JTAM (Jurnal Teori dan Aplikasi Matematika)
http://journal.ummat.ac.id/index.php/jtam
p-ISSN 2597-7512 | e-ISSN 2614-1175
Vol. 4, No. 2, October 2020, pp. 107-114

On Properties of the $(2n+1)$-Dimensional Heisenberg Lie Algebra

Edi Kurniadi
Department of Mathematics of FMIPA of Universitas Padjadjaran, Bandung, West Java, Indonesia
edi.kurniadi@unpad.ac.id

Article History:
Received : 08-06-2020
Revised 1 : 22-06-2020
Revised 2 : 08-09-2020
Accepted : 09-09-2020
Online : 03-10-2020

Keyword:
Heisenberg Lie algebra;
Heisenberg Lie group;
Frobenius Lie algebra;
Generalized character;
Unitary dual;
Plancherel measure.

ABSTRACT

In the present paper, we study some properties of the Heisenberg Lie algebra of dimension $2n+1$. The main purpose of this research is to construct a real Frobenius Lie algebra from the Heisenberg Lie algebra of dimension $2n+1$. To achieve this, we exhibit how to compute the derivation of the Heisenberg Lie algebra by following Oom's result. In this research, we use a literature review method to some related papers corresponding to a derivation of a Lie algebra, Frobenius Lie algebras, and Plancherel measure. Determining a conjecture of a real Frobenius Lie algebra is obtained. As the main result, we prove that conjecture. Namely, for the given the Heisenberg Lie algebra, there exists a commutative subalgebra of dimension one such that its semi direct sum is a real Frobenius Lie algebra of dimension $2n+2$. Futhermore, in the notion of the Lie group of the Heisenberg Lie algebra which is called the Heisenberg Lie group, we compute the generalized character of its group and we determine the Plancherel measure of the unitary dual of the Heisenberg Lie group. As our contributions, we complete some examples of Frobenius Lie algebras obtained from a nilpotent Lie algebra and we also give alternative computations to find the Plancherel measure of the Heisenberg Lie group.

https://doi.org/10.31764/jtam.v4i2.2339

A. INTRODUCTION

The notion of the Heisenberg Lie algebra and its Lie group were studied by many researchers in many areas and many perspectives. For example we can find a new structure of irreducible modules defined over the Heisenberg Lie algebra (Bekkert & et al, 2013) and (Lu & Zhao 2015). In the context of a representation theory of a Lie group, we can see that the Heisenberg Lie group can be realized in the quaternionic stage (Balachandirin & et al, 2017) and modular representations over a Heisenberg algebras (Szechtman, 2014). Furthermore, the structure of generalization of the Heisenberg Lie algebra were studied in (Cantuba & Merciales, 2020), (Cantuba, 2019), (Souza, 2019), (Niroomand & Johari, 2018), (Nirooman & Parvizi, 2017), and (Liu & et al, 2012).

The Heisenberg Lie algebra is a familiar example of nilpotent Lie algebras and it can be important model in other Lie algebra types. For example we introduce the notion of Frobenius Lie algebras which can be obtained from a split torus (Ooms, 2009). Even though, the Heisenberg Lie algebra is never to be a Frobenius Lie algebra, but the semidirect sum of the Heisenberg Lie algebra and a split torus can be a Frobenius Lie algebra. For 4-dimensional Frobenius Lie algebra constructed from 3-dimensional Heisenberg Lie algebra can be found in (Kurniadi, 2020).

Moreover, we have some facts that the Lie group of the Heisenberg Lie algebra is unimodular (Hilgert & Neeb, 2012) and its irreducible unitary representation of that group is square-integrable. It is well known in some classical books of harmonic analysis as well that its Duflo-Moore operator for the square-integrable representation of Heisenberg Lie group is a multiple of a scalar. As intance, the Duflo-Moore operator for a square-integrable representation of a Lie group of Filiform Lie algebra of dimension four is a multiple scalar (Kurniadi, 2020). Finally, our main purpose in this present paper is to construct a real Frobenius Lie algebra of dimension $2n + 2$ corresponding to the Heisenberg Lie algebra of dimension $2n + 1$ by proving the following conjecture.

