**JTAM (Jurnal Teori dan Aplikasi Matematika)**
http://journal.ummat.ac.id/index.php/jtam
p-ISSN 2597-7512 | e-ISSN 2614-1175
Vol. 9, No. 1, January 2025, pp. 68-81

# Metric Coloring of Pencil Graphs

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**Article History:**
Received : 14-10-2024
Revised : 16-12-2024
Accepted : 21-12-2024
Online : 01-01-2025

**Keywords:**
Metric Dimension;
Resolving Set;
Metric Coloring;
Pencil Graph.

**ABSTRACT**

A graph is defined as an ordered pair $(V, E)$, where $V$ is a non-empty set of elements called vertices, and $E$ is a set of edges that are finite and may be empty. Each edge connects two distinct vertices from $V(G)$. Let $f: V(G) \rightarrow \{1,2,3, \dots, k\}$ be a coloring of the vertices of graph $G$, where two adjacent vertices can be colored with the same color. Considering the set of color classes $\Pi = \{C_1, C_2, \dots, C_k\}$, for a vertex $v$ in $G$, the color representation of $v$ is a $k$-vector $r(\Pi) = (d(v, C_1), d(v, C_2), \dots, d(v, C_k))$,, where $d(v, C_1) = min\{d(v, c) : c \in C_1\}$. If $r(u \mid \Pi) \neq r(v \mid \Pi)$ for every two adjacent vertices $u$ and $v$ in $G$, the coloring is called a metric coloring of $G$. Thus, it can be concluded that two adjacent vertices u and v can be colored with the same color if their metric code conditions are different. The minimum number of the metric coloring is called as metric chromatic number. The goal of this research is analizing the metric chromatic number of the pencil graph. This graph was chosen because no previous research had been carried out on this graph. The proof begins by determining the lower bound, then determining the upper bound by checking coloring function and checking the metric coloring function and the metric code function of each vertex. In this research, we got the exact value of metric chromatic number of several type of pencil graph.

https://doi.org/10.31764/jtam.v9i1.27242


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## A. INTRODUCTION

Graph theory is branch of mathematics which is part of the discrete mathematics (Putri et al., 2021). Graph itself can be defined as $G(V, E)$, where graph $G$ consist of a set of $V(G)$ and $E(G)$, where $V(G)$ is a non- empty set of elements called vertices, while $E(G)$ is a set of elements called edges. $E(G)$ is a set that can be empty of unordered pairs $u, v$ of vertices $u, v$ elements of $V(G)$ called edges. $V(G)$ is called the set of vertices of $G$ and $E(G)$ is called the set of edges of $G$ (Slamin, 2023). The number of vertices in the graph $G$ is called the order of $G$ denoted by $|V(G)|$ and the number of edges in the graph $G$ is called the size of $G$ denoted by $|E(G)|$ (Firman et al., 2022). Then the degree of a vertex $v$ in $G$ is the number of neighbouring vertices $v$ which is denoted by $d(v)$.

Graph theory is divided into several topics, including graph coloring and metric dimension (Daswa & Riyadi, 2017). Graph coloring is the assignment of colors to vertices, edges, or regions so that neighbouring elements do not have the same color (Adawiyah et al., 2023). Coloring a graph can be a positive number (Saputra et al., 2024). Graph coloring itself includes 3 (three) types, namely vertex coloring, side coloring, and region coloring (Afriantini et al., 2019). In the current study, researchers will discuss vertex coloring. Vertex coloring in graphs can be interpreted as giving colors to neighboring vertices in the graph with different colors so that, there are no neighboring vertices with the same color (Ma'arif et al., 2021).

The next topic to be discussed is metric dimension. The problem of metric dimension was first researched by Harary and Melter (Raj & George, 2017). The metric dimension of a graph is a concept from graph theory with practical applications in various domains. It is used to identify the location of a vertex in a graph uniquely based on its distances to a set of selected vertices, called the resolving set. The metric dimension is the minimum size of such a resolving set. This concept is widely applied in many field such as in navigation and routing, chemical graph theory, sensor network, etc.

The topic of metric dimension discusses the distance between vertices in a graph (Febrianti et al., 2019). Let $\Pi = \{c_1, c_2, c_3, \dots, c_k\}$ be a set of vertex in $G$. For the set $\Pi = \{c_1, c_2, c_3, \dots, c_k\} \subset$ V(G) (Estrada-Moreno et al., 2013). The vertex representation of $v_i$ respect to W is an ordered k tuple $r(v_i|\Pi) = (d(v_i, c_1), d(v_i, c_2), \dots, d(v_i, c_k)$. The set $W$ is called resolving set if every vertices in $G$ has different code representation. The metric dimension of a graph is the minimum cardinality of $W$ in graph $G$ (Epstein et al., 2015;Tillquist et al., 2023). The metric dimension of G can be denoted by $dim\,(G)$ (Kelenc et al., 2018; Knor et al., 2021). The combination of these two topics results in a new topic, namely metric coloring. It combines the concept of coloring and metric dimension where we should used the conpect of coloring in giving color of the vertices in graph $G$ and then by considering each vertex color of the graph, we determine the vertex representation of each vertex respect to each color.

