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Dynamical Analysis of Discrete-Time Modified Leslie-Gower Predator-Prey with Fear Effect

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ABSTRACT

It has been studied that fear plays a significant role in establishing ecological communities, influencing biodiversity, and preserving ecological balance in predator-prey interactions. In this study, it is proposed a discrete-time predator-prey model that takes the fear effect into account that is derived by using Euler method. Objective of this study is analyzing the model by linearization. Similar to the continuous model properties, the trivial fixed point and the predator-free fixed point are both unstable. The discrete model differs from the continuous model in that the stability of the interior fixed point and the free prey fixed point is affected by the time step size. Using numerical methods, we examine period-doubling bifurcations related to interior fixed point and prey-free point that are impacted by time step size.

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A. INTRODUCTION

In the last few decades, numerous scientists from various backgrounds have conducted in-depth research on the dynamics of the predator-prey interaction, a key topic in ecology and evolutionary biology. Developing hypotheses and learning more about the dynamics, persistence, and structure of biological communities are the main goals of population ecology. A population model that is both mathematically tractable and physiologically relevant must be carefully studied (Beretta & Kuang, 2002). The predator-prey model, which forecasts the rate of prey consumption depending on the number (density) of predator and prey groups, is a crucial part of such a model (Alaoui & Okiye, 2003). Lotka (1920) presented a fundamental model for this phenomenon in the early twentieth century. This model accurately depicted the fluctuating behaviour of the size predator-prey communities. Recently, this Lotka-Volterra model has been development by many researchers. In 2022, the dynamical of discrete two-predators one-prey Lotka–Volterra model has been studied (Khaliq et al., 2022). By involving fear effect and linear harvesting, Panigoro et al. (2023) developed the basic Lotka-Volterra model.

Leslie proposed a proportionate relationship between the predator's carrying capacity and the number of prey (Leslie, 1958). Leslie emphasized that the thriving ability of predators and prey is limited. The Lotka-Volterra model did not reflect this reality. The main disadvantage of this model is that the predator cannot alter to another food source if the prey population is at low concentrations (Huang et al., 2014). This model was corrected by Aziz-Alaoui & Daher Okiye (2003) including a temporary alternate food supply parameter. This last model has undergone extensive development, as can be seen in (Junior & Maidana, 2021; Chen et al., 2021; Rahmi et al., 2021; Singh & Malik, 2021) and the references therein.

Several pieces of research have studied dynamics modified Leslie-Gower in the discrete model. Sun et al. (2023) studied the dynamics of the discrete modified Leslie-Gower predator-prey model with a Holling type II functional response. In that work, they investigated the fold, 1:1 strong resonance, fold-flip, and 1:2 strong resonance bifurcation. In 2021, Singh et al. (2023) applied Michaelis Menten-type functional response and prey harvesting in a modified Leslie-Gower model. By using center manifold and bifurcation theory, Singh and Malik show multiple bifurcations of codimension 1, including transcritical bifurcation in the discrete-time model, Neimark-Sacker bifurcation, flip bifurcation, fold bifurcation, and fold-flip bifurcation.

The majority of discrete models discussed previously only took into consideration a case in which the predator kills the prey directly. Recently, researchers have increasingly observed that, in addition to actual hunting, the mere sight of predators may fear prey which has an impact on various aspects of the prey, such as habitat, pace of reproduction, how they forage, etc. A predator-prey model with a fear effect was proposed by Wang et al. (2016) using the Holling type II functional response. Wang and Zou examined a prey-predator model with adaptive predator avoidance by incorporating age-stage structure (Wang & Zou, 2017). Mondal, et al. examined a prey-predator model that took the fear effect and extra food into account (Mondal et al., 2018). Furthermore, several researchers have shown that the fear effect has an even greater impact on prey populations than does direct hunting. Pal et al. (2019) consider a modified Leslie-Gower predator-prey model in which prey population exhibit anti-predator behavior out of dread of potential predation and predators cooperate while hunting. It is known that the fear factor might stabilize the predator-prey system (Pal et al., 2019). In 2021, Chen et al. (2021) studied a discrete-time predator-prey system that included the fear effect of the predator on the prey with other food resources for the predator. Their research shows that, whereas predator species tend to be stable since there are alternative sources of food available, prey species will be pushed to fade away due to fear of predators and under specific conditions, the prey-free equilibrium may even be globally stable.

Some research of the fear effect on the discrete predator predator-prey model has been established. Santra has studied the impact of the effect of fear on the discrete predator-prey model with square root functional and the step size. The suitable conditions for the existence of Neimark-Sacker, flip, and fold bifurcation have also been obtained analytically (Santra, 2021). Pal et al. (2023) studied dynamics of a predator–prey system with fear and memory in the presence of two discrete delays. It is found that the dynamics of the system are found to be destabilized by the fear parameter and the predation rate, whereas the system is stabilized by the parameter that represents the strength of fading memory. In 2024, Din et al. (2024) conducted the stability and bifurcation dynamics analysis on logistic-type discrete-time model for two species' competitive interactions with fear effect. Li et al. (2024) and Mondal et al. (2024) proposed the discrete predator-prey model with fear effects and refuges. According to their research, the existence of bifurcation is driven by the fear effect and prey refuge, which both influence solution behavior. Pal et al. (2019) has considered a continuous modified Leslie–Gower model with fear effect and hunting cooperation. We obtain the following model by eliminating hunting cooperation.