Conjecture 1. *Let h be a $(2n + 1)$-dimensional Heisenberg Lie algebra then there exists a 1-dimensional commutative subalgebra Γ of derivaton of h such that a semi direct sum $h \oplus \Gamma$ is a Frobenius Lie algebra.*

Furthermore, inspired by computations in (Kirillov, 2004) page 59—61 about the Plancherel measure in the unitary of the 3-dimensional Heisenberg Lie group, we shall generalize this result to the Plancherel measure in the unitary dual of $(2n + 1)$-dimensional Heisenberg Lie group corresponding to its irreducible unitary representation presented in (Corwin & Greenleaf, 1990) page 48. In the previous work we can see the notions and some applications of unitary dual groups (Bagarello & Russo, 2018), (Baraquin, 2017), and (Cebron & Ulrich, 2016).

B. METHODS

The method of this research is the literature review. Especially, we survey two main results as follows. First, we generalize the result in paper (Ooms, 2009) about the construction of a Frobenius Lie algebra from the Heisenberg Lie algebra. The second, we compute the generalized character corresponding to the irreducible unitary representation of the Heisenberg Lie group of dimension $2n + 1$. Moreover, we continue the Plancherel measure computation of the Heisenberg Lie group of dimension 3 as stated in (Kirillov, 2004) to the case of dimension $2n + 1$.

In the this section, we shall first briefly review some basic notions such as the Heisenberg Lie algebra and its Lie group, Frobenius Lie algebras, the unitary irreducible representation of the Heisenberg Lie group, the unitary dual of the Heisenberg Lie group, generalized characters, and the Plancherel measure.

Let Δ and ∇ be vectors in $\mathbb{R}^n$ with $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_n)$ and $\nabla = (\nabla_1, \nabla_2, \dots, \nabla_n)$. We denote the inner product in $\mathbb{R}^n$ by formula:

$$(\Delta|\nabla) \coloneqq \sum_{\alpha=1}^{n} \Delta_\alpha \nabla_\alpha. \qquad (1)$$

Definition 1(Corwin & Greenleaf, 1990)**.** The $(2n+1)$-dimensional Lie algebra h with basis $S \coloneqq \{\Delta, \nabla, t\} = \{\Delta_1, \Delta_2, \dots, \Delta_n, \nabla_1, \nabla_2, \dots, \nabla_\mathrm{n}, t\}$ is said to be the Heisenberg Lie algebra if its nonzero bracket is given by the following formula

$$[\Delta_k, \nabla_k] = t, \qquad (1 \le k \le n). \qquad (2)$$

We realize the elements of the Heisenberg Lie algebra h as a matrix algebra of dimension $(n+2) \times (n+2)$ as follows.

$$h(\Delta, \nabla, t) = \begin{pmatrix} 0 & \Delta & t \\ 0 & 0 & \nabla \\ 0 & 0 & 0 \end{pmatrix}. \qquad (3)$$

Using the exponential map we obtain the corresponding Heisenberg Lie group H whose elements are given by

$$H(\Delta, \nabla, t) = \begin{pmatrix} 1 & \Delta & t \\ 0 & 1 & \nabla \\ 0 & 0 & 1 \end{pmatrix}. \qquad (4)$$

Proposition 1(Kirillov, 2004)**.** *Let H be the Heisenberg Lie group whose the Heisenberg Lie algebra is h and h^* be a dual vector space of h. Then the coadjoint orbits of H in h^* are the hyperplane Θ_λ with $\lambda \neq 0$ and the points.*

The unitary irreducible representations of H in the carrier space $\mathrm{L}^2(\mathbb{R}^n)$ that is in the Hilbert space of square-integrable functions on $\mathbb{R}^n$ corresponding to the hyperplane Θ_λ can be found in (Corwin & Greenleaf, 1990) and rewrite as follows.

$$\pi_\lambda\big(H(\Delta, \nabla, t)\big)\theta(\zeta) = e^{2\pi i\lambda\left(t + (\zeta|\nabla) + \left(\frac{1}{2}\right)(\Delta|\nabla)\right)}\theta(\zeta + \Delta), \qquad (5)$$

where $\Delta, \nabla, \zeta \in \mathbb{R}^n, t \in \mathbb{R}, \theta \in \mathrm{L}^2(\mathbb{R}^n)$ and $(\zeta|\nabla)$ is the usual inner product on $\mathbb{R}^n$.