Metric coloring is a vertex coloring on a graph where neighboring vertices can have the same color but have different metric code representations that produce metric chromatic numbers (Kristiana et al., 2023). The purpose of metric coloring is to find the metric chromatic number (Evalia et al., 2019). Metric chromatic number is the minimum cardinality of the metric coloring in graph $G$. Some previous studies about metric coloring was conducted by Alfarisi at al with the title of the research "Metric Chromatic Number of Unicyclic Graphs" (Alfarisi et al., 2019). Anoher research entitled " The metric coloring of related wheel graphs" was conducted by Waluyo et al in 2022. This reseach is still widely open. Thus, in this research we anlalyze the metric coloring of another graph which is not investigated yet.

In this research we focussed on the metric coloring of pencil graph (Simamora & Salman, 2015). Pencil graph is formed from $K_3$ and the graph ladder (Admadja, 2020; Makalew et al., 2020). This graph was chosen because there is no previous research that discusses the metric coloring of this graph. The results of this research perform in some theorem. The proof starts by determining the lower bound, then determining the upper bound with the coloring function, and checking the metric coloring function and metric code function of each vertex (Rohmatulloh et al., 2021). This rearch is forter the innovation and development of graph theory widely, especially in metric coloring topic.

## B. METHODS

The research methods used in this research are the pattern recognition method and the axiomatic deductive method. The pattern recognition method is a method used to formulate the metric chromatic number pattern on the graph under study ((Adawiyah et al., 2020). It can be a method for determining a pattern of the metric coloring by examining and interpreting distance-based patterns among its vertices. This technique involves extracting features like distance matrices, adjacency connections, and unique distance vectors (referred to as metric coordinates) for each vertex in relation to a selected subset of reference vertices. By analyzing these patterns, the method identifies the minimum resolving set, capable of uniquely distinguishing every vertex in the graph.

The axiomatic deductive method is a method using deductive principles that already exist in mathematical logic by using existing axioms, lemmas, and theorems so that they can be applied to metric chromatic numbers (Adawiyah et al., 2019). The axiomatic deductive method in graph theory is a rigorous mathematical approach used to develop and explore concepts, theorems, and properties of graphs based on a set of foundational axioms. This method starts with a well-defined set of basic axioms, lemmas, and theorems. Using logical reasoning and deductive processes, more complex results are derived step by step, ensuring that every conclusion is consistent with the established axioms. By building from fundamental principles, the axiomatic deductive method ensures that the proof remains a coherent and structured field, with its results universally accepted. We used some previous definition and lemmas which was proven in some relevant researches. To prove some theorems in this article we use the definition of metric coloring and a lemma related to the lower bound of metric chromatic number. The definition and lemma are as follows:

**Definition 1** (Alfarisi et al., 2019) *Suppose $f : V(G) \rightarrow \{1,2,3,\ldots,k\}$ is a vertex coloring of a graph G where two neighboring vertices can be colored with the same color. Considering the color class $= \{C_1, C_2, \ldots, C_k\}$. For a vertex $v$ in $G$, the color representation of $v$ is the $k$-vector $r(v|\Pi)=(d(v,C_1), d(v,C_2),\ldots,d(v, C_k))$, where $d(v,C_1) = min(d(v,c) : c \in C_1$. If $r(u \mid \Pi) \neq r(v \mid \Pi)$ for any two neighboring vertices u and v in G is called metric coloring. So it can be concluded that two vertices of the adjacent graph G can be colored with the same color but provided that $r(u \mid \Pi) \neq r(v \mid \Pi)$.*

**Lemma 1** (Zhang, 2016)*Suppose G is a simple and connected graph of order n, then $2 \leq \mu(G) \leq \chi(G) \leq n$.* Based on **Lemma 1**, to prove the lower bound of the metric chromatic number. Then we can adjust whether the metric coloring conforms to **Definition 1** of metric coloring. We can consider the following steps to prove the metric coloring theorem of pencil graph:

1. Determine the pencil graph to be studied;
2. Determine the vertex set and edge set of the *type I* pencil graph ($Pc_nI$) with $\geq 3$, then determine the cardinality of its vertices and edges;
3. Assign a color to each vertex in the *type I* pencil graph ($Pc_nI$) so that the two vertices that are neighbors do not have the same color;
4. Determine the color class of *type I* pencil graph ($Pc_nI$);
5. Determine the metric code on *type I* pencil graph ($Pc_nI$);

6. Determine the metric code pattern on *type I* pencil graph $(Pc_nI)$;
7. Determine the metric coloring pattern of *type I* pencil graph $(Pc_nI)$;
8. Checking whether the pattern result of metric coloring matches the definition of metric coloring on *type I* pencil graph $(Pc_nI)$, if it matches the definition, then proceed to determine the function, but if it does not match then please repeat step 3;
9. Determine the metric coloring function on *type I* pencil graph $(Pc_nI)$;
10. Determine the metric coloring result theorem on *type I* pencil graph $(Pc_nI)$;
11. Perform steps 1-10 on *type II, III*, and *IV* pencil graphs;
12. Proving the theorem that has been obtained;
13. Finish.