$$\begin{aligned} \frac{dx}{dt} &= \left(\frac{r_1}{1+ky} - e - \beta x - \frac{c_1 y}{x+K_1}\right)x, \\ \frac{dy}{dt} &= \left(r_2 - \frac{c_2 y}{x+K_2}\right)y, \end{aligned} \tag{1}$$

where predator and prey population sizes are represented by the variables x and y, respectively. The initial condition is $x(0) \geq 0$ and $y(0) \geq 0$. The parameters $r_1, k, e, \beta, c_1, K_1, r_2, c_2$, and K_2 represent the intrinsic growth rate of the prey population, the degree of fear inhibiting the growth of the prey population, the prey mortality rate, the degree of rivalry between members of the same prey, the highest value that an x per capita decrease rate may achieve, the extent to which environment protects prey, the intrinsic growth rate of the predator population, the competition rate among individuals of predator, and the extent to which environment protects predator. In this article, we discretize model (1) using the Euler scheme, i.e.

$$\begin{aligned} x_{n+1} &= x_n + h\left(\frac{r_1}{1+ky_n} - e - \beta x_n - \frac{c_1 y_n}{x_n+K_1}\right)x_n = f_1(x_n, y_n), \\ y_{n+1} &= y_n + h\left(r_2 - \frac{c_2 y_n}{x_n+K_2}\right)y_n = f_2(x_n, y_n), \end{aligned} \tag{2}$$

where h is time step size. The objective of this research is to analyze the dynamics of the discrete model (2), then we determine the fixed point of model (2) and their local stability. The numerical simulations are also excited to demonstrate the stability and the effect of time step size (h) on the dynamics of solutions behaviour, especially on the existence of period-doubling bifurcations.

B. METHODS

This research is a analysis dynamics of the discrete modified Leslie-Gower predator-prey model. The steps involved in conducting the research are as follows.

1. Review the Dynamics Properties of Model (1)

There are four equilibrium points for model (1), i.e. E_0 (0,0), E_1 $\left(\frac{r_1 - e}{\beta}, 0\right)$, E_2 $\left(0, \frac{r_2 K_2}{c_2}\right)$, and E_3 (x^*, y^*), where x^* are (is) the root(s) of the following cubic equation

$$\eta_3 (x^*)^3 + \eta_2 (x^*)^2 + \eta_1 (x^*) + \eta_0 = 0, \tag{3}$$

with

$$\begin{aligned}
\eta_3 &= kr_2\beta c_2\\
\eta_2 &= ekr_2c_2 + \beta\left(c_2^2 + kr_2c_2(K_1 + K_2)\right) + c_1kr_2^2\\
\eta_1 &= e\left(c_2^2 + kr_2c_2(K_1 + K_2)\right) + \beta(c_2^2K_1 + kr_2c_2K_1K_2) + c_1r_2(c_2 + 2kr_2K_2) - c_2^2r_1\\
\eta_0 &= e(c_2^2K_1 + kr_2c_2K_1K_2) + c_1r_2(c_2K_2 + kr_2K_2^2) - c_2^2r_1K_1.
\end{aligned}$$

The first two equilibria are always unstable, while $E_2\left(0,\frac{r_2K_2}{c_2}\right)$ is stable when $e - \frac{r_1c_2}{c_2+kr_2K_2} + \frac{c_1r_2K_2}{c_2K_1} > 0$, and $E_3(x^*,y^*)$ is stable when $\frac{bu^*v^*}{(u^*+K_1)^2} < \min\{\kappa_1,\kappa_2\}$ where $\kappa_1 = a_1u^* + r_2$ and $\kappa_2 = \frac{1}{r_2}\left[\frac{a_2(v^*)^2}{(u^*+K_2)^2}\left(\frac{mr_0u^*}{(1+mv^*)^2} + \frac{bu^*}{u^*+K_1}\right)\right] + r_2a_1u^*$.

2. Determining the Fixed Points of the Model (2)

To find the fixed point $E\hat{}(x\hat{},y\hat{})$ of the system (1), we solve the following system.

$$\begin{aligned}
\hat{x} &= \hat{x} + h\left(\frac{r_1}{1+k\hat{y}} - e - \beta\hat{x} - \frac{c_1\hat{y}}{\hat{x}+K_1}\right)\hat{x},\\
\hat{y} &= \hat{y} + h\left(r_2 - \frac{c_2\hat{y}}{\hat{x}+K_2}\right)\hat{y}.
\end{aligned} \tag{4}$$

3. Examine the Local Stability of Each Fixed Point

The first step in figuring out stability is to linearize the model resulting the Jacobian matrix as follow.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial y_n} \end{bmatrix}$$

we determine all eigenvalues for each fixed point. A fixed point is locally asymptotically stable if all of its eigenvalues in the Jacobian matrix have modulus smaller than 1.

4. Numerical Simulation

To demonstrate the analytical results, numerical simulations are carried out using Matlab software, with parameter values that meet the stability conditions of fixed point. We investigate with various time step size values for every set of parameter values to demonstrate how time step size affects solution behavior.

