JTAM (Jurnal Teori dan Aplikasi Matematika)
http://journal.ummat.ac.id/index.php/jtam
p-ISSN 2597-7512 | e-ISSN 2614-1175
Vol. 6, No. 1, January 2022, pp. 185-193

Brown-McCoy Radical in Restricted Graded Version

Puguh Wahyu Prasetyo
Mathematics Education Department, Universitas Ahmad Dahlan, Yogyakarta, Indonesia
puguh.prasetyo@pmat.uad.ac.id

Article History:
Received : 21-11-2021
Revised : 25-12-2021
Accepted : 27-12-2021
Online : 22-01-2022

Keywords:
Radical ring;
Graded ring;
Brown-McCoy radical;
Graded Brown-McCoy Radical;

ABSTRACT

Some conjectures related to the radical theory of rings are still open. Hence, the research on the radical theory of rings is still being investigated by some prominent authors. On the other hand, some results on the radical theory of rings can be implemented in another branch or structure. In radical theory, it is interesting to bring some radical classes into graded versions. In this chance, we implement a qualitative method to conduct the research to bring the Brown-McCoy radical class to the restricted graded Brown-McCoy radical class as research objective. We start from some known facts on the Brown-McCoy radical class and furthermore, let G be a group, we explain the Brown-McCoy radical restricted with respect to the group G. The result of this paper, we describe the Brown-McCoy radical in restricted graded version and it is denoted by $\mathcal{G}^G$. Furthermore, we also give the fact by explaining $\mathcal{G}^G(A) = (\mathcal{G}(A))_G$, for any ring A, as the final outcome of this paper.

https://doi.org/10.31764/jtam.v6i1.6112

A. INTRODUCTION

In this paper, we use $I \lhd R$ to express I ideal of R. Let R be a ring and let a be a nonzero element of R. The element a is called a nilpotent element if $a^n = 0$ for some $n \in \mathbb{Z}^+$. If $\forall a \in R, a^n = 0$ for some $n \in \mathbb{Z}^+$, R is called a nil ring. Based on its history, the concept of radical class was observed by Köethe by investigating the property of the class $\mathcal{N} = \{R \mid \forall r \in R, \exists n \in \mathbb{Z}^+ \ni r^n = 0\}$ of rings. In other words, the class $\mathcal{N}$ consists of all nil rings (Gardner & Wiegandt, 2004). Köethe also discovered some facts on the class $\mathcal{N}$ of all nil rings. For all rings $R \in \mathcal{N}$ then $/I \in \mathcal{N}, \forall I \lhd R$. Furthermore, the largest ideal I of a ring R which consists of nilpotent elements, is also the member of the class $\mathcal{N}$. Finally, if there exists ring R and $I \lhd R \ni I \in \mathcal{N}$ and $R/I \in \mathcal{N}$ implies $R \in \mathcal{N}$. These properties of the class $\mathcal{N}$ of all nil rings motivated Amitsur and Kurosh to define a radical class of rings. A class of rings γ is said to be radical if $A/I \in \gamma, \forall 0 \neq I$ proper ideal of A, for every ring $R, \gamma(R) = \Sigma\{I \lhd R | I \in \gamma\} \in \gamma$ and for every ring R, there exist an ideal I of R and $I, R/I \in \gamma$ implies $R \in \gamma$ (Gardner & Wiegandt, 2004). Directly, we can infer that the class $\mathcal{N}$ of all nil rings is a radical class. However, the class $\mathcal{N}_0 = \{R$ is a ring $| R^n = \{0\}$ for some $n \in \mathbb{Z}^+\}$ is not a radical class. In the development of radical theory, there are two types radical based on the construction. The lower radical and the upper radical. The lower radical $\mathcal{L}\mathcal{N}_0$ of the class $\mathcal{N}_0$ is being investigated by Baer, and it is denoted by β. Furthermore, since it follows from the fact that the radical class β is precisely the upper radical class of the class of all prime rings, the Baer radical class β is also called the prime radical class of rings. On the other hand, the class $\mathcal{L}$ of all locally nilpotent rings forms a radical class of rings and it is called the Levitzki radical. The structure of Levitzki radical can be seen in (Gardner & Wiegandt, 2004), and the implementation of Levitzki radical in a skew polynomial ring can be seen in (Hong & Kim, 2019).