The Figure 1 is a schematic of the research design:

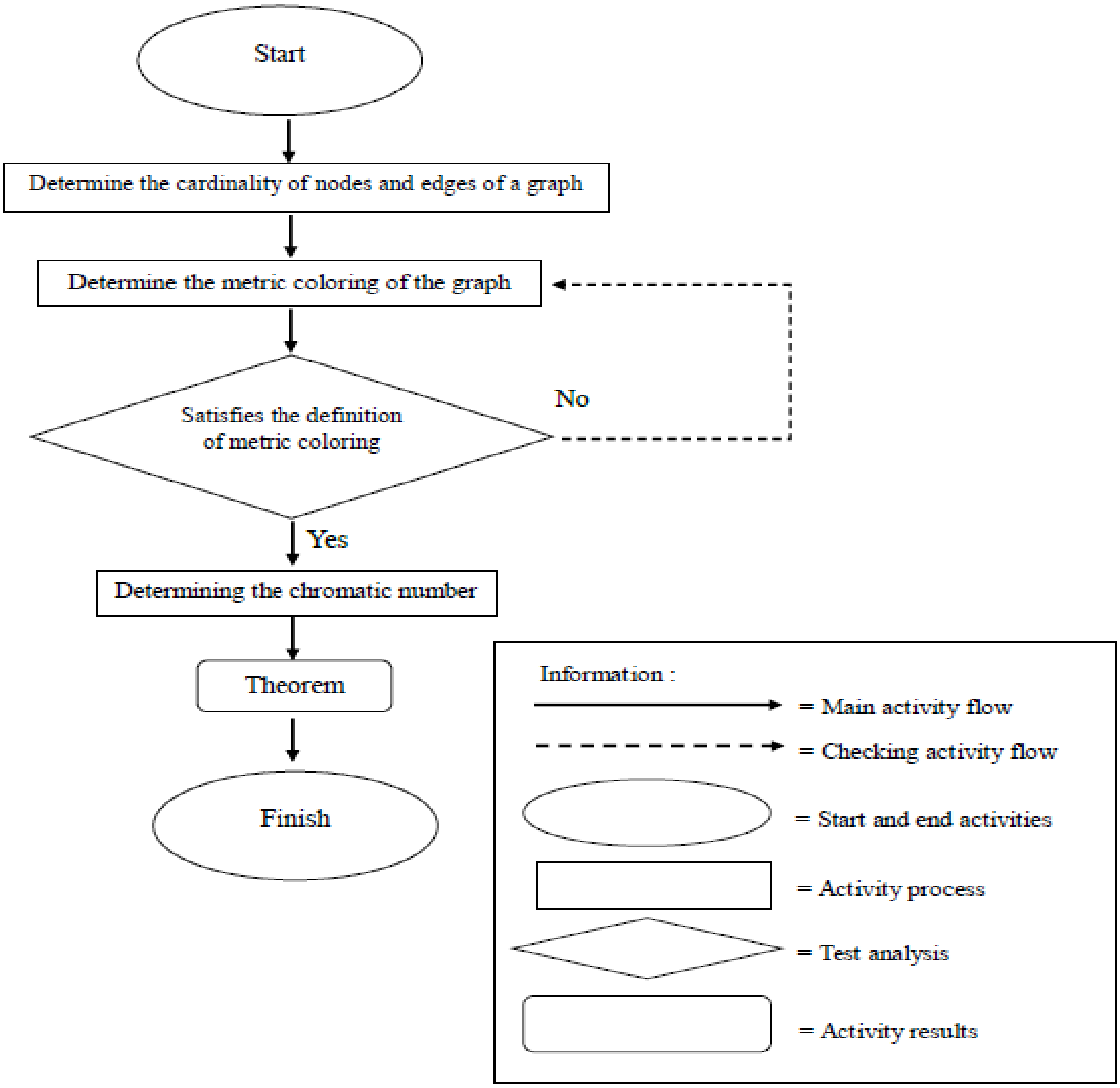


**Figure 1**: Research flowchart

## C. RESULT AND DISCUSSION

This research produces four theorems about metric coloring on pencil graphs. The following is the result of the theorem and its proof regarding the metric coloring of *type I, II, III, and IV* pencil graphs

**Theorem 1** Given a *type I* pencil graph $(Pc_nI)$ for $n \geq 3$, the metric chromatic number of the *type I* pencil graph is

$$\mu(Pc_nI) = 3$$

**Proof.** A *type I* pencil graph $(Pcn\,I)$ is a graph formed from the result of the edge comb operation of the graph $K_3$ with ladder graph. A type I pencil graph $(Pc_nI)$ has a vertex set $(Pc_nI) = \{z\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and the set of edges $E(Pc_nI) = \{x_1z\} \cup \{y_1z\} \cup \{x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{y_iy_{i+1}; 1 \leq i \leq n-1\} \cup \{x_iy_i; 1 \leq i \leq n\}$. Based on the set of vertices and the set of edges is obtained as the cardinality of $|V\,(Pc_nI)| = 2n + 1$ and the cardinality of edge $|E\,(Pc_nI)| = 3n$ respectively.