C. RESULT AND DISCUSSION

1. Fixed Point and Local Stability

We solve the same system equations to get the fixed point of model (2) and the equilibrium point of model (1). Model (2)'s fixed points and model (1)'s equilibrium point are the same as follows.

a. The trivial fixed point, $E_0(0,0)$ that always exists.
b. The predator-free fixed point, $E_1\left(\frac{r_1-e}{\beta},0\right)$ that exists when $r_1 > e$.
c. The prey-free fixed point, $E_2\left(0,\frac{r_2K_2}{c_2}\right)$ that always exists.
d. The coexistence fixed point, $E_3\ (x^*, y^*)$, where x^* are (is) the root(s) of the following cubic equation.

$$\eta_3(x^*)^3 + \eta_2(x^*)^2 + \eta_1(x^*) + \eta_0 = 0, \tag{4}$$

with

$$\eta_3 = kr_2\beta c_2$$
$$\eta_2 = ekr_2c_2 + \beta\left(c_2^2 + kr_2c_2(K_1+K_2)\right) + c_1kr_2^2$$
$$\eta_1 = e\left(c_2^2 + kr_2c_2(K_1+K_2)\right) + \beta(c_2^2K_1 + kr_2c_2K_1K_2) + c_1r_2(c_2 + 2kr_2K_2) - c_2^2r_1$$
$$\eta_0 = e(c_2^2K_1 + kr_2c_2K_1K_2) + c_1r_2(c_2K_2 + kr_2K_2^2) - c_2^2r_1K_1.$$

Similarly, (4) can be expressed as

$$(x^*)^3 + 3p_2(x^*)^2 + 3p_1(x^*) + p_0 = 0, \tag{5}$$

with

$$p_2 = \frac{\eta_2}{3\eta_3},$$
$$p_1 = \frac{\eta_1}{3\eta_3},$$
$$p_0 = \frac{\eta_0}{\eta_3}.$$

Furthermore, we apply the transformation $z = x^* + p_2$ on (5) to obtain as follows.

$$z^3 + 3\gamma_1 z + \gamma_0 = 0, \tag{6}$$

with

$$\gamma_1 = p_1 - p_2^2$$
$$\gamma_0 = p_0 - 3p_1p_2 + 2p_2^3.$$

The fixed point $E_3(x^*, y^*)$ with $y^* = \frac{r_2(x^*+K_2)}{c_2}$ exist(s) if there are (is a) $x^* > 0$, with $x^* = z^* - p_2$. In other words, E_3 exist(s) if there are (is a) solution of (6), i.e. z^* and $p_2 < z^*$. The existence of $z^* > 0$ can be determined by using Cardano's criterion (Cai et al., 2015).

Lemma 1 (Cardano's criterion) (Cai et al., 2015)

i. If $\gamma_0 < 0$, (6) has a single positive root.
ii. Suppose $\gamma_0 > 0$ and $\gamma_1 < 0$, then:
   a. if $\gamma_0^2 + 4\gamma_1^3 = 0$, (6) has a positive root of multiplicity two;
   b. if $\gamma_0^2 + 4\gamma_1^3 < 0$, (6) has two positive roots;
iii. If $\gamma_0 = 0$ and $\gamma_1 < 0$, (6) has a unique positive root.

The fear level influences the existence of E_3. The fixed points of model (2) and their existence condition are same with equilibrium points of model (1). The linearization system (2) around any fixed point $\hat{E}(\hat{x}, \hat{y})$ produced the following Jacobian matrix.

$$J(\hat{E}) = \begin{pmatrix} 1 + \frac{hr_1}{1+k\hat{y}} - eh - 2\beta\hat{x}h - \frac{hc_1\hat{x}\hat{y}}{(\hat{x}+K_1)^2} - \frac{hc_1\hat{y}}{\hat{x}+K_1} & -\frac{hkr_1\hat{x}}{(1+k\hat{y})^2} - \frac{c_1h\hat{x}}{\hat{x}+K_1} \\ \frac{c_2h\hat{y}^2}{(\hat{x}+K_2)^2} & 1 + r_2h - 2\frac{c_2h\hat{y}}{\hat{x}+K_2} \end{pmatrix}.$$

By substituting $E_0(0,0)$ to the Jacobian matrix, we have

$$J(E_0) = \begin{pmatrix} 1 + r_1h - eh & 0 \\ 0 & 1 + r_2h \end{pmatrix}.$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = 1 + r_1h - eh$ and $\lambda_2 = 1 + r_2h$. Since $|\lambda_2| > 1$, $E_0(0,0)$ is unstable. This outcome is meet for stability in a system (1) as well. The Jacobian matrix for prey-free fixed point is

$$J(E_1) = \begin{pmatrix} 1 + r_1h - eh & h(r_1 - e)\left(\frac{kr_1}{\beta} + \frac{c_1}{r_1 - e + \beta K_1}\right) \\ 0 & 1 + r_2h \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $\lambda_1 = 1 + r_1h - eh$ and $\lambda_2 = 1 + r_2h$. Clearly, $|\lambda_2| > 1$. Furthermore, the fixed point $E_1\left(\frac{r_1-e}{\beta}, 0\right)$ is unstable. The stability of a continuous system also yields the same result. The Jacobian matrix for $E_2\left(0, \frac{r_2K_2}{c_2}\right)$ is as follows.

$$J(E_2) = \begin{pmatrix} 1 + \frac{r_1c_2h}{c_2 + kr_2K_2} - eh - \frac{c_1r_2K_2h}{c_2K_1} & 0 \\ \frac{hr_2^2}{a_2} & 1 - r_2h \end{pmatrix}.$$