The nilpotent property of skew generalized power series ring has been discussed in (Ouyang & Liu, 2013). Moreover, a nilpotent derived from the implementation Jacobson radical class $\mathcal{J} = \{A | (A,\circ)$ forms a group, where $a \circ b = a + b - ab$ for every $a, b \in A\}$ of graded group ring is described in (Ilić-Georgijević, 2021). The definition of a graded ring will be given later in Definition 7 in this section. In fact, it follows from (Prasetyo & Melati, 2020) that the Jacobson radical $\mathcal{J}(A)$ of a ring A is two-sided brace. Some property of nilpotent group and the property of skew left brace of nilpotent type are described in (Smoktunowicz, 2018) and (Cedó et al., 2019).

On the other hand, $0 \neq J \lhd R, J$ is essential if $J \cap K \neq \{0\}, \forall 0 \neq K \lhd R$ and it will be denoted by $J \lhd\circ R$. Moreover, μ is special if μ consists of prime rings, for every ring $R \in \mu$ then every nonzero ideal I of R is also contained in μ, and for every essential ideal J of R such that $J \in \mu$ implies $R \in \mu$. An upper radical class $U(\mu)$ of a special class μ is special. It follows from the fact that β is precisely $U(\pi)$, where π is the class of all prime rings. Hence, β is special. Some properties related to special classes of rings and their generalization and their implementation in the development of the radical theory of rings and modules can be seen in (France-Jackson et al., 2015; Prasetyo et al., 2017, 2020; Prasetyo, Setyaningsih, et al., 2016; Prasetyo, Wijayanti, et al., 2016; Wahyuni et al., 2017).

Furthermore, the class of all simple rings with unity is denoted by $\mathcal{M}$. The upper radical $U(\mathcal{M})$ was being observed by Brown and McCoy, and it is called the Brown-McCoy radical class, and it is denoted by $\mathcal{G}$ (Gardner & Wiegandt, 2004). On the other hand, Emil Ilić-Georgijević in his paper (Ili'c-Georgijevi'c, 2016) introduce a large graded Brown-McCoy radical of a graded ring and compare with the classical graded Brown-McCoy of a graded ring. In this paper, for a fixed group G, we scrutinize the restricted G −graded Brown-McCoy radical which is denoted by $\mathcal{G}^G$ by using fundamental concept of radical class of ring for graded. Moreover for any ring A, we explain what $\mathcal{G}^G(A)$ is.

We provide some examples of simple rings with unity and their counterexamples.

Example 1.

Consider the following concrete simple rings with unity

1. Every field is a simple ring with unity.
2. Let F be a field. Then the set $M_n(F)$ of all matrices of the size $n \times n$ over F forms a simple ring with unity

We give simple concrete rings which do not contain unity.

Example 2.

1. Let $M_\infty(R)$ be the ring of all infinite matrices which are row infinite over a ring R, that is, every matrix in $M_\infty(R)$ has a countably infinite number of rows, but almost all entries in each row are equal to 0. In the case, R is a field, then $M_\infty(R)$ is simple. Clearly, the center of $M_\infty(R)$ is $\{0\}$. Therefore, the simple ring $M_\infty(R)$ does not contain the identity element.
2. The ideal $2\mathbb{Z}_4 = \{0,2\}$ of $\mathbb{Z}_4 = \{0,1,2,3\}$ is a simple ring, and it does not have unity.
   The Cayley tables of the addition and multiplication modulo 4 of $\mathbb{Z}_4$ respectively are shown in Table 1 and Table 2 below.

Table 1. Addition modulo 4 of $\mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 2. Multiplication modulo 4 of $\mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

We further use the set $\mathbb{Z}_4$ of all integer numbers modulo 4 to construct a graded ring in Example 9. In fact, for every special class μ of rings, μ is essentially closed. However, every class of prime rings does not necessarily to be essentially closed. We provide the following examples to make these conditions clear.

Example 3.
The class π is special (Gardner & Wiegandt, 2004).