The proof of metric coloring on type I pencil graph $(Pc_nI)$ starts with prove the lower bound of metric coloring and then proceed to prove the upper bound. Based on **Lemma 1**, it is known that $\mu(Pc_nI) \geq 2$. Next, it is proved that, $\mu(Pc_nI) = 2$ is impossible. Assume $\mu(Pc_nI) = 2$, then $f: V(Pc_nI) \to \{1,2\}$. In such a way up to $\Pi = \{C_1, C_2\}$. If the vertex coloring of the graph $K_3$ in the *type I* pencil graph $(Pc_nI)$ using 2 colors, then there are two neighboring vertices that have the same color. As a result, the two neighboring vertices have the same metric code, so it contradicts **Definition 1.** Suppose the vertex $z$ that neighbors the vertex $x_1$and $y_1$. The vertex $z$ is colored red and the vertices $x_1$ and $y_1$are colored blue and then we find the representation of the code metric of vertex $z$, $x_1$ and $y_1$then at obtained code metric vertex $z\,(0,1,1), x_1(1,0,1)$ *and* $y_1(1,0,1)$. So, it is known that there are two vertices that have the same metric code, namely vertex $x_1$ and $y_1$. Based on the fact, then we get the lower bound $\mu(Pc_nI) \geq 3$. Next, it will be proven the upper bound of the metric coloring on the *type I* pencil graph which is $\mu(Pc_nI) \geq 3$. Suppose $f$ is a metric coloring, we get the set of color classes $\Pi = \{C_1, C_2, C_3\}$ where

$$C_1 = \{z\}$$
$$C_2 = \{x_i; 1 \leq i \leq n,\ i\ odd\} \cup \{y_i; 1 \leq i \leq n,\ i\ even\}$$
$$C_3 = \{x_i; 1 \leq i \leq n,\ i\ even\} \cup \{y_i; 1 \leq i \leq n,\ i\ odd\}$$

The metric coloring function is as follows.

$$f(v) = \begin{cases} 1, & \text{if } v = z \\ 2, & \text{if } v = x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq n,\ \ i \text{ odd},\ \ j \text{ even} \\ 3, & \text{if } v = x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq n,\ \ i \text{ even},\ \ j \text{ odd} \end{cases}$$

Based on the color label of $f$ in the graph $(P_{cn}I)$, the vertex representation of any vertives in $P_{cn}I$ respect to $\Pi$ can be seen in Table 1 as follow:

**Table 1.** Metric code representation on *type I* pencil graphs

| $\boldsymbol{v}$ | $\boldsymbol{r(v\|\Pi)}$ | **Conditions** |
|---|---|---|
| $z$ | $(0,1,1)$ | $-$ |
| $x_i$ | $(i,0,1)$ | $1 \leq i \leq n, i\ odd$ |
| $x_i$ | $(i,1,0)$ | $1 \leq i \leq n, i\ even$ |
| $y_j$ | $(j,0,1)$ | $1 \leq j \leq n, j\ even$ |
| $y_j$ | $(j,1,0)$ | $1 \leq j \leq n, j\ odd$ |

Based on Table 1 we can see that any two neighboring vertices have different metric code representations. For example, the vertex $z$ that neighbors the vertex $x_1$and $y_1$. Vertex $z$ has a metric code representation (0,1,1) while vertices $x_1$and $y_1$ has metric code representations (1,0,1) and (1,1,0). Thus, it is found that the upper bound of *type I* pencil graph is $\mu(P_{cn}I) \leq 3$. Based on the upper and lower bounds of the *type I* pencil graph, $3 \leq \mu(P_{cn}I) \leq 3$and the metric coloring function and metric code representation are in accordance with **Definition 1.** then it is proven that the metric chromatic number of the *type I* pencil graph is 3. Hence $\mu(P_{cn}I) \leq 3$. ■ An illustration of metric coloring on *type I* pencil graph $(Pc_nI)$ can be seen in Figure 2.

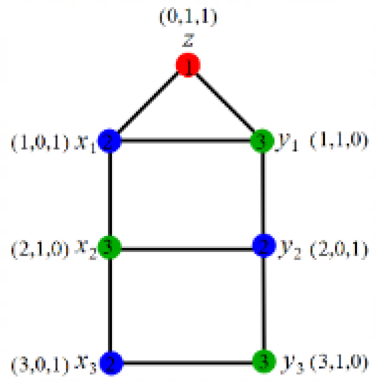


**Figure 2.** Metric coloring of *type I* pencil graph $(Pc_nI)$

**Theorem 2** Given a *type II* pencil graph $(Pc_nII)$ for $n \geq 3$, the metric chromatic number of the *type II* pencil graph is