The eigenvalues of $J(E_2)$ are $\lambda_1 = 1 + \frac{r_1 c_2 h}{c_2 + k r_2 K_2} - eh - \frac{c_1 r_2 K_2 h}{c_2 K_1}$ and $\lambda_2 = 1 - r_2 h$. If $0 < h < \frac{2}{e - \frac{r_1 c_2}{c_2 + k r_2 K_2} + \frac{c_1 r_2 K_2}{c_2 K_1}}$ then $|\lambda_1| < 1$. If $0 < h < \frac{2}{r_2}$ then $|\lambda_2| < 1$. Obviously, the fixed point $E_2\left(0, \frac{r_2 K_2}{c_2}\right)$ stable if $0 < h < \min\left\{\frac{2}{r_2}, \frac{2}{e - \frac{r_1 c_2}{c_2 + k r_2 K_2} + \frac{c_1 r_2 K_2}{c_2 K_1}}\right\}$. In continuous system, E_2 stable if $e - \frac{r_1 c_2}{c_2 + k r_2 K_2} + \frac{c_1 r_2 K_2}{c_2 K_1} > 0$, but in discrete system there is a condition for time step size h. In both discrete and continuous systems, fear has an impact on E_2 stability. The Jacobian matrix for coexistence fixed point $E_3(x^*, y^*)$ is

$$J(E_3) = \begin{pmatrix} 1 + \frac{h r_1}{1 + k y^*} - eh - 2\beta x^* h - \frac{h c_1 x^* y^*}{(x^* + K_1)^2} - \frac{h c_1 y^*}{x^* + K_1} & -\frac{h k r_1 x^*}{(1 + k y^*)^2} - \frac{c_1 h x^*}{x^* + K_1} \\ \frac{c_2 h (y^*)^2}{(x^* + K_2)^2} & 1 + r_2 h - 2\frac{c_2 h y^*}{x^* + K_2} \end{pmatrix}.$$

Since $\frac{h r_1}{1 + k y^*} - eh - \beta x^* h - \frac{h c_1 y^*}{x^* + K_1} = 0$ and $h r_2 - \frac{h c_2 y^*}{x^* + K_2} = 0$, we obtain this following matrix.

$$J(E_3) = \begin{pmatrix} 1 - \beta x^* h - \frac{h c_1 x^* y^*}{(x^* + K_1)^2} & -\frac{h k r_1 x^*}{(1 + k y^*)^2} - \frac{c_1 h x^*}{x^* + K_1} \\ \frac{c_2 h (y^*)^2}{(x^* + K_2)^2} & 1 - r_2 h \end{pmatrix}.$$

The characteristic equation $J(E_3)$ is as follows.

$$\lambda^2 - \text{trace}\left(J(E_3)\right) + \det\left(J(E_3)\right) = 0.$$

If all three of the following criteria are met, the coexistence fixed point $E_3\ (x^*, y^*)$ is locally asymptotically stable:

i. $1 + \text{trace}\left(J(E_3)\right) + \det\left(J(E_3)\right) > 0$,
ii. $1 - \text{trace}\left(J(E_3)\right) + \det\left(J(E_3)\right) > 0$,
iii. $\det\left(J(E_3)\right) - 1 < 0$.

The stability of E_3 is depend on time step size h and fear rate k.
Remark. Similar to local stability in continuous system, the local stability of fixed point $E_0(0,0)$ and $E_1\left(\frac{r_1 - e}{\beta}, 0\right)$ are always unstable. In continuous system, $E_2\left(0, \frac{r_2 K_2}{c_2}\right)$ stable if $e - \frac{r_1 c_2}{c_2 + k r_2 K_2} + \frac{c_1 r_2 K_2}{c_2 K_1} > 0$. Different with the stability condition in continuous system, in discrete system there is an additional condition that depend on time step size h. The local stability for coexistence fixed point in discrete system depend on the time step size h.

2. Numerical Simulation

In this section, for systems (1) and (2), we execute some numerical simulations by choosing the time step size which satisfy the stability condition. The values of the parameters used to perform the numerical simulations in this article are based on theoretical assumptions because the real data is not available, as shown in Table 1 Figure 1, Figure 2, and Figure 3.

Table 1. Parameter value

Parameter	Simulation 1	Simulation 2	Simulation 3
r_1	0.32	0.4	0.7
k	4	0.5	0.64
β	0.25	0.25	0.01
c_1	0.5	0.2	0.5
K_1	0.3	0.7	1
e	0.2	0.2	0.2
r_2	0.4	0.1	0.1
c_2	0.4	0.45	0.1
K_2	0.6	0.3	0.2

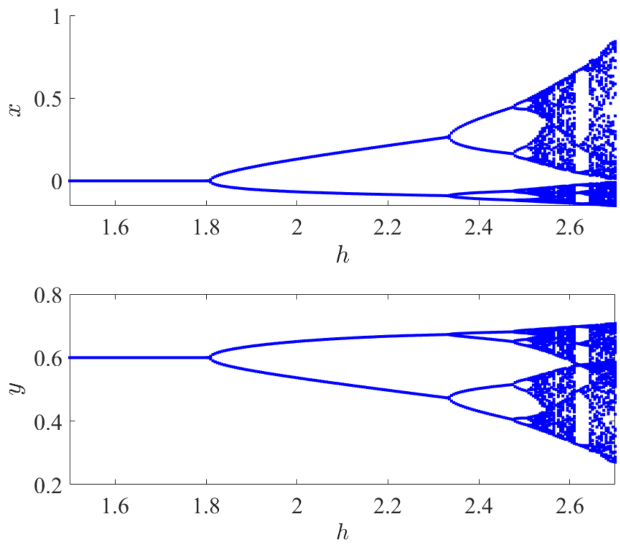

Figure 1. Bifurcation diagram in (a) $(h - x)$-plane and (b) $(h - y)$-plane for the discrete model with parameter of simulation 1