Example 4.
Let R be a prime ring. The prime ring R is called a $*-$ring if $R/I \in \beta$ for every nonzero proper ideal I of R. The definition of $*-$ring was introduced by Halina Korolczuk in 1981 (Prasetyo et al., 2017). Let $*$ be the symbol to express the class of all $*-$rings. However, $*$ is not closed under essential extension.

It follows from Example 4 that there is a class of prime rings, but it is not essentially closed. This condition motivated the existence of the definition of essential cover and essential closure.

Definition 5.
Let δ be any class of rings. The essential cover of δ is denoted by $\varepsilon(\delta)$, and it is defined as $\varepsilon(\delta) = \{A | \exists B \lhd\circ A \ni B \in \delta\}$. Moreover, if δ is closed under essential extension, then $\varepsilon(\delta) = \delta$. Furthermore, the set $\delta_k = \bigcup_{t=0}^{\infty} \delta^t = 0$ is the essential closure of δ, where $\delta^{(0)} = \delta$ and $\delta^{(t+1)} = \varepsilon(\delta^{(t)})$.

In general, the essential closure μ_k of a special class of rings μ is μ itself since $\mu^{(t)} = \mu$ for every $t \in \{0,1,2,\dots\}$. However, in the case of nonspecial class of ring, the essential closure δ_k of δ strictly contains δ. We provide the following example.

Example 6.
It follows from Example 4 that the class $*$ is not special. The essential closure of $*_k$ of $*$ strictly contains $*$.

A graded ring is one of the kinds of rings such that its structure is being investigated by prominent authors. We give highlight some research outcomes related to the existence of graded rings. In the point of view of an epsilon category, the multiplicity of graded algebras and epsilon-strongly groupoid graded can be found in (Das, 2021) and (Nystedt et al., 2020), respectively. The studies on graded ring related to Leavitt path algebra can be seen in (R. Hazrat, 2014; Roozbeh Hazrat et al., 2018; Lännström, 2020; Vaš, 2020a, 2020b). Furthermore, the studies on graded ring related to the specific structure of rings and modules, namely weakly prime ring, non-commutative rings, prime spectrum, unique factorization rings, positively graded rings, simple rings, $S-$Noetherian ring, and Dedekind rings can be accessed in (Abu-Dawwasb, 2018; Al-Zoubi & Jaradat, 2018; Alshehry & Abu-Dawwasb, 2021; Çeken & Alkan, 2015; Ernanto et al., 2020; Kim & Lim, 2020; Nystedt & Öinert, 2020; Wahyuni et al., 2020; Wijayanti et al., 2020) respectively. Thus, it is interesting to investigate some further properties and structures related to graded rings. On the other hand, the purpose of this research is to determine what $\mathcal{G}^G(R)$ is. Finally, in the following definition, we provide the definition a graded ring.

Definition 7.
Let G be a monoid. A ring R is a G −graded ring if $R = \bigoplus_{g \in G} R_g$ where the set $\{R_g | g \in G\}$ is the collection of additive subgroups of $R \ni R_g R_h \subseteq R_{gh} \forall\, g, h \in G$ (Kim & Lim, 2020). In order to make the reader be clear in understanding the concept of a graded ring, we provide the following examples.

Example 8.
Let $R(x)$ be the set of all infinite sequences $(a_0, a_1, a_2, a_3, \dots)$ where $a_i \in R$ and R is a ring for every $i \in \{0,1,2,3, \dots\}$ and where there exists $n \in G = \{0,1,2,3 \dots\}$ such that for all integers $k \geq n, a_k = 0$. Directly we can infer that the polynomial ring $R(x) = \bigoplus_{g \in G} R_g$ where $R_0 = (a_0, 0,0, \dots), R_1 = (0, a_1, 0,0, \dots), \dots R_g = (0,0, \dots, a_g, 0,0, \dots)$ such that $R_g R_h \subseteq R_{gh}$ for every $g, h \in G$. The polynomial $R(x)$ is graded, and it is graded by its degree. Therefore, for every ring R, the graded ring naturally exists. We also provide another example of a graded ring as follows.

