressible NSE can be described in the following (Shibata & Shimizu, 2009).

$$\begin{cases} \mathbf{u_t} - (\mathrm{u} \cdot \nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla \mathrm{p} = \mathbf{0} & \text{in } \Omega_t \\ \qquad \mathrm{div} \quad \mathbf{u} = 0 & \text{in } \Omega_{t.} \end{cases} \tag{5}$$

The linearized of the NSE which called compressible Stokes equation written in equation (1). In this paper, we consider the Stokes equations of incompressible fluid flow with Robin boundary condition in half-space case. The generalized Stokes resolvent problem for incompressible fluid motion in half-space can be written in the following (Shibata & Shimada, 2007).

$$\begin{cases} \lambda\mathbf{u} - \mathrm{Div}\, \mathbf{T}(\mathbf{u}, p) = \mathbf{f} & \text{in } \mathbb{R}^n_+ \\ \qquad \mathrm{div} \quad \mathbf{u} = 0 & \text{in } \mathbb{R}^n_+ \end{cases} \tag{6}$$

with Robin Boundary condition

$$\begin{cases} \mathbf{n} \cdot \mathbf{u} = 0 & \text{on } \mathbb{R}^n_0 \\ \boldsymbol{\alpha}\mathbf{u} + \beta(\mathbf{T}(\mathbf{u}, p)\mathbf{n} - \langle \mathbf{T}(\mathbf{u}, p)\mathbf{n}, \mathbf{n}\rangle\mathbf{n}) = \mathbf{h} & \text{on } \mathbb{R}^n_0 \end{cases} \tag{7}$$

where $\mathbf{u}(\mathbf{x}) = \big(u_1(\mathbf{x}), u_2(\mathbf{x}), \cdots, u_n(\mathbf{x})\big)$ and p are velocity and pressure, respectively. Meanwhile, $\mathbf{f}(\mathbf{x}) = \big(f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x})\big)$ and $\mathbf{h}(\mathbf{x}) = \big(h_1(\mathbf{x}), h_2(\mathbf{x}), \cdots, h_n(\mathbf{x})\big)$ are unknown function and $\mathbf{n} = (0, \ldots, 0, -1)$ is a unit outer normal. We define the stress tensor $\mathbf{T}(\mathbf{u}, p)$ as

$$\mathbf{T}(\mathbf{u}, p) = \mathbf{D}(\mathbf{u}) - p\mathbf{I} \tag{8}$$

with $\mathbf{D}(\mathbf{u})$ is a deformation tensor which defined as

$$\mathrm{D}(\mathbf{u})_{jk} = \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}, \tag{9}$$

and $\mathbf{I}$ is an $n \times n$ identity matrix. A half-space is one of the two divisions of the three-dimensional Euclidean space that a plane makes. Half-spaces are referred to as half-planes (open or closed) in two-dimensional spaces. A half-line or ray is a half-space in a one-dimensional space. A half-space is more broadly defined as one of the two sections that an affine space is divided into by a hyperplane. As a result, any subspace connecting a point in one set to a position in the other must intersect the hyperplane. To put it another way, the points that are not incident to the hyperplane are divided into two convex sets. For the half-space case, we define $\mathbb{R}^n_+$ and its boundary which noted by $\mathbb{R}^n_0$ are in the following:

$$\mathbb{R}^n_+ = \{ \mathbf{x} = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

and

$$\mathbb{R}^0_+ = \{ \mathbf{x} = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \mid x_n = 0\}. \tag{10}$$

The primary objective of this study is to determine the solution formula of equation (1) with Robin boundary condition in equation (2) in half-space case by using partial Fourier transform. Before we state the solution formula of the equation (1) with the Robin boundary condition, firstly we introduce the definition of Sobolev space $W^m_q(\Omega)$ and main theorem in the following:

Definition 1. (Adams & Fournier, 2003)
Let $k \in \mathbb{N} \cup \mathbb{N}_0$ and $p \in [1, \infty)$ then the Sobolev Space $W^m_q(\Omega)$ is defined by

$$W^m_q(\Omega) \coloneqq \{\mathbf{u} \in L_q(\Omega) \mid \mathrm{D}^\alpha \mathbf{u} \in L_q(\Omega), \forall \alpha \text{ with } |\alpha| \leq \mathrm{m}\}$$