$$\mu(Pc_nII) = 3$$

**Proof.** Pencil graph type *type II* $(Pc_nII)$ is a graph formed from the comb operation of the edge graph $K_3$ with a ladder graph with an additional straight line in the middle of vertices $z_1$ *and* $z_2$. Pencil graph *type II* $(Pc_nII)$ has the set $V\ (Pc_nII)$ $=\{z_1\} \cup \{z_2\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and the edge set $E(Pc_nII)$=$\{x_1z_1\} \cup \{y_1z_1\} \cup \{x_nz_2\} \cup \{y_nz_2\} \cup \{z_1z_2\} \cup \{x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{y_iy_{i+1}; 1 \leq i \leq n-1\} \cup \{x_iy_i;\ 1 \leq\ i\ \leq\ n-1\}$. Based on the set of vertices and the set of edges, the cardinality of $|V\ (Pc_nII)|\ =\ 2n\ +\ 2$ and the cardinality of $|E\ (Pc_nII)|\ =\ 3n\ +\ 2$ are obtained, respectively.

The proof of metric coloring on pencil graph of *type II* $(Pc_nII)$ starts with prove the lower bound of the metric coloring and then proceed to prove the upper bound. Based on **Lemma 1**, it is known that $\mu(Pc_nII) \geq 2$. Next, it is provedthat, $\mu(Pc_nII) = 2$ is impossible. Assume $\mu(Pc_nII) = 2$, then $f: V(Pc_nII) \rightarrow 1,2$. Such that $\Pi = \{C_1, C_2\}$. If the vertex coloring of the graph $K_3$in the pencil graph *type II* $(Pc_nII)$ uses 2 colors, then there are two neighboring vertices that have the same color. As a result, there are two neighboring vertices that have the same metric code, so it contradicts **Definition 1**. Suppose the vertex $z_1$is neighbor to the vertex $x_1$and $y_1$. Vertex $z_1$ is colored red and vertices $x_1$ and $y_1$are colored blue then we find the metric code representation of vertices $z_1$, $x_1$and $y_1$then we get the metric code of vertices $z_1(0{,}1{,}1)$, $x_1(1{,}0{,}1)$ and $y_1(1{,}0{,}1)$. So it is known that there are two vertices that have the same metric code, namely vertex $x_1$ and $y_1$. Then we get the lower bound $\mu(Pc_nII) \geq 3$. Next is to prove the upper bound of the metric coloring on *type II* pencil graph $\mu(Pc_nII) \leq 3$. Suppose $f$ is a metric coloring, we get the set of color classes $\Pi = \{C_1, C_2, C_3\}$ where

$$C_1 = \{z\}$$
$$C_2 = \{x_i; 1 \leq i \leq n,\ i\ odd\} \cup \{y_i; 1 \leq i \leq n,\ i\ even\} \cup \{z_2\}$$
$$C_3 = \{x_i; 1 \leq i \leq n,\ i\ even\} \cup \{y_i; 1 \leq i \leq n,\ i\ odd\}$$

The metric coloring function is as follows.

$$f(v) = \begin{cases} 1, & \text{if } v = z_1 \\ 2, & \text{if } v = x_i, y_j, z_2; 1 \leq i \leq n, 1 \leq j \leq n,\ \ i \text{ odd},\ \ j \text{ even} \\ 3, & \text{if } v = x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq n,\ \ i \text{ even},\ \ j \text{ odd} \end{cases}$$

Based on the color label of $f$ in the graph $(P_{cn}II)$, we have the vertex representation in pencil graph type II respect to $\Pi$ in Table 2 as follow:

**Table 2.** Metric code representation on *type II* pencil graphs

| $v$ | $r(v\|\Pi)$ | **Conditions** |
|---|---|---|
| $z_i$ | $(0, i, i)$ | $1 \leq i \leq 1$ |
| $z_i$ | $(i, 0, i)$ | $1 \leq i \leq 1$ |
| $x_i$ | $(i, 0{,}1)$ | $1 \leq i \leq n, i\ odd$ |
| $x_i$ | $(i, 1{,}0)$ | $1 \leq i \leq n, i\ even$ |
| $y_j$ | $(j, 0{,}1)$ | $1 \leq j \leq n, j\ even$ |
| $y_j$ | $(j, 1{,}0)$ | $1 \leq j \leq n, j\ odd$ |

Based on Table 2 we can see that any two neighboring vertices have different metric code representations. For example, the vertex $z_1$which neighbors the vertex $x_1$ and $y_1$. Vertex $z_1$ has a metric code representation of $(0{,}1{,}1)$ while vertices $x_1$ and $y_1$has metric code representations $(1{,}0{,}1)$ and $(1{,}1{,}0)$. Thus, it is found that the upper bound of *type II* pencil graph is $\mu(Pc_nII) \leq 3$. Based on the upper and lower bounds of the *type II* pencil graph, we get $3 \leq \mu(Pc_nII) \leq 3$and the metric coloring function and metric code representation are in accordance with **Definition 1**, it is proved that the metric chromatic number of the *type II* pencil graph is 3. Hence

$\mu(Pc_nII) = 3$. ■ An illustration of metric coloring on *type II* pencil graph $(Pc_nII)$ can be seen in Figure 3.