On the other hand, the irreducible unitary representations of H corresponding to 0-dimensional orbit is 1-dimensional representation and its Plancherel measure is equal to zero. In addition, in (Kirillov, 2004) we have that every irreducible unitary representation of H is exactly unitarily equivalent to one of above both representations. Therefore, as a set we can deduce that the unitary dual $\widehat{H}$ of H equals $\mathbb{R}\backslash\{0\} \cup \mathbb{R}^{2n}$.

Let θ be a function in the space $\mathrm{L}^1(G)$ of integrable functions on topological group G and let dx be left-invariant measure on G. Fix a representative representation π_λ for any $\lambda \in \widehat{G}$. The Fourier transform $\hat{\theta}$ as an operator-valued function on $\widehat{G}$ is defined as follows (see (Kirillov, 2004) page 370).

$$\hat{\theta}(\lambda) = \int_G \theta(x)\pi_\lambda(x)dx. \qquad (6)$$

The function θ can be constructed by the Fourier inversion formula, we have

$$\theta(x) = \int_{\hat{G}} \operatorname{tr}\left(\hat{\theta}(\lambda)\pi_\lambda(x)^*\right)d\mu(\lambda) \qquad (7)$$

where μ is a mesure on $\hat{G}$ which is called the Plancherel measure and the Planchere formula is given in the terms of

$$\|\theta\|^2 = \int_{\hat{G}} \operatorname{tr}\left(\hat{\theta}(\lambda)^*\hat{\theta}(\lambda)\right)d\mu(\lambda) = \|\hat{\theta}\|^2. \qquad (8)$$

Now we turn to introduce the notion of a Frobenius Lie algebra. The recent research of Frobenius Lie algebras can be found in (Alvarez & et al, 2018), (Pham, 2016), (Diatta & Manga, 2014), (Stachura, 2013), and (Zeitlin, 2012). Let $\mathfrak{g}$ be a Lie algebra with basis $S' \coloneqq \{y_1, y_2, y_3, \dots, y_{2n}\}$. Let $\mathrm{Mt}([y_i, y_j])$ $(1 \leq i, j \leq 2n)$ be matrix of dimension $2n \times 2n$ consisting of brackets $[y_i, y_j]$ as its elements. Then $\mathfrak{g}$ is said to be Frobenius if determinant of matrix $\mathrm{Mt}([y_i, y_j])$ is not equal to zero.
For example, let $\mathfrak{g}$ be Lie algebra with basis $\{y_1, y_2\}$. The non zero bracket is given by $[y_1, y_2] = y_2$. We see that the matrix

$$\mathrm{Mt}([y_i, y_j]) = \begin{bmatrix} [y_1, y_1] & [y_1, y_2] \\ [y_2, y_1] & [y_2, y_2] \end{bmatrix} = \begin{bmatrix} 0 & y_2 \\ -y_2 & 0 \end{bmatrix} \qquad (9)$$

has determinat $y_2^2 \neq 0$. Therefore, $\mathfrak{g}$ is Frobenius Lie algebra.

C. RESULT AND DISCUSSION

In this section we devide our argument into two part as follows:

1. Construction of a Frobenius Lie algebra from the Heisenberg Lie Algebra.

The first argument is how to prove the existence of a 1- dimensional commutative subalgebra Γ of derivaton of the Heisenberg Lie algebra h. For our purpose, we claim the Conjecture 1 above and we prove the following Proposition.

Proposition 2. *Let h be a $(2n+1)$-dimensional Heisenberg Lie algebra then there exists a 1-dimensional commutative subalgebra Γ of derivaton of h such that a semi direct sum $h \oplus \Gamma$ is a Frobenius Lie algebra.*

Proof.