The following theorem is the main result of this article

Theorem 2. Let $p(x,t)$ be a pressure and $\mathbf{u}(\mathrm{x},\mathrm{t})$ velocity in N-dimensional Euclidean space $\mathbb{R}^N$, $N \geq 2$ and set $\boldsymbol{x}' = (x_1, \dots, x_{N-1})$ and $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$ then the equation system of (1) with the Robin Boundary condition which defined as (2) has a unique solution formula of $(p, \mathbf{u}) \in W^{1,2}_q(\mathbb{R}^N_+)$ with

$$p = \mathcal{F}^{-1}_{\xi'}\left[-\frac{(A+B)e^{-Bx_N}}{B(\alpha+\beta(A+B))}\sum_{k=1}^{N-1} i\xi_k h_k(\boldsymbol{\xi}', 0)\right](\boldsymbol{x}', x_N)$$

and

$$\begin{aligned} u_j(\mathbf{x}) &= \mathcal{F}^{-1}_{\xi'}\left[\frac{\hat{h}_j(\boldsymbol{\xi}',0)e^{-Ax_N}}{\alpha+\beta A}\right](\boldsymbol{x}', x_N) \\ &\quad - \mathcal{F}^{-1}_{\boldsymbol{\xi}'}\left[\frac{\alpha+\beta A}{\alpha+\beta B}\frac{\xi_j e^{-Ax_n}}{(A-B)\big(\alpha+\beta(A+B)\big)}\sum_{k=1}^{n-1}\xi_k h_k(\boldsymbol{\xi}',0)\right](\mathbf{x}', x_n) \\ &\quad + \mathcal{F}^{-1}_{\boldsymbol{\xi}'}\left[\frac{\xi_j e^{-Bx_n}}{B(A-B)\big(\alpha+\beta(A+B)\big)}\sum_{k=1}^{n-1}\xi_k h_k(\boldsymbol{\xi}',0)\right](\mathbf{x}', x_n) \end{aligned}$$

for $j = 1, \dots, N-1$ and also

$$u_N(\mathbf{x}) = \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-Ax_n} - e^{-Bx_n}}{(A-B)\big(\alpha+\beta(A+B)\big)}\sum_{k=1}^{n-1}\xi_k h_k(\boldsymbol{\xi}',0)\right](\boldsymbol{x}', x_N).$$

where

$$A = \sqrt{\lambda + |\xi'|^2} \text{ and } B = |\xi'|.$$

For proving the main Theorem 2, first of all we explain the Robin boundary condition. Dirichlet and Neumann boundary conditions are combined in a weighted manner to form Robin boundary conditions. In contrast, mixed boundary conditions are boundary conditions of several types defined on various boundary subsets. Because of the way they are used in electromagnetic problems, Robin boundary conditions are also known as impedance boundary conditions or convective boundary conditions (Hahn & Özisik, 2012). In the following section, we state the steps of the proof. First of all, we transform the equation of (1) by using partial Fourier transform (8). Using similar method, we transform the boundary condition (2). Finally, finding all coefficients of equation (18), we got the solution formula of equation system (1).

2. The proof of Theorem (2)

a. Resolvent problem

In this subsection, we investigate the application of partial Fourier transform to the model problem (1) and also to the Robin boundary condition of equation (2). First of all, substituting second equation to the first equation of (1), we have

$$\lambda\mathbf{u} + \nabla p - \Delta\mathbf{u} = \mathbf{f}. \tag{11}$$

Using definition of boundary in (5), the first equation of equation (3) can be write in the following

$$u_n|_{x_n=0} = 0, \tag{12}$$

while second equation of equation (3) described in the following form

$$\begin{cases} \alpha u_j - \beta\big(\partial_n u_j + \partial_j u_n\big)\big|_{x_n=0} = h_j|_{x_n=0}; \quad j = 1,2,\cdots,n-1 \\ \alpha u_n|_{x_n=0} = h_n|_{x_n=0}. \end{cases} \tag{13}$$

Let $\mathbf{f} = \mathbf{0}$ then for incompressible fluid flow and equation (11)-(13), we have new homogenous equation system

$$\begin{cases} \lambda\mathbf{u} + \nabla p - \Delta\mathbf{u} = \mathbf{0}, & \text{in } \mathbb{R}_+^n \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \mathbb{R}_+^n \\ u_n = 0, & \text{on } \mathbb{R}_0^n \\ \alpha u_j - \beta\partial_n u_j = h_j,\ j = 1,\cdots,n-1, & \text{on } \mathbb{R}_0^n. \end{cases} \tag{14}$$

b. Partial Fourier Transform

By using partial Fourier transform as defined in (8) and applying to equation system of (14), we have

$$\begin{cases} (\lambda + |\xi'|^2)\hat{u}_j(\boldsymbol{\xi}', x_n) + i\xi_j\hat{p}(\boldsymbol{\xi}', x_n) - \partial_n^2\hat{u}_j(\boldsymbol{\xi}', x_n) = 0, \; j = 1, \cdots, n-1 \\ (\lambda + |\xi'|^2)\hat{u}_n(\boldsymbol{\xi}', x_n) + \partial_n\hat{p}(\boldsymbol{\xi}', x_n) - \partial_n^2\hat{u}_n(\boldsymbol{\xi}', x_n) = 0, \\ \sum_{j=1}^{n-1} i\xi_j\hat{u}_j(\boldsymbol{\xi}', x_n) + \partial_n\hat{u}_n(\boldsymbol{\xi}', x_n) = 0, \\ \hat{u}_n(\boldsymbol{\xi}', 0) = 0, \\ \alpha\hat{u}_j(\boldsymbol{\xi}', 0) - \beta\partial_n\hat{u}_j(\boldsymbol{\xi}', 0) = \hat{h}_j, \; j = 1, \cdots, n-1. \end{cases} \tag{15}$$