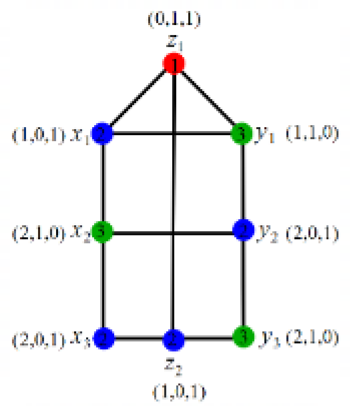


**Figure 3.** Metric coloring of *type II* pencil graph $(Pc_nII)$

**Theorem 3** Given a *type III* pencil graph $(Pc_nIII)$ for $n \geq 3$, the metric chromatic number of the *type III* pencil graph is

$$\mu(Pc_nIII) = 3$$

**Proof.** Pencil graph *type III* $(Pc_nIII)$ is a graph formed from the two-sided comb operation of graph $K3$ with ladder graph. Pencil graph *type III* $(Pc_nIII)$ has a set of vertices $V(Pc_nIII)$ = $\{z_1\} \cup \{z_2\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_i;\ 1 \leq i \leq n\}$ and the set of edges $(Pc_nIII)$ = $\{x_1z_1\} \cup \{y_1z_1\} \cup \{x_nz_2\} \cup \{y_nz_2\} \cup \{x_ix_{i+1};\ 1 \leq i \leq n-1\} \cup \{y_iy_{i+1};\ 1 \leq i \leq n-1\} \cup \{x_iy_i;\ 1 \leq i \leq n\}$. Based on the set of vertices and the set of edges, the cardinality is obtained as $|V(Pc_nIII)| = 2n + 2$ and the cardinality of the edge is $|E\ (Pc_nIII)| = 3n + 2$.

The proof of metric coloring on pencil graph of *type III* $(Pc_nIII)$ starts with proving the lower bound of metric coloring then proceed proves its upper bound. Based on **Lemma 1**, it is known that $\mu(Pc_nIII) \geq 2$. Next, it will be proved that, $\mu(Pc_nIII) = 2$ is impossible. Assume $\mu(Pc_nIII) = 2$, then $f: V(Pc_nIII) \to \{1,2\}$. Such that $\Pi = \{C_1, C_2\}$. If the vertex coloring on the graph $K_3$on the pencil graph of *type III* $(Pc_nIII)$ uses 2 colors, then there are two neighboring vertices that have the same color. As a result, there are two neighboring vertices that have the same metric code, thus contradicting **Definition 1**. Suppose the vertex $z_1$ which neighbors the vertices $x_1$and $y_1$. Vertex $z_1$ is colored red and vertices $x_1$and $y_1$ are colored blue color then we find the metric code representation of vertices $z_1$, $x_1$and $y_1$then we get the metric code of vertices $z_1(0,1,1)$, $x_1(1,0,1)$ and $y_1(1,0,1)$. So it is known that there are two vertices that have the same metric code, namely vertices $x_1$and $y_1$. Then the lower bound is obtained $\mu(Pc_nIII) \geq 3$. Next is to prove the upper bound of the metric coloring on the *type I* pencil graph which is $(Pc_nIII) \leq 3$. . Suppose $f$ is a metric coloring, we get the set of color classes $\Pi = \{C_1, C_2, C_3\}$ where

$$C_1 = \{z_1, z_2\}$$
$$C_2 = \{x_i;\ 1 \leq i \leq n,\ i\ odd\} \cup \{y_i;\ 1 \leq i \leq n,\ i\ even\}$$
$$C_3 = \{x_i;\ 1 \leq i \leq n,\ i\ even\} \cup \{y_i;\ 1 \leq i \leq n,\ i\ odd\}$$

The metric coloring function is as follows.

$$f(v) = \begin{cases} 1, & \text{if } v = z_1, z_2 \\ 2, & \text{if } v = x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq n, \ \ i \text{ odd}, \ \ j \text{ even} \\ 3, & \text{if } v = x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq n, \ \ i \text{ even}, \ \ j \text{ odd} \end{cases}$$

Based on the color label of $f$ in the graph $(P_{cn}III)$, the vertex representation of any vertices in $P_{cn}III$ respect to $\Pi$ can be seen in Table 3 as follow:

**Table 3.** Metric code representation on *type III* pencil graphs

| $v$ | $r(v\|\Pi)$ | **Conditions** |
|---|---|---|
| $z_i$ | $(0, i, i)$ | $1 \leq i \leq 1$ |
| $z_i$ | $(i, 0, i)$ | $1 \leq i \leq 1$ |
| $x_i$ | $(i, 0, 1)$ | $1 \leq i \leq \frac{n}{2}, i\ odd$ |
| $x_i$ | $(n - i + 1, 0, 1)$ | $\frac{n}{2} \leq i \leq n, i\ odd$ |
| $x_i$ | $(i, 1, 0)$ | $1 \leq i \leq \frac{n}{2}, i\ even$ |
| $x_i$ | $(n - i + 1, 1, 0)$ | $\frac{n}{2} \leq i \leq n, i\ even$ |
| $y_j$ | $(j, 0, 1)$ | $1 \leq i \leq \frac{n}{2}, i\ even$ |
| $y_j$ | $(n - j + 1, 0, 1)$ | $\frac{n}{2} \leq i \leq n, i\ even$ |
| $y_j$ | $(j, 1, 0)$ | $1 \leq i \leq \frac{n}{2}, i\ odd$ |
| $y_j$ | $(n - j + 1, 1, 0)$ | $\frac{n}{2} \leq i \leq n, i\ odd$ |