Example 9.
Consider the set $\mathbb{Z}_4 = \{0,1,2,3\}$ and let A be any ring. Now the set $M_{2\times2}(A)$ is the set of all 2×2 matrices over ring A. Define $A_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We therefore have $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$ and $A_g A_h \subseteq A_{gh}$ for every $g, h \in Z_4$. Thus, we can infer that A is a $\mathbb{Z}_4$ −graded ring.

Some further and nontrivial examples of graded rings can be accessed in (Alshehry & Abu-Dawwasb, 2021) and (Pratibha et al., 2017). Moreover, the monoid G can be strictly replaced by any arbitrary group. Let R and S be graded rings with respect to G, the symbol R^u will denote the underlying ungraded ring. A ring homomorphism f which maps R to S is called a graded homomorphism of degree (h, k) if $f(R_g) \subseteq S_{hgk}, \forall g \in G$. The existence of this graded homomorphism motivated the existence of the definition of graded radical. For simply, the definition of graded radical class can be seen in Definition 2 in (Fang & Stewart, 1992), which is similar to the definition of radical class in the graded version.

Furthermore, we shall follow the construction of the restricted graded radical introduced by Hongjin Fang and Patrick Stewart in their paper (Fang & Stewart, 1992). Let γ be radical, $\gamma^G = \{R | R \text{ is a } G \text{ −graded ring and } A^u \in \gamma\}$. Moreover, for further consideration, it will be called a graded radical γ^G of γ.

Some properties of graded radical related to the normality and specialty of the graded radical can be seen in (Fang & Stewart, 1992). The previous work on Brown-McCoy has been developed by Emil Ilić-Georgijević in 2016. He introduced a large Brown-McCoy radical for graded ring (Ili'c-Georgijevi'c, 2016). Moreover, the aim of this research is to describe what restricted graded Brown-McCoy radical $\mathcal{G}^G$ is by following the work of (Fang & Stewart, 1992) and for every ring A, we also describe what $\mathcal{G}^G(A)$ is. The flowchart of this research can be seen in the Figure 1 below.