Furthermore, let $A = \sqrt{\lambda + |\xi'|^2}$ and $B = |\xi'|$, then equation system of (15) can be written in the following

$$\begin{cases} A^2\hat{u}_j(\boldsymbol{\xi}', x_n) + i\xi_j\hat{p}(\boldsymbol{\xi}', x_n) - \partial_n^2\hat{u}_j(\boldsymbol{\xi}', x_n) = 0, \; j = 1, \cdots, n-1 \\ A^2\hat{u}_n(\boldsymbol{\xi}', x_n) + \partial_n\hat{p}(\boldsymbol{\xi}', x_n) - \partial_n^2\hat{u}_n(\boldsymbol{\xi}', x_n) = 0, \\ \sum_{j=1}^{n-1} i\xi_j\hat{u}_j(\boldsymbol{\xi}', x_n) + \partial_n\hat{u}_n(\boldsymbol{\xi}', x_n) = 0, \\ \hat{u}_n(\boldsymbol{\xi}', 0) = 0, \\ \alpha\hat{u}_j(\boldsymbol{\xi}', 0) - \beta\partial_n\hat{u}_j(\boldsymbol{\xi}', 0) = \hat{h}_j(\boldsymbol{\xi}', 0), \; j = 1, \cdots, n-1. \end{cases} \tag{16}$$

Moreover, letting the solution formula of equation system (15) are

$$\begin{cases} \hat{u}_j(\boldsymbol{\xi}', x_n) = P_j e^{-Ax_n} + Q_j e^{-Bx_n}, \;\; j = 1,2, \cdots, n-1, \\ \hat{u}_n(\boldsymbol{\xi}', x_n) = P_n e^{-Ax_n} + Q_n e^{-Bx_n}, \\ p(\boldsymbol{\xi}', x_n) = Re^{-Bx_n}. \end{cases} \tag{17}$$

Substituting the equation (17) to (16), then we have

$$\begin{cases} (A^2 - B^2)Q_j + i\xi_j R = 0, \; j = 1,2, \cdots, n-1 \\ (A^2 - B^2)Q_n - BR = 0, \\ \sum_{j=1}^{n-1} i\xi_j P_j - AP_n = 0, \\ \sum_{j=1}^{n-1} i\xi_j Q_j - BQ_n = 0, \\ P_n + Q_n = 0, \\ \alpha\left(P_j + Q_j\right) + \beta\left(AP_j + BQ_j\right) = \hat{h}_j(\boldsymbol{\xi}', 0). \;\; j = 1,2, \cdots, n-1. \end{cases} \tag{18}$$

Solving equation system of (18), we have coefficients P_j, P_N, Q_j, Q_N and R for $j = 1,2, \cdots, n-1$ which written as follow

$$P_j = \frac{\hat{h}_j(\boldsymbol{\xi}', x_n)}{\alpha + \beta A} - -\frac{\alpha + \beta B}{\alpha + \beta A}\frac{\xi_j}{B(A-B)\left(\alpha + \beta(A+B)\right)}\sum_{k=1}^{n-1}\xi_k h_k(\boldsymbol{\xi}', x_n), \tag{19}$$

$$Q_j = \frac{\xi_j}{B(A-B)\big(\alpha+\beta(A+B)\big)} \sum_{k=1}^{n-1} \xi_k h_k(\boldsymbol{\xi}', x_n), \tag{20}$$

$$P_n = \frac{1}{(A-B)\big(\alpha+\beta(A+B)\big)} \sum_{k=1}^{n-1} i\xi_k h_k(\boldsymbol{\xi}', x_n), \tag{21}$$

$$Q_n = -\frac{1}{(A-B)\big(\alpha+\beta(A+B)\big)} \sum_{k=1}^{n-1} i\xi_k h_k(\boldsymbol{\xi}', x_n), \tag{22}$$

$$R = -\frac{(A+B)}{B\big(\alpha+\beta(A+B)\big)} \sum_{k=1}^{n-1} i\xi_k h_k(\boldsymbol{\xi}', x_n). \tag{23}$$

Substituting equation (19)-(23) to (18) and applying inverse partial Fourier transform, we complete the main **Theorem 2.** ■

D. CONCLUSION AND SUGGESTIONS

The velocity and pressure model problem's solution formula are constructed using multipliers. These multipliers are known as solution operator families. The operator families for the model problem's solutions being bounded. Future research will focus on the boundedness of the solutions operator with surface tension, as was stated at the conclusion of this paper. Be aware that the results, including the proofs, can be used to further the study of fluid dynamics. From a purely mathematical perspective, the solution regularity of the model problem is a critical factor. Furthermore, this finding is a key first step in proving boundlessness in bent-half space and the general domain. Therefore, result in this article become important part to consider the maximal Lp-Lq regularity class.

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