Let $S \coloneqq \{\Delta, \nabla, t\} = \{\Delta_1, \Delta_2, \dots, \Delta_n, \nabla_1, \nabla_2, \dots, \nabla_n, t\}$ be basis for h. We apply the formula in (Ayala, Kizil, & Tribuzy, 2012) to consider the matrix derivation $\Pi \coloneqq [\alpha_{ij}]^T$ of h where $[\alpha_{ij}]^T$ is transpose of $[\alpha_{ij}]$. We have that Π is the matrix derivation if and only the equation below is satisfied.

$$\sum_{k=1}^{2n+1} C_{ij}^k \alpha_{kp} = \sum_{k=1}^{2n+1} (d_{ik} C_{kj}^p + \alpha_{jk} C_{ik}^p). \qquad (10)$$

where $1 \leq i, j, p \leq 2n+1$ and $C_{ij}^k \in \mathbb{R}$ is a structure constants of brackets $[\Delta_i, \nabla_j]$.
Furthermore, we follow the detail computations for the Heisenberg Lie algebra of dimension 3 in (Kurniadi, 2020a) to the Heisenberg Lie algebra of dimension $2n+1$ case in order to find a 1- dimensional commutative subalgebra Γ of derivaton of h. Namely we have $\Gamma \coloneqq \langle s \rangle$ where

$$s \coloneqq \mathrm{diag}\{p_1, p_2, p_3, \dots, p_n, 1-p_1, 1-p_2, 1-p_3, \dots, 1-p_n, 1\}. \tag{11}$$

and $\mathrm{diag}\{p_1, \dots, 1\}$ is the diagonal matrix of the matrix derivation Π.
To prove that the semi direct sum $h \oplus \Gamma$ with basis
$S'' \coloneqq \{\Delta, \nabla, t, s\} = \{\Delta_1, \Delta_2, \dots, \Delta_n, \nabla_1, \nabla_2, \dots, \nabla_n, t, s\}$ is a Frobenius Lie algebra, we prove that the determinant of matrix $\mathrm{Mt}([\varepsilon_i, \varepsilon_j])$, $(\varepsilon_i, \varepsilon_j \in S'')$ is not equal to zero.

For simpler computations, we choose $s_0 \coloneqq \mathrm{diag}\{1,1,1,\dots,1,0,0,0,\dots,0,1\}$. Then we have the following matrix of the form.

$$\mathrm{Mt}([\varepsilon_i, \varepsilon_j]) \coloneqq \begin{bmatrix} 0 & \Delta_1 & \Delta_2 & \dots & \Delta_n & 0 & 0 & \dots & 0 & t \\ -\Delta_1 & 0 & 0 & \dots & 0 & t & 0 & \dots & 0 & 0 \\ -\Delta_2 & 0 & 0 & \dots & 0 & 0 & t & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -t & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -t & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ -t & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{12}$$

By interchange between two coloumns we have that $\det\left(\mathrm{Mt}([\varepsilon_i, \varepsilon_j])\right) = t^{2n+2} \neq 0$. Thus, the semi direct sum $h \oplus \Gamma$ *is* Frobenius Lie algebra as desired.

■

2. The Plancherel Measure in the Unitary Dual of the Heisenberg Lie Group.

Let π_λ be the unitary irreducible representations of the Heisenberg Lie group H in the carrier space $L^2(\mathbb{R}^n)$ corresponding to the hyperplane Θ_λ as written in eqs. (5). We have

Proposition 3. *The Plancherel measure on the* $\widehat{H} = \mathbb{R}\backslash\{0\}$ *corresponding to the unitary irreducible representations* π_λ *of the Heisenberg Lie group* H *of dimension* $2n+1$ *is of the form* $|\lambda|^n d\lambda$.

Proof.