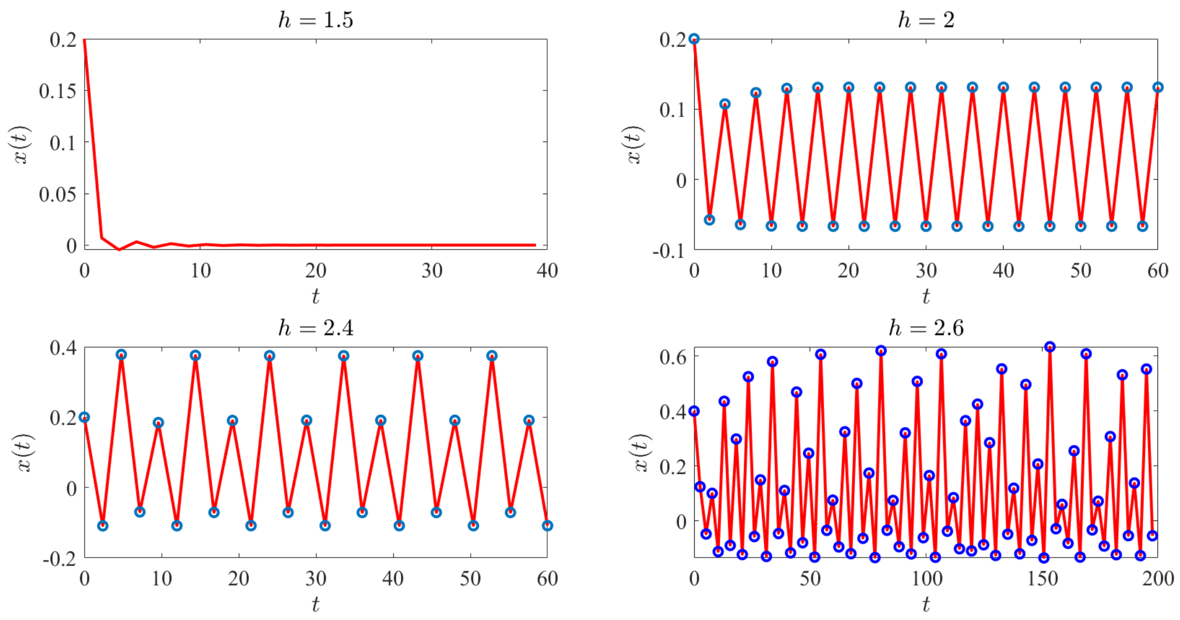

Figure 2. Numerical solutions $x(t)$ for the discrete model with parameter of simulation 1 with various value of time step size

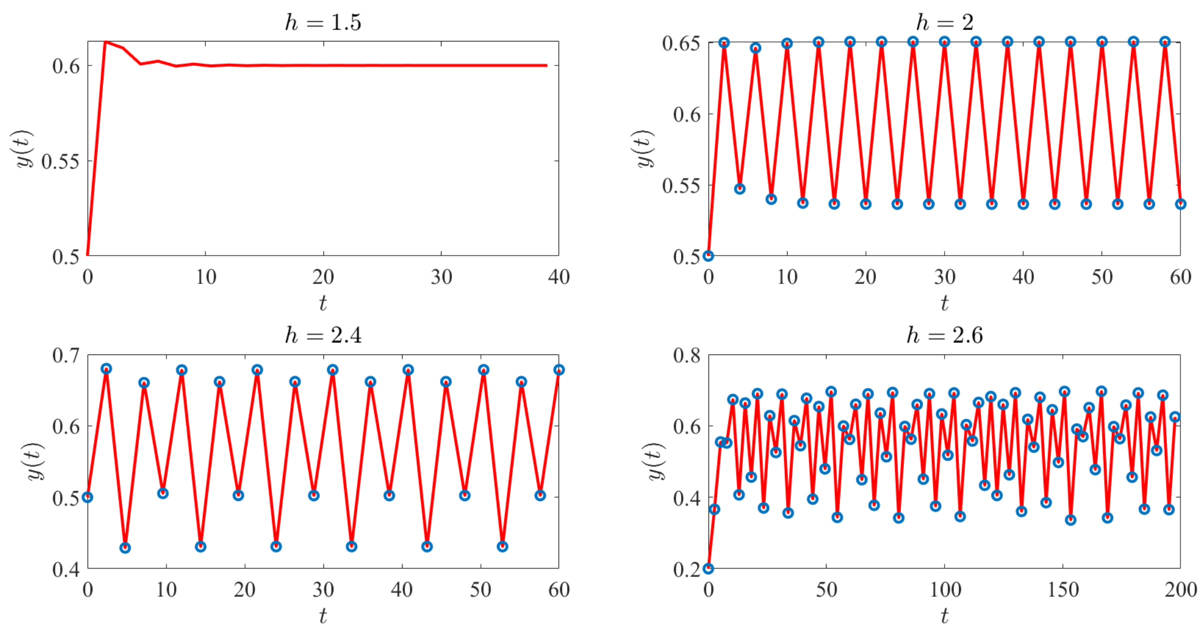

Figure 3. Numerical solutions $y(t)$ for the discrete model with parameter of simulation 1 with various value of time step size