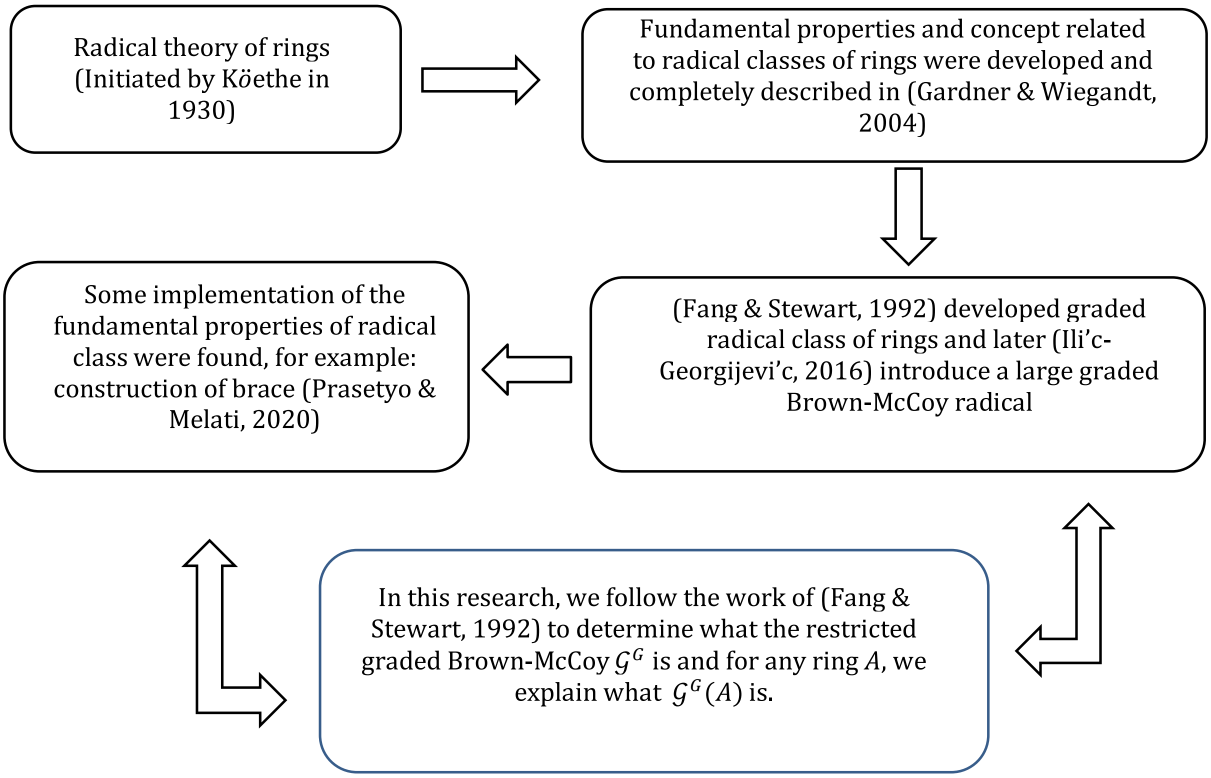

Figure 1. The Research Flowchart

B. METHODS

This research is conducted using a qualitative method derived from facts and known concepts from a literature study. To gain some properties of radical of rings, we follow some concepts and radical construction (Gardner & Wiegandt, 2004). There are lower radical class and upper radical class. In this part, we focus on the upper radical class construction. We post the known result in the early part of Section C. We start with the structure of what the Brown-McCoy radical of a ring actually is and prove the property completely. In the development of graded radicals, the graded version of the radical class of rings has been being investigated. Gardner and Plant, in their paper (Gardner & Plant, 2009), investigated and compared the Jacobson radical and graded Jacobson radical. The homogeneity radicals defined by the nilpotency of a graded semigroup ring are described in (Hong et al., 2018). Some further fundamental structures of graded radical of rings can also be studied from (Lee & Puczylowski, 2014), (Hong et al., 2014) and (Mazurek et al., 2015). However, we shall follow the concept of restricted graded radical which was introduced by (Fang & Stewart, 1992) to construct the restricted graded Brown-McCoy radical. Finally, in virtue of the construction of the restricted graded Brown-McCoy radical; we can determine the structure of the restricted graded Brown-McCoy radical.

C. RESULT AND DISCUSSION

We separate this part into two subsections. In the first part, we describe what the Brown-McCoy of a ring actually is. Moreover, in the second part, we describe the construction of restricted graded Brown-McCoy radical and give some of its properties.

1. Brown-McCoy Radical

We start this part with this proposition as a reminder and we give complete proof.

Proposition 10. (Gardner & Wiegandt, 2004)
Let A be any ring, $\mathcal{G}(A) = \cap \{I_\lambda | A/I_\lambda \in \mathcal{M}\}$

Proof.

In fact, $\mathcal{G} = \mathcal{U}(\mathcal{M}) = \{A |$ there is no $A/I \in \mathcal{M}$ for every I ideal of $A\}$y. Now let $A \in \mathcal{M}$. Suppose I and K be ideals of A such that $IK = \{0\}$. Since the ring A is a simple ring, in the case of $K = A$ implies $I = \{0\}$ or in the case of $I = A$ implies $K = \{0\}$. Hence, $\{0\}$ is a prime ideal of A. Thus, A is prime. So, we may deduce that $\mathcal{M}$ consists of prime rings. Furthermore $\mathcal{M}$ is hereditary, and it is essentially closed. This gives $\mathcal{M}$ is special. Thus $\mathcal{G}$ is hereditary. Furthermore, since $\mathcal{M}$ is essentially closed, the essential cover $\mathcal{EM}$ of $\mathcal{M}$ coincide with $\mathcal{M}$. It follows from Theorem 3.7.2 in (Gardner & Wiegandt, 2004) that $\mathcal{G}(A) = \cap \{I_\lambda | A/I_\lambda \in \mathcal{M}\}$. ■

2. Graded Brown-McCoy Radical

Let G be any group. It follows from the construction of restricted $G-$graded radical described in (Fang & Stewart, 1992) that a restricted $G-$graded Brown-McCoy radical class can be defined as $\mathcal{G}^G = \{A$ is a $G-$graded ring $| A^u \in \mathcal{G}\}$, where A^u is underlying ungraded of the ring A. For further consideration, we call the class of rings $\mathcal{G}^G$ by graded Brown-McCoy radical. In the next theorem, we will show that the graded Brown-McCoy radical $\mathcal{G}^G$ is hereditary.