Let $\psi \in L^1(H)$ and $\theta \in L^2(\mathbb{R}^n)$. Using eqs.(5) and eqs. (6) we have

$$\begin{aligned} \hat{\psi}(\pi_\lambda)\theta(\zeta) &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(\Delta, \nabla, t)\, \pi_\lambda\big(H(\Delta, \nabla, t)\big)\theta(\zeta)\, d\Delta d\nabla dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(\Delta, \nabla, t)\, e^{2\pi i\lambda\left(t + (\zeta|\nabla) + \left(\frac{1}{2}\right)(\Delta|\nabla)\right)} \theta(\zeta + \Delta)\, d\Delta d\nabla dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(\xi - \zeta, \nabla, t)\, e^{2\pi i\lambda\left(t + (\zeta|\nabla) + \left(\frac{1}{2}\right)(\{\xi - \zeta\}|\nabla)\right)} \theta(\xi)\, d\xi d\nabla dt, \\ &\qquad (\quad \xi \coloneqq \zeta + \Delta \quad) \end{aligned} \tag{13}$$

which is an integral operator with the kernel

$$K_\psi(\xi,\zeta) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \psi(\xi-\zeta,\nabla,t) e^{2\pi i\lambda\left(t+(\zeta|\nabla)+\left(\frac{1}{2}\right)(\{\xi-\zeta\}|\nabla)\right)} d\nabla dt \quad (14)$$

and the trace is given by the formula

$$\operatorname{tr} \pi_\lambda(\psi) = \int_{\mathbb{R}^n} K_\psi(\zeta,\zeta)\, d\zeta \quad (15)$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \psi(0,\nabla,t)\, e^{2\pi i\lambda(t+(\zeta|\nabla))}\, d\nabla dt d\zeta$$

$$= |\lambda|^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \psi(0,\nabla,t)\, e^{2\pi i\lambda t+2\pi i(y|\nabla)}\, d\nabla dt dy$$

$$(\quad y \coloneqq \lambda\zeta \quad) \quad (16)$$

$$= |\lambda|^{-n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \psi(0,\nabla,t) e^{2\pi i\lambda t} \delta(\nabla) d\nabla dt$$

$$(\delta(\nabla) \coloneqq \int_{\mathbb{R}^n} e^{2\pi i(y|\nabla)}\, dy\ )$$

$$= |\lambda|^{-n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \psi(0,0,t) e^{2\pi i\lambda t} dt.$$

Using Fourier inversion formula then we have

$$\psi(\Delta,\nabla,t) = \int_{\widehat{H}} \operatorname{tr}\ \left(\hat{\psi}(\pi_\lambda)\pi_\lambda\big(H(\Delta,\nabla,t)\big)^*\right)|\lambda|^n d\lambda \quad (17)$$

Therefore, we have that the Plancherel measure on $\widehat{H}$ is of the form $d\mu(\lambda) \coloneqq |\lambda|^n d\lambda$.

■

Furthermore, in the terms of the Fourier transform we have that its generelized character is of the measure $\frac{d\Delta d\nabla}{t}$.

In the end of this section, we achieved our main purpose to show the existence of a real Frobebius Lie algebra constructed from the Heisenberg Lie algebra of dimension $2n+1$ as can be seen in Proposition 2 and its proof.

D. CONCLUSION AND SUGGESTIONS

From our discussion above, we conclude that For a given Heisenberg Lie algebra then there exists a commutative subalgebra Γ of dimension one of derivation h such that the semidirect sum $h \oplus \Gamma$ is a Frobenius Lie algebra. Furthermore, The Plancherel measure on the $\widehat{H} = \mathbb{R} \backslash \{0\}$ corresponding to the unitary irreducible representations π_λ of Heisenberg Lie group H of dimension $2n + 1$ is of the form $|\lambda|^n d\lambda$. It is suggessted for future research to construct and classify the isomorphism classes of Frobenius Lie algebras of dimension ≥ 8. These problems are still open to study.

ACKNOWLEDGEMENT

We sincerely thank to Universitas Padjadjaran who has funded the work through Riset Percepatan Lektor Kepala (RPLK) year 2020 with contract number 1427/UN6.3.1/LT/2020.

REFERENCES

Alvarez, M. A., & et al. (2018). Contact and Frobenius solvable Lie algebras with abelian nilradical. *Comm. Algebra*, *46*, 4344–4354.

Ayala, V., Kizil, E., & Tribuzy, I. D. A. (2012). On an algorithm for finding derivations of lie algebras. *Proyecciones Journal of Mathematics*, *31*(1), 81–90. https://doi.org/10.4067/S0716-09172012000100008

Bagarello, F., & Russo, F. G. (2018). A description of pseudo-bosons in the terms of nilpotent Lie algebras. *Journal of Geometry and Physics*, *125*, 1--11.

Balachandirin,Muraleetharan, & et al. (2017). A representation of Weyl_Heisenber Lie algebra in the quaternionic setting. *Annal.of Physics*, *385*, 180--213.

Baraquin, I. (2017). Stocastics aspects of the unitary dual group,. *Comptes Rendus Mathematique*, *357*, 180--185.

Bekkert,Victor, & et al. (2013). New irreducible modules for Heisenberg and affine Lie algebras. *Journal of Algebra*, *373*, 284--298.

Cantuba, R. R. S. (2019). Lie polynomials in q-deformed Heisenberg algebras. *Journal of Algebra*, *522*, 101--123.

Cantuba, R. R. S., & Merciales, Mark Anthony C. (2020). An extension of a q-deformed Heisenberg algebra and its Lie polynomials. *Expositiones Mathematics*.

Cebron,G, & Ulrich,M. (2016). Haar states and Levi processes on the unitary dual group,. *Journal of Functional Analysis*, *270*, 2769--2811.

Corwin, L. ., & Greenleaf, F. . (1990). *Representations of nilpotent Lie groups and their application. Part I. Basic theory and examples,*. Cambridge: Cambridge University Press, Cambridge.

Diatta, A., & Manga, B. (2014). On properties of principal elements of frobenius lie algebras. *J. Lie Theory*, *24*(3), 849–864.

Hilgert, J., & Neeb, K.-H. (2012). *Structure and Geometry of Lie Groups*. New York: Springer Monographs in Mathematics, Springer.

Kirillov, A. A. (2004). Lectures on the Orbit Method, Graduate Studies in Mathematics. *American Mathematical Society,Providence*, *64*.

Kurniadi, E. (2020a). Construction of Real Frobenius Lie Algebras from Non Abelian Nilpotent Lie Algebras of Dimension ≤4. *Submitted to Journal Ilmu Dasar UNEJ*.

Kurniadi, E. (2020b). On Square-Integrable Representations of A Lie Group of 4-Dimensional Standard Filiform Lie Algebra. *Cauchy:Jurnal Matematika Murni Dan Aplikasi*.

Liu, D., & et al. (2012). Lie bialgebras structures on the twisted Heisenberg-Virasoro algebra. *Journal of Algebra*, *359*, 35--48.

Lu,Rencai, & Zhao,Kaiming. (2015). Finite-dimensional simple modules over generalized Heisenberg algebras. *Linear Algebra and Its Applications*, *475*, 276--291.

Nirooman,Peyman, & Parvizi, M. (2017). 2-capability and 2-nilpotent multiplier of finite dimensional nilpotent Lie algebras. *Journal of Geometry and Physics*, *121*, 180--185.

Niroomand, P., & Johari, F. (2018). The structure, capability, and the Schur multiplier of the generelized Heisenberg Lie algebra. *Journal of Algebra*, *505*, 482--489.

Ooms, A. I. (2009). Computing invariants and semi-invariants by means of Frobenius Lie algebras. *J. Algebra*, *321*, 1293--1312. https://doi.org/10.1016/j.jalgebra.2008.10.026

Pham, D. N. (2016). G-Quasi-Frobenius Lie Algebras. *Archivum Mathematicum*, *52*(4), 233–262. https://doi.org/10.5817/AM2016-4-233

Souza, J. . (2019). Sufficient conditions for dispersiveness of invariant control affine system on the Heisenberg group. *System &Control Letters*, *124*, 68--74.

Stachura, P. (2013). On the quantum ax+b group. *Journal of Geometry and Physics*, *73*, 125--149.

Szechtman, F. (2014). Modular representations of Heisenberg algebras. *Linear Algebra and Its Applications*, *457*, 49--60.

Zeitlin,A.M. (2012). Unitary representations of a loop ax+b group, Wiener measure and \Gamma -function. *Journal of Functional Analysis*, *263*(3), 529--548.