The initial value for the first simulation is $x(0) = 0.2, y(0) = 0.5$. With the parameter values listed in Table 1 for Simulation 1, we found that there are only three fixed points which exist, namely the trivial fixed point $E_0(0,0)$ which is unstable, the predator-free fixed point $E_1(0.48,0)$ which also unstable, and the prey-free fixed point or $E_2(0,0.6)$. The time series for $x(t)$ and $y(t)$ are displayed in Figures 2 and 3, respectively. According to Figure 1, Figure 2, and Figure 3, if $h < h^* = 1.81$, the solution will converge to the $E_2(0,0.6)$. For the greater time-step sizes, the solution may show chaotic behavior ($h = 2.6$) or maybe stable periodic solution of period two ($h = 2$) and four ($h = 2.4$). The bifurcation diagram Figure 1 illustrates this phenomenon by illustrating how different values of time step size h impact a changing value of the fixed point. In the second simulation, $x(0) = 0.4$ and $y(0) = 0.2$ are the initial value. Figures 5 and 6 show the time series for $x(t)$ and $y(t)$, respectively. Figures 4(a) and 4(b) bifurcation diagrams illustrate that E_3 is locally asymptotically stable for $h < 12.56$, loses stability at $h = 12.56$, and exhibits an attractive invariant curve for $h > h^* = 12.56$. The solution stable periodic solution of period three is found at $h = 12.9$, while the solution stable periodic solution of period six is found at $h = 13.1$. Chaotic behavior can be seen by the solution at $h = 13.4$.

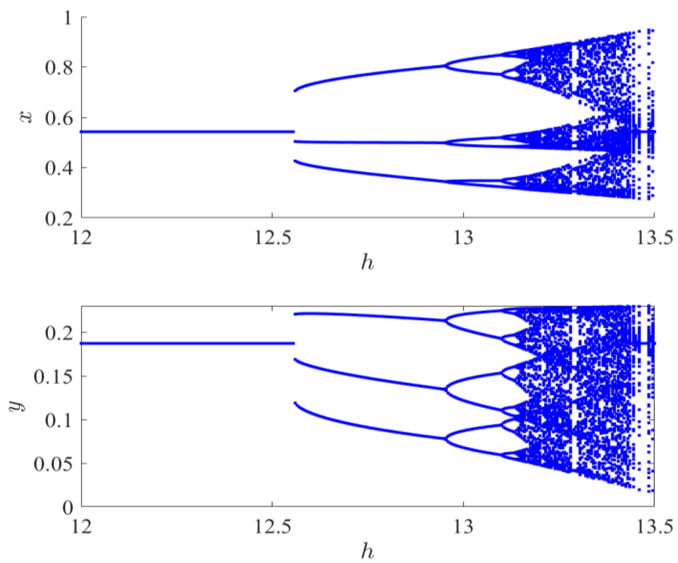

Figure 4. Bifurcation diagram in (a) $(h - x)$-plane and (b) $(h - y)$-plane for the discrete model with parameter of simulation 2

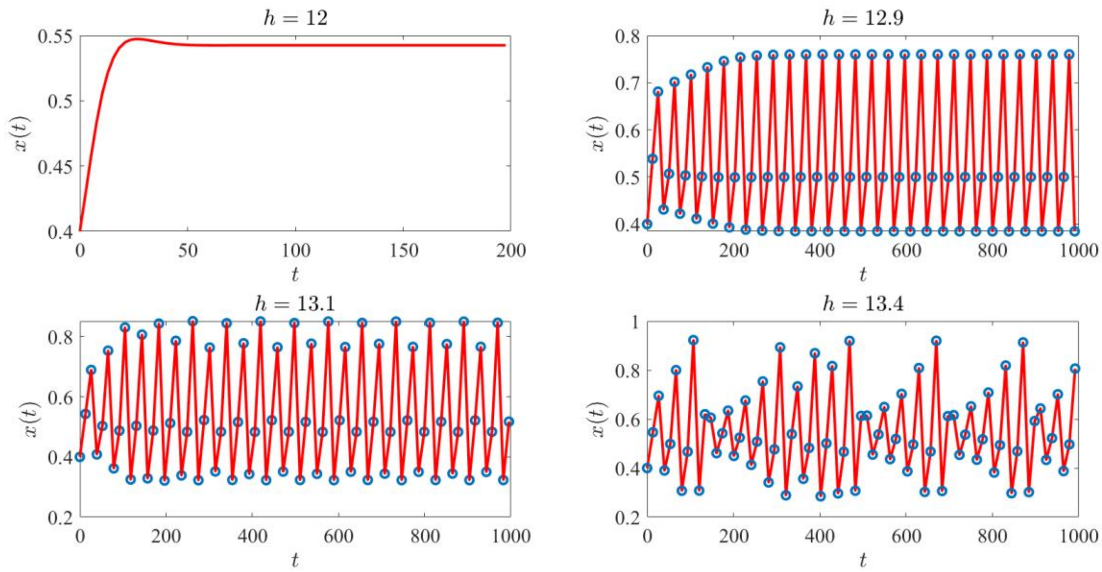

Figure 5. Numerical solutions $x(t)$ for the discrete model with parameter of simulation 2 with various value of time step size

Figure 7 illustrates how the time step size affects the system's solution. As step size h rises, the size of the limit cycle also increases. As the time step size, h, rises, so does the radius of the limit cycle. The biological implications of these numerical findings demonstrate that, although the coexistence fixed point. We present two phase portraits pictures at Figure 8 to illustrate the dynamics for each case: Figure 9 shows a stable coexistence fixed point when $h = 0.6$, and in Figure 10, where the solution converges to the limit-cycle when $h = 0.05$.

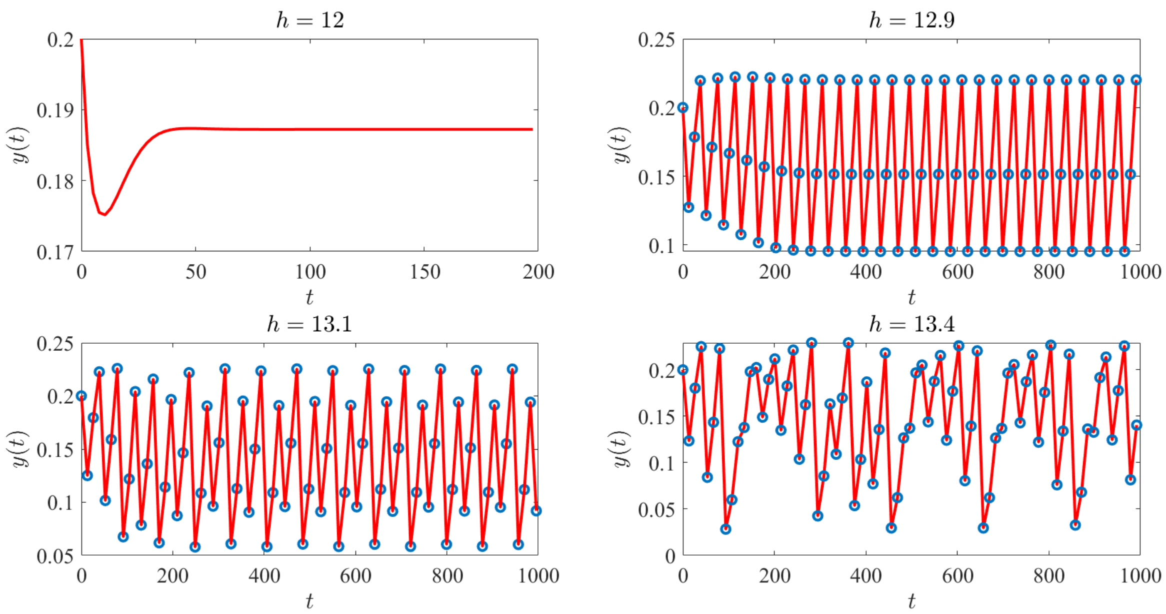

Figure 6. Numerical solutions $y(t)$ for the discrete model with parameter of simulation 2 with various value of time step size

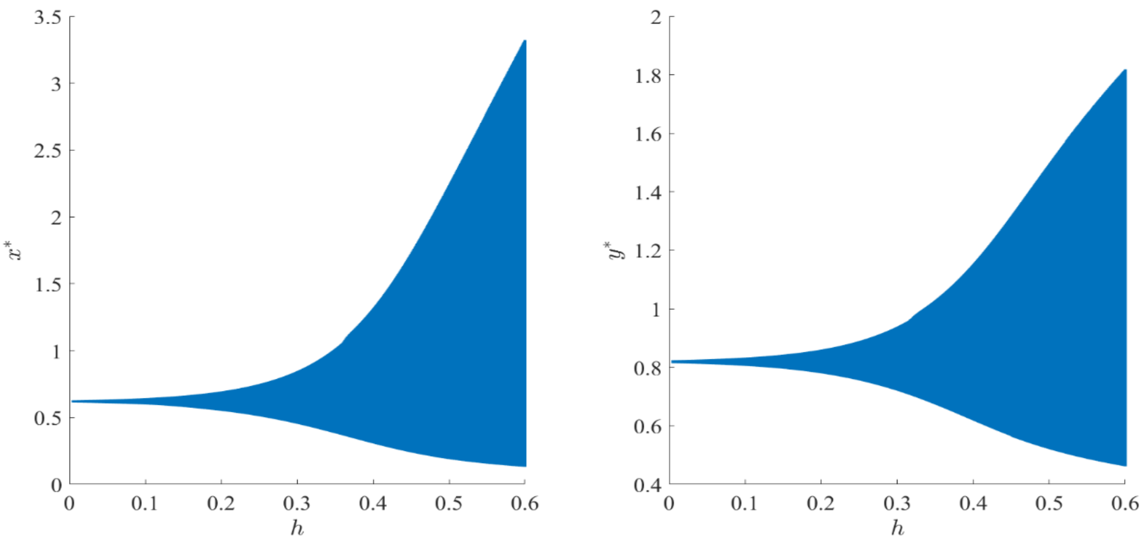

Figure 7. Impact of h to the stability of $E_3(x^*, y^*)$

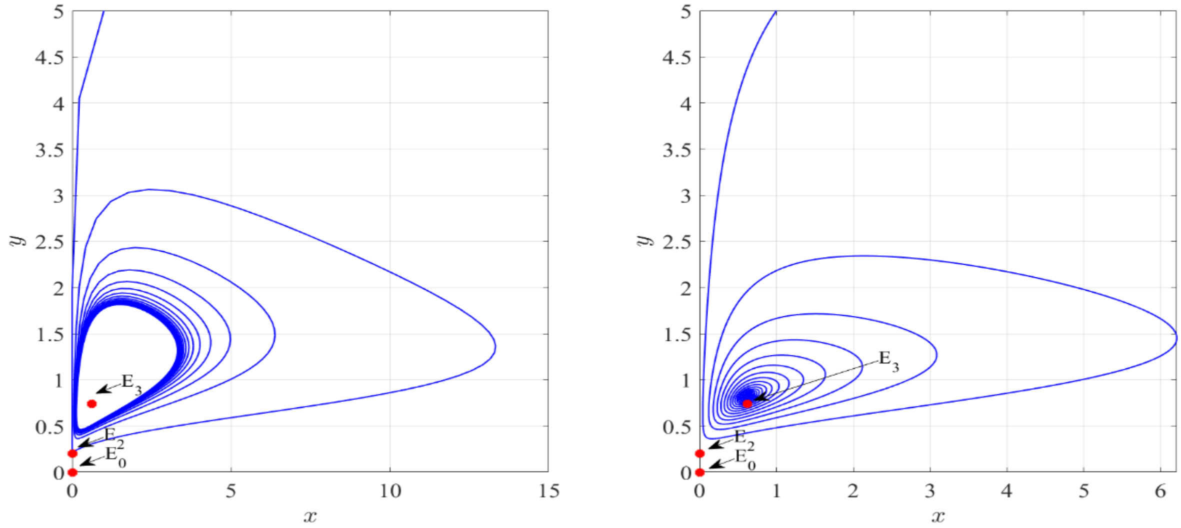

Figure 8. Phase portraits of simulation 3 at (a) $h = 0.6$ and (b) $h = 0.05$

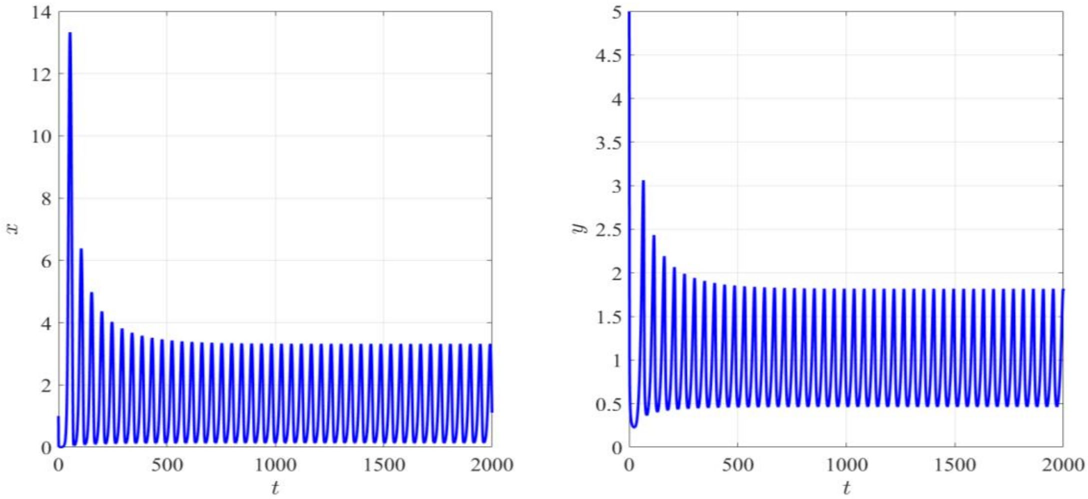

Figure 9. Solution $x(t)$ and $y(t)$ of simulation 3 at $h = 0.6$

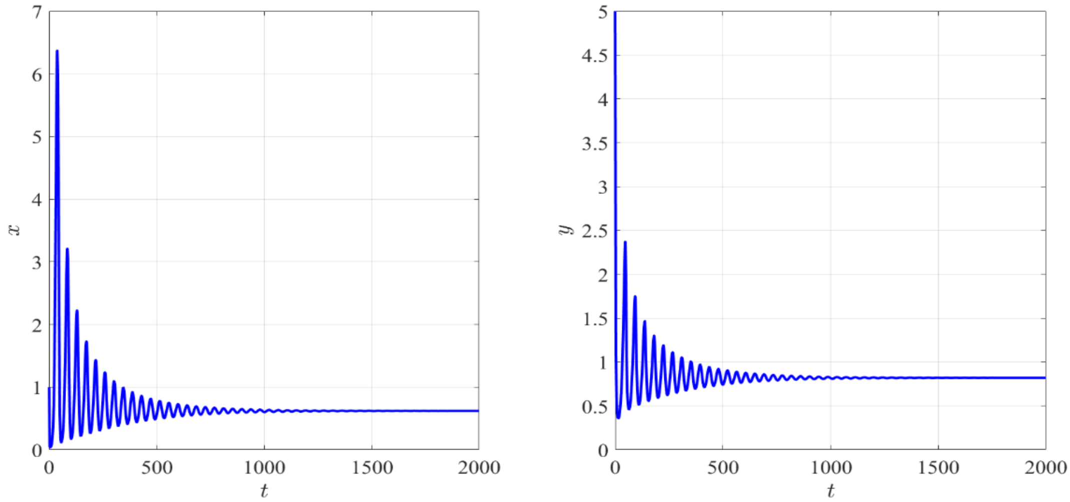

Figure 10. Solution $x(t)$ and $y(t)$ of simulation 3 at $h = 0.05$

D. CONCLUSION AND SUGGESTIONS

We investigated a discrete-time predator-prey system that includes the fear effect in this paper. The outcome in a continuous system is also unstable; both of the trivial fixed point and the predator-free fixed point. We observe a sufficient number of parameters for the local stability of the coexistence fixed point and the prey-free fixed point. Furthermore, it is demonstrated that the stability of the coexistence fixed point and the free prey fixed point is affected by the time step size and fear effect. In contrast to numerical simulations of continuous models, which employ small time step sizes to provide precise results, time step sizes for numerical simulations of discrete models can be selected based on stability conditions. The system exhibits interesting behaviors in numerical simulations through two bifurcations: a flip bifurcation that comprises orbits of periods 2, 4, 6, and 8, and an invariant cycle, and chaotic sets, respectively. These show that the fixed points are unstable when chaos reigns. Future studies can use the global Lyapunov function to determine the global stability of prey-free and coexistence fixed point. By applying the center manifold and normal forms theory of bifurcation, future research on this topic should yield more analytical conclusions concerning bifurcation.

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