Based on Table 3 we can see that any two neighboring vertices have different metric code representations. For example, the vertex $z_1$which neighbors the vertex $x_1$ and $y_1$. Vertex $z_1$ has a metric code representation of $(0,1,1)$ while vertices $x_1$ and $y_1$has metric code representations $(1,0,1)$ and $(1,1,0)$. Thus, it is found that the upper bound of *type III* pencil graph is $\mu(Pc_nIII) \leq 3$. Based on the upper and lower bounds of the *type III* pencil graph, we get $3 \leq \mu(Pc_nIII) \leq 3$and the metric coloring function and metric code representation are in accordance with **Definition 1**, it is proved that the metric chromatic number of the *type III* pencil graph is 3. Hence $\mu(Pc_nIII) = 3$. ■ An illustration of metric coloring on *type III* pencil graph $(Pc_nIII)$ can be seen in Figure 4.

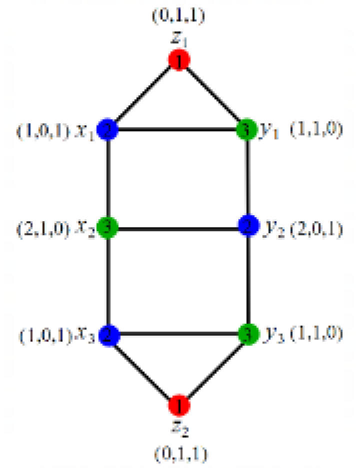


**Figure 4.** Metric coloring of *type III* pencil graph $(Pc_nIII)$

**Theorem 4** Given a *type IV* pencil graph $(Pc_nIV)$ for $n \geq 3$, the metric chromatic number of the *type IV* pencil graph is

$$\mu(Pc_nIV) = 3$$

**Proof.** Pencil graph type *type IV* $(Pc_nIV)$ is a graph formed from the comb operation of the edge graph $K_3$ with a ladder graph with an additional straight line in the middle of vertices $z_1$ *and* $z_2$. Pencil graph *type IV* $(Pc_nIV)$ has the set $V\ (Pc_nIV)$ =$\{z_1\} \cup \{z_2\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and the edge set $E(Pc_nIV)$=$\{x_1z_1\} \cup \{y_1z_1\} \cup \{x_nz_2\} \cup \{y_nz_2\} \cup \{z_1z_2\} \cup \{x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{y_iy_{i+1}; 1 \leq i \leq n-1\} \cup \{x_iy_i;\ 1 \leq\ i\ \leq\ n-1\ \}$. Based on the set of vertices and the set of edges, the cardinality of $|V\ (Pc_nIV)|\ =\ 2n\ +\ 2$ and the cardinality of $|E\ (Pc_nIV)|\ =\ 3n\ +\ 3$ are obtained, respectively.

The proof of metric coloring on pencil graph of *type IV* $(Pc_nIV)$ starts with prove the lower bound of the metric coloring and then proceed to prove the upper bound. Based on **Lemma 1**, it is known that $\mu(Pc_nIV) \geq 2$. Next, it is provedthat, $\mu(Pc_nIV) = 2$ is impossible. Assume $\mu(Pc_nIV) = 2$, then $f: V(Pc_nIV) \to \{1,2\}$. Such that $\Pi = \{C_1, C_2\}$. If the vertex coloring of the graph $K_3$in the pencil graph *type IV* $(Pc_nIV)$ uses 2 colors, then there are two neighboring vertices that have the same color. As a result, there are two neighboring vertices that have the same metric code, so it contradicts **Definition 1**. Suppose the vertex $z_1$is neighbor to the vertex $x_1$and $y_1$. Vertex $z_1$ is colored red and vertices $x_1$ and $y_1$are colored blue then we find the metric code representation of vertices $z_1$, $x_1$and $y_1$then we get the metric code of vertices $z_1(0,1,1)$, $x_1(1,0,1)$ and $y_1(1,0,1)$. So it is known that there are two vertices that have the same metric code, namely vertex $x_1$ and $y_1$. Then we get the lower bound $\mu(Pc_nII) \geq 3$. Next is to prove the upper bound of the metric coloring on *type II* pencil graph $\mu(Pc_nII) \leq 3$. Suppose $f$ is a metric coloring, we get the set of color classes $\Pi = \{C_1, C_2, C_3\}$ where

$$C_1 = \{z_1\}$$
$$C_2 = \{x_i; 1 \leq i \leq n,\ i\ odd\} \cup \{y_i; 1 \leq i \leq n,\ i\ even\} \cup \{z_2\}$$
$$C_3 = \{x_i; 1 \leq i \leq n,\ i\ even\} \cup \{y_i; 1 \leq i \leq n,\ i\ odd\}$$