Theorem 11.
Let G be any group. The restricted $G-$graded Brown-McCoy radical $\mathcal{G}^G$ is hereditary.

Proof.
It can be directly inferred by the hereditaries of the Brown-McCoy radical $\mathcal{G}$ and Proposition 2 in (Fang & Stewart, 1992) that $\mathcal{G}^G$ is hereditary. However, we will provide the proof in detail for the reader. Now let $A \in \mathcal{G}^G$ and let I be any nonzero homogenous proper ideal of A. Then $A = \bigoplus_{g \in G} A_g$ and $I = \bigoplus_{h \in G} I \cap A_h$. Hence, I is a $G-$graded ring. It is clear that I^u is a ideal of $A^u \in \mathcal{G}$. Since $\mathcal{G}$ is hereditary, $I^u \in \mathcal{G}$. So, we can deduce that $I \in \mathcal{G}^G$. Thus, $\mathcal{G}^G$ is hereditary. ■

The hereditaries of $\mathcal{G}^G$ explained in Theorem 11 implies the following property.

Theorem 12.
Let G be any group, $\mathcal{G}^G(A)$ is the intersection of all $G-$graded ideals I of A such that A/I is $G-$graded simple ring with unity, where A is a $G-$graded ring.

Proof.
Let A be any $G-$graded ring. In virtue of Theorem 2, $\mathcal{G}^G$ is hereditary. Furthermore, since $\mathcal{G}^G$ is hereditary and it follows from Proposition 2 in (Fang & Stewart, 1992) that $\mathcal{G}^G(A) = (\mathcal{G}(A))_G$, where $(\mathcal{G}(A))_G = \bigoplus_{g \in G} \{\mathcal{G}(A) \cap A_g | g \in G\}$. Now let $\{I_\lambda\}, \lambda \in \Lambda$, where Λ is index, be the collection of all ideals of A such that $A/I_\lambda \in \mathcal{M}$. Define $I_\lambda^G = \bigoplus_{g \in G} I_\lambda \cap A_g$ for every $\lambda \in \Lambda$. It is clear that I_λ^G is a $G-$graded ideal of A such that I_λ^G is simple $G-$graded with unity. Now the intersection of all $\{I_\lambda\}, \lambda \in \Lambda$ is

$$\begin{aligned}\cap_{\lambda \in \Lambda} I_\lambda^G &= \cap_{\lambda \in \Lambda} \left\{\bigoplus_{g \in G} I_\lambda \cap A_g\right\} \\ &= \bigoplus_{g \in G} \left\{(\cap_{\lambda \in \Lambda} I_\lambda) \cap A_g\right\} \\ \cap_{\lambda \in \Lambda} I_\lambda^G &= \bigoplus_{g \in G} \left\{\mathcal{G}(A) \cap A_g \middle| g \in G\right\} \qquad (1)\end{aligned}$$

On the other hand,

$$\mathcal{G}^G(A) = (\mathcal{G}(A))_G \tag{2}$$

It follows from equations (1) and (2) that $(\mathcal{G}(A))_G$ is precisely the intersection of all G −graded ideals I of A such that A/I is G −graded simple ring with unity which completes the proof. ■

On the other hand, a radical class is called an N −radical if it is normal and supernilpotent. The detail of the definition of normal and supernilpotent can be accessed in (Gardner & Wiegandt, 2004). In fact, the Brown-McCoy radical is not an N −radical as also explained in (Gardner & Wiegandt, 2004) which implies that the property explained in Theorem 3.18.14 in (Gardner & Wiegandt, 2004) does not hold for a ring of Morita context T. However, it is interesting to investigate how about the restricted graded version for the Brown-McCoy radical. If the property also does not hold for the restricted version, then a counter-example should be exist.

D. CONCLUSION AND SUGGESTIONS

It follows from the research outcomes of this research that for a fixed group G, we can contruct the restricted graded Brown-McCoy and let A be ring, $\mathcal{G}^G(A) = \cap \{I_\lambda | A/I_\lambda$ is a G −graded which is a member of $\mathcal{M}\}$. For further research, we can be continue to investigate the restricted G −graded for Levitzki radical, Thierrin radical, anti simple radical, and Behrens radical.

