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ABSTRACT

An inclusive distance vertex irregular labelling of a simple graph G is a function of the vertex set of G to positive integer set such that the sum of its vertex label and the labels of all vertices adjacent to the vertex are distinct. The minimum of maximum label of the vertices is said to be inclusive distance irregularity of strength of G, denoted by $\widehat{\text{dis}}(G)$. The purpose of this research is showing that $\widehat{\text{dis}}(T_{n,2}) = \left\lceil \frac{n^2+1}{2} \right\rceil$, where $T_{n,2}$ is a complete n-ary tree to level two.

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A. INTRODUCTION

Graph labeling is one of the most popular research areas of graph theory. Graph labeling was first introduced in the mid 1960s (Rosa, 1967). In more than 60 years nearly 200 graph labeling techniques have been studied in over 3200 papers (Gallian, 2022). In general, there are two kind of labelling, namely regular and irregular labelings. A regular labeling of graphs means that the labelling satisfying injective function, such as graceful (Wang et al., 2015), harmonious (Lasim et al., 2022), elegant (Elumalai & Sethuraman, 2010), felicitous (Manickam et al., 2012), and so on. In this paper, we focus on irregular labeling. All graphs discussed here are simple, undirected, and finite. For a general terminology of graph-theoretic, we follow (Chartrand & Zhang, 2012) and (Ringel & Hartsfield, 1990).

The distance vertex irregular labeling was introduced by (Slamin, 2017). This labeling was inspired by (Miller et al., 2003) who introduced a distance magic labeling and (Chartrand et al., 1988) who introduced an irregular assignment. Detail survey of distance magic labeling can be studied in (Arumugam et al., 2011). Furthermore, (Bong et al., 2017) generalized the concept of labeling introduced by (Slamin, 2017) to inclusive and non-inclusive vertex irregular d-distance vertex labeling, for any distance d up to the diameter. (Bong et al., 2017) defined the inclusive vertex irregular d-distance vertex labeling. For $d = 1$, (Bača et al., 2018) called this labeling as an inclusive distance vertex irregular labeling.

An inclusive distance vertex irregular labelling of a graph G is a function $f: V(G) \longrightarrow \{1, 2, \dots, k\}$ such that each vertex of G has distinct weight. The weight of a vertex $u \in V(G)$ under the labeling f is defined as $wt(u) = f(u) + \sum_{uv \in E(G)} f(v)$. Not all graphs can be applied this labelling, namely graph which contains at least two vertices with the same closed neighbourhood. For example, a complete graph with three vertices, since all vertices will have the same closed neighbourhood. It is not difficult to apply this labelling for a graph. That is why, the problem of this area is find the minimum k for the vertex label such that a graph G admits this labelling is called the inclusive distance vertex irregularity strength of G and is denoted by $\widehat{\text{dis}}(G)$. If such k does not exist we say that $\widehat{\text{dis}}(G) = \infty$.

The exact value of the inclusive distance vertex irregularity strength of many graphs have been widely studied in the literature (see (Bača et al., 2018), (Utami et al., 2018), (Bong et al., 2020), (Halikin et al., 2020), (Utami et al., 2020), (Susanto et al., 2021), (Majid et al., 2023), and (Windartini et al., 2014)). Furthermore, (Cichacz et al., 2021) gave the upperbound of the inclusive distance vertex irregularity strength for a simple graph G on n vertices in which no two vertices have the same closed neighbourhood that $\widehat{\text{dis}}(G) \leq n^2$. Next, (Susanto, et al., 2022) studied inclusive distance vertex irregularity strength for the join product of graphs. (Santoso et al., 2022) using genetic algorithm for the inclusive labelling of a graph. (Bong et al., 2020) gave a lower bound for the inclusive distance vertex irregularity strengths for a graph G of order n, the maximum degree Δ, and the minimum degree δ,

$$\widehat{dis}(G) \geq \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil.$$

(Susanto et al., 2021) developed a new lower bound for the inclusive distance vertex irregularity strength of graphs that generalizes the lower bound by (Bong et al., 2020), that a graph G with the maximum degree Δ, the minimum degree δ,

$$\widehat{dis}(G) \geq \max_{\delta \leq r \leq \Delta} \left\{ \left\lceil \frac{\delta + \sum_{j=\delta}^{r} n_j}{r + 1} \right\rceil \right\}, \tag{1}$$

where n_r is the number of vertices of degree r in G for every $\delta \leq r \leq \Delta$.

In this paper, we give the exact value of inclusive distance vertex irregularity strengths of a complete n-*ary* tree to level two. Denoted by $T_{n,2}$, a complete n-ary tree is a rooted tree such that each vertex of degree greater than one has exactly n children and all degree-one vertices are of equal distance (height) to the root (Li et al., 2010). Therefore, $T_{n,2}$ has $n(n+1)$ vertices and n^2 leaves.

B. METHODS

Let $T_{n,2}$ be a complete n-ary tree. To find the minimum k for the vertex label such that a graph G admits the inclusive distance vertex irregular labelling, are as follows: (i) Give the notation of all vertex of $T_{n,2}$. Let $V(T_{n,2}) = \{c_0, c_i, v_i^j | i, j \in [1, n]\}$ be the vertex set of $T_{n,2}$, where c_0 is the root vertex of $T_{n,2}$ of degree n, the vertex c_i has degree $n + 1$, and the vertex v_i^j has degree 1. Let again $E(T_{n,2}) = \{v_i^j c_i, c_i c_0 | i, j \in [1, n]\}$ be the edge set of $T_{n,2}$. The illustration of the notation of vertices of $T_{n,2}$ is given in Figure 1; (ii) Define function $f: V(T_{n,2}) \rightarrow$

$\{1, 2, \ldots, k\}$, by the considering the Inequality (1) for the value of k, (iii) Count the weight of each vertex of $V(T_{n,2})$; (iv) Show that each vertex of $V(T_{n,2})$ has distinct weight.

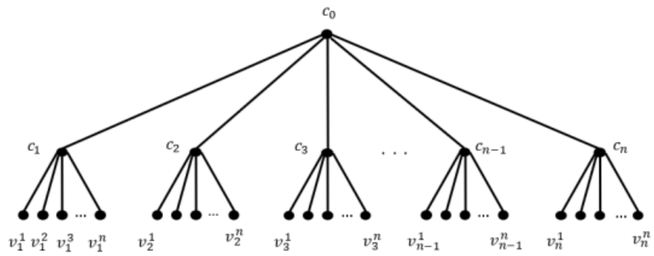

Figure 1. The vertex notation of $T_{n,2}$

C. RESULT AND DISCUSSION

In this section, we discuss the exact value of inclusive distance irregularity strength of a graph n-*ary* tree to level 2. There are two lemmas about the upper bound of $\widehat{dis}(T_{n,2})$. Indeed the lower bound of $\widehat{dis}(T_{n,2})$ following the Theorem 1.

Lema 1. Let $n \geq 3$ be an odd integer and and $T_{n,2}$ be an n-*ary* tree to level 2. The upper bound of inclusive distance irregularity strength of $T_{n,2}$ is

$$\widehat{dis}(T_{n,2}) \leq \left\lceil \frac{n^2 + 1}{2} \right\rceil.$$

Proof. For prove this by labelling a graph $T_{n,2}$ using inclusive vertex irregular. For $n = 3$, an inclusive distance vertex irregular labelling of $T_{3,2}$ can be seen in Figure 1, where the weight of the vertex shown by a red number.

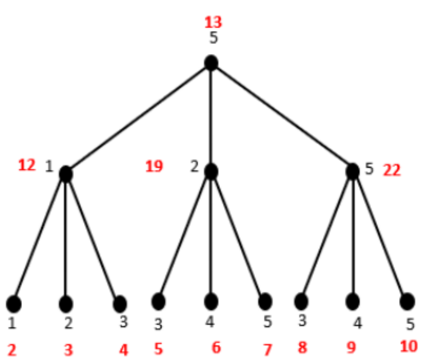

Figure 2. An inclusive distance vertex irregular labelling of graph $T_{3,2}$

For $n \geq 5$, define an inclusive distance vertex irregular labelling of $T_{n,2}$ as follow. For odd n, we have $\left\lceil \frac{n^2+1}{2} \right\rceil = \frac{n^2+1}{2}$. Define $f: V(T_{n,2}) \rightarrow \left\{1, 2, \ldots, \frac{n^2+1}{2}\right\}$ as follow.

$$f(c_i) = \begin{cases} \frac{n^2+1}{2} & \text{for } i = 0, n, \\ 1 & \text{for } i = 1, \\ \left(\frac{n+1}{2}\right) i & \text{for } i = 2, 3, \ldots, n-1. \end{cases} \quad (1)$$

For $j = 1, 2, \ldots, n$,

$$f(v_i^j) = \begin{cases} j & \text{for } i = 1, 2, \\ \left(\frac{n-1}{2}\right) i - n + j + 1 & \text{for } i = 3, 4, \ldots, n-1, \\ \frac{n^2+1}{2} - n + j & \text{for } i = n. \end{cases} \quad (2)$$

The illustration of vertex labelling can be seen in Figure 2.

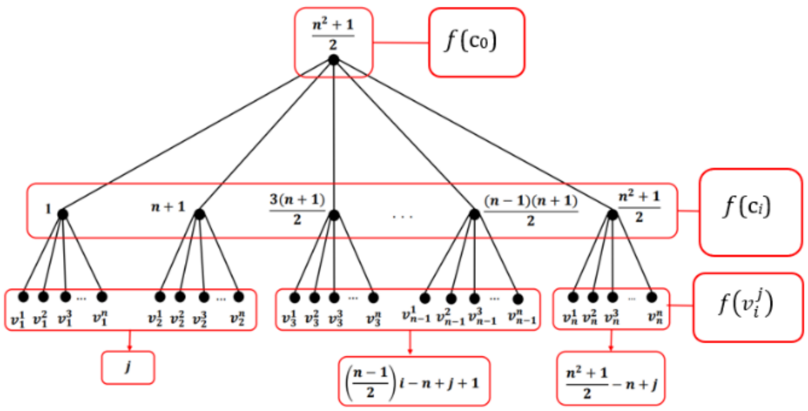

Figure 3. The vertex labelling of $T_{n,2}$ by Equation (1) and (2)

By (1) and (2) we obtain the vertex weight as follow.

$$wt(c_i) = \begin{cases} \frac{n^3+4n^2-3n+6}{4} & \text{for } i = 0, \\ \frac{2n^2+n+3}{2} & \text{for } i = 1, \\ \frac{(n^2+1)i+3n+1}{2} & \text{for } i = 2, 3, \ldots, n-1, \\ \frac{n^3+n^2+2n+2}{2} & \text{for } i = n. \end{cases}$$

$$wt(v_i^j) = \begin{cases} j+1, & \text{for } i = 1, \text{and } j = 1, 2, \ldots, n, \\ ni - n + j + 1, & \text{for } i = 2, 3, \ldots, n, \text{and } j = 1, 2, \ldots, n. \end{cases}$$

We next show that all vertices weight are distinct. For $i = 2, 3, \ldots, n-1$, we have $\frac{2n^2+3n+3}{2} \leq wt(c_i) \leq \frac{n^3-n^2+4n}{2}$. We can check easily that $wt(c_1) < wt(c_i) < wt(c_n)$ and also $wt(c_1) < wt(c_0) < wt(c_n)$. Now, we assume that $wt(c_0) = wt(c_i)$, for $i = 2, 3, \ldots, n-1$. Then

$$\frac{n^3+4n^2-3n+6}{4} = \frac{(n^2+1)i+3n+1}{2}$$

$$\frac{n^3+4n^2-3n+6}{4} - \left(\frac{3n+1}{2}\right) = \frac{(n^2+1)i}{2}$$

$$\frac{n^3+4n^2-9n+4}{4} = \frac{(n^2+1)i}{2}$$

$$i = \frac{n^3 + 4n^2 - 9n + 4}{2n^2 + 2}.$$

So, i is not integer, a contradiction. We consider now, the weight of the vertex v_i^j. We can see that $2 \le wt(v_1^j) \le n + 1 < n + 2 \le wt(v_2^j) \le n^2 + 1 < \frac{2n^2+n+3}{2} = wt(c_1)$ for each $j = 1, 2, \dots, n$. Therefore, all vertices of $T_{n,2}$ have different weight. ■

In the Figure 4.4, we can see an inclusive distance vertex irregular labelling of graph $T_{5,2}$.

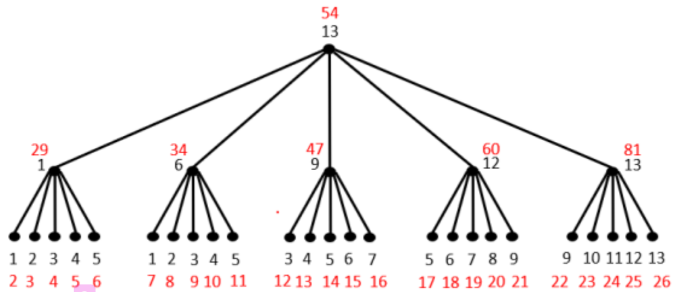

Figure 4. An inclusive distance vertex irregular labelling of graph $T_{5,2}$

Lema 2 Let $n \ge 2$ be an even integer and and $T_{n,2}$ be a complete n-*ary* tree to level 2. The upper bound of inclusive distance irregularity strength of $T_{n,2}$ is

$$\widehat{dis}(T_{n,2}) \le \left\lceil \frac{n^2 + 1}{2} \right\rceil.$$

Proof. Let n be an even integer. For $n = 2$, an inclusive vertex irregular labeling of $T_{2,2}$ can be seen in Figure 3, where the weight of the vertex shown by a red number.

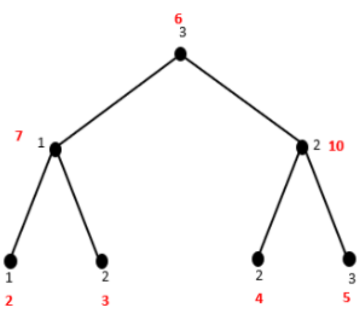

Figure 5. An inclusive distance vertex irregular labelling of