The metric coloring function is as follows.

$$f(v) = \begin{cases} 1, & \text{if } v = z_1 \\ 2, & \text{if } v = x_i, y_j, z_2; 1 \le i \le n, 1 \le j \le n, \ \ i \text{ odd}, \ \ j \text{ even} \\ 3, & \text{if } v = x_i, y_j; 1 \le i \le n, 1 \le j \le n, \ \ i \text{ even}, \ \ j \text{ odd} \end{cases}$$

Based on the color label of $f$ in the graph $(P_{cn}IV)$, we have the vertex representation in pencil graph type IV respect to $\Pi$ in Table 4 as follow:

**Table 4.** Metric code representation on *type IV* pencil graphs

| $\boldsymbol{v}$ | $\boldsymbol{r(v\|\Pi)}$ | **Conditions** |
|---|---|---|
| $z_i$ | $(0, i, i)$ | $1 \le i \le 1$ |
| $z_i$ | $(i, 0, i)$ | $1 \le i \le 1$ |
| $x_i$ | $(i, 0,1)$ | $1 \le i \le n, i\ odd$ |
| $x_i$ | $(i, 1,0)$ | $1 \le i \le n, i\ even$ |
| $y_j$ | $(j, 0,1)$ | $1 \le j \le n, j\ even$ |
| $y_j$ | $(j, 1,0)$ | $1 \le j \le n, j\ odd$ |

Based on Table 4 we can see that any two neighboring vertices have different metric code representations. For example, the vertex $z_1$which neighbors the vertex $x_1$ and $y_1$. Vertex $z_1$ has a metric code representation of $(0,1,1)$ while vertices $x_1$ and $y_1$has metric code representations $(1,0,1)$ and $(1,1,0)$. Thus, it is found that the upper bound of *type IV* pencil graph is $\mu(Pc_nIV) \le 3$. Based on the upper and lower bounds of the *type IV* pencil graph, we get $3 \le \mu(Pc_nIV) \le 3$and the metric coloring function and metric code representation are in accordance with **Definition 1**, it is proved that the metric chromatic number of the *type IV* pencil graph is 3. Hence $\mu(Pc_nIV) = 3$. ■ An illustration of metric coloring on *type IV* pencil graph $(Pc_nIV)$ can be seen in Figure 5.

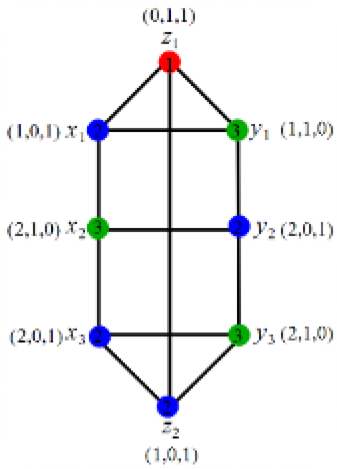


**Figure 5.** Metric coloring of *type IV* pencil graph $(Pc_nIV)$

## D. CONCLUSION AND SUGGESTIONS

Based on the results of the above research, a new theorem related to metric coloring on pencil graph *type I* ($P_{cn}I$), pencil graph *type II* ($P_{cn}II$), pencil graph *type III* ($P_{cn}III$), and pencil graph *type IV* ($P_{cn}IV$) is obtained as follows:

**Theorem 1** Given a *type I* pencil graph ($P_{cn}I$)for $n \geq 3$, the metric chromatic number of the *type I* pencil graph is

$$\mu(P_{cn}I) = 3$$

**Theorem 2** Given a *type II* pencil graph ($P_{cn}II$) for $n \geq 3$, the metric chromatic number of the *type II* pencil graph is

$$\mu(P_{cn}II) = 3$$

**Theorem 3** Given a pencil graph of *type III* ($P_{cn}III$) for $n \geq 3$, the metric chromatic number of the pencil graph of *type III* is

$$\mu(P_{cn}III) = 3$$

**Theorem 4** Given a *type IV* pencil graph ($P_{cn}IV$) for $n \geq 3$, the metric chromatic number of the *type IV* pencil graph is

$$\mu(P_{cn}IV) = 3$$

This new theorem contributes to the development of metric coloring theory or applications, especially in the context of pencil graphs or other graphs. This research can serve as a reference for other researchers who wish to conduct similar studies. The research results can serve as a guideline for conducting similar research. Open Problem: Based on previous research, the topic of metric coloring is still relatively new, so many graphs have not been explored. Readers can continue research on this metric coloring topic by using other graphs and and analyze the metric coloring of graph operation.

## ACKNOWLEDGEMENT

We would like to express our deepest gratitude for the support from COMED research group and the Faculty of Teacher Training and Education, University of Jember Indonesia in 2024.

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