ACKNOWLEDGEMENT

This research is original research in the fundamental research category supported by LPPM Universitas Ahmad Dahlan under the contract number PD-185/SP3/LPPM-UAD/VI/2021.

REFERENCES

Abu-Dawwasb, R. (2018). Graded prime submodules over non-commutative rings. *Vietnam Journal of Mathematics*, *46*(3), 681–692. https://doi.org/10.1007/s10013-018-0274-2.

Al-Zoubi, K., & Jaradat, M. (2018). On the graded classical prime spectrum of a graded module. *Proyecciones*. https://doi.org/10.4067/S0716-09172018000300519

Alshehry, A. S., & Abu-Dawwasb, R. (2021). Graded weakly prime ideals of non-commutative rings. *Communications in Algebra*. https://doi.org/10.1080/00927872.2021.1928151

Cedó, F., Smoktunowicz, A., & Vendramin, L. (2019). Skew left braces of nilpotent type. *Proceedings of the London Mathematical Society*, *118*(6), 1367–1392. https://doi.org/10.1112/plms.12209

Çeken, S., & Alkan, M. (2015). On Graded Second and Coprimary Modules and Graded Secondary Representations. *Bulletin of the Malaysian Mathematical Sciences Society*. https://doi.org/10.1007/s40840-014-0097-6

Das, S. (2021). Epsilon multiplicity for graded algebras. *Journal of Pure and Applied Algebra*, *225*(10).

Ernanto, I., Marubayashi, H., Ueda, A., & Wahyuni, S. (2020). Positively Graded Rings which are Unique Factorization Rings. *Vietnam Journal of Mathematics*. https://doi.org/10.1007/s10013-020-00407-1

Fang, H., & Stewart, P. (1992). Radical Theory for Graded Rings. *Journal of the Australian Mathematical Society*, *52*, 143–153.

France-Jackson, H., Wahyuni, S., & Wijayanti, I. E. (2015). Radical related to special atoms revisited. *Bulletin of the Australian Mathematical Society*. https://doi.org/10.1017/S0004972714000811

Gardner, B. J., & Plant, A. (2009). The graded Jacobson radical of associative rings. *Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica*, *59*(1), 31–36.

Gardner, B. J., & Wiegandt, R. (2004). *Radical Theory of Rings*. Marcel Dekker, Inc.

Hazrat, R. (2014). Leavitt path algebras are graded von neumann regular rings. *Journal of Algebra*. https://doi.org/10.1016/j.jalgebra.2013.10.037

Hazrat, Roozbeh, Rangaswamy, K. M., & Srivastava, A. K. (2018). Leavitt path algebras: Graded direct-finiteness and graded Σ-injective simple modules. *Journal of Algebra*. https://doi.org/10.1016/j.jalgebra.2018.01.041

Hong, C. Y., & Kim, N. K. (2019). Descriptions of radicals of skew polynomial and skew Laurent polynomial rings. *Journal of Pure and Applied Algebra*, *223*, 3413–3424.

Hong, C. Y., Kim, N. K., Madill, B. W., Nielsen, P. P., & Ziembowski, M. (2018). Homogeneous nilradicals over semigroup graded rings. *Journal of Pure and Applied Algebra*, *222*, 1513–1528.

Hong, C. Y., Kim, N. K., & Nielsen, P. P. (2014). Radicals in skew polynomial and skew Laurent polynomial rings. *Journal of Pure and Applied Algebra2*, *218*, 1916–1931.

Ili'c-Georgijevi'c, E. (2016). On Graded Brown–McCoy Radicals of Graded Rings. *Journal of Algebra and Its Applications*, *15*(8), 1650143:(1-13). https://www.worldscientific.com/doi/10.1142/S0219498816501437

Ilić-Georgijević, E. (2021). On the Jacobson radical of a groupoid graded ring. *Journal of Algebra*. https://doi.org/10.1016/j.jalgebra.2021.01.007

Kim, D. ., & Lim, J. W. (2020). When are graded rings graded S-Noetherian rings. *Mathematics*, *8*(9).

Lännström, D. (2020). Induced quotient group gradings of epsilon-strongly graded rings. *Journal of Algebra and Its Applications*. https://doi.org/10.1142/S0219498820501625

Lee, P.-H., & Puczylowski, E. R. (2014). On prime ideals and radicals of polynomial rings and graded rings. *Journal of Pure and Applied Algebra*, *218*, 323–332.

Mazurek, R., Nielsen, P. P., & Ziembowski, M. (2015). The upper nilradical and Jacobson radical of semigroup graded rings. *Journal of Pure and Applied Algebra2*, *219*, 1082–1094.

Nystedt, P., & Öinert, J. (2020). Simple graded rings, nonassociative crossed products and Cayley-Dickson doublings. *Journal of Algebra and Its Applications*. https://doi.org/10.1142/S021949882050231X

Nystedt, P., Öinert, J., & Pinedo, H. (2020). Epsilon-Strongly Groupoid-Graded Rings, The Picard Inverse Category and Cohomology. *Glasgow Mathematical Journal*. https://doi.org/10.1017/S0017089519000065

Ouyang, L., & Liu, J. (2013). Nilpotent Property of Skew Generalized Power Series Rings. *Advances in Mathematics*, *42*(6), 782–794.

Prasetyo, P. W., & Melati, C. Y. (2020). Konstruksi Brace Dua Sisi Dengan Menggunakan Ring Jacobson. *Limits*, *17*(123–137).

Prasetyo, P. W., Setyaningsih, R., Wijayanti, I. E., & France-Jackson, H. (2016). Some properties of prime essential rings and their generalization. *AIP Conference Proceedings*. https://doi.org/10.1063/1.4958543

Prasetyo, P. W., Wijayanti, I. E., & France-Jackson, H. (2016). Some special classes μ whose upper radical U (μ) determined by μ coincides with the upper radical U (∗μ)k determined by (∗μ)k. *AIP Conference Proceedings*. https://doi.org/10.1063/1.4940814

Prasetyo, P. W., Wijayanti, I. E., & France-Jackson, H. (2017). ∗p -modules and a special class of modules determined by the essential closure of the class of all ∗-rings. *JP Journal of Algebra, Number Theory and Applications*. https://doi.org/10.17654/NT039010011

Prasetyo, P. W., Wijayanti, I. E., France-Jackson, H., & Repka, J. (2020). Weakly special classes of modules. *Mathematics and Statistics*. https://doi.org/10.13189/ms.2020.081304

Pratibha, Mishra, R. K., & Mohan, R. (2017). A Report on Graded Rings and Graded Modules.

Global Journal of Pure and Applied Mathematics, *13*(9), 6827–6853.
Smoktunowicz, A. (2018). On Engel groups, nilpotent groups, rings, braces and the Yang-Baxter equation. *Transactions of the American Mathematical Society*, *370*(9), 6535–6564. https://doi.org/10.1090/tran/7179
Vaš, L. (2020a). Graded Cancellation Properties of Graded Rings and Graded Unit-regular Leavitt Path Algebras. *Algebras and Representation Theory*. https://doi.org/10.1007/s10468-020-09963-z
Vaš, L. (2020b). Realization of graded matrix algebras as Leavitt path algebras. *Beitrage Zur Algebra Und Geometrie*. https://doi.org/10.1007/s13366-020-00487-7
Wahyuni, S., Marubayashi, H., Ernanto, I., & Sutopo. (2020). Strongly graded rings which are generalized Dedekind rings. *Journal of Algebra and Its Applications*. https://doi.org/10.1142/S0219498820500437
Wahyuni, S., Wijayanti, I. E., & France-Jackson, H. (2017). A PRIME ESSENTIAL RING THAT GENERATES A SPECIAL ATOM. *Bulletin of the Australian Mathematical Society*. https://doi.org/10.1017/S000497271600085X
Wijayanti, I. E., Marubayashi, H., Ernanto, I., & Sutopo. (2020). Finitely generated torsion-free modules over integrally closed domains. *Communications in Algebra*. https://doi.org/10.1080/00927872.2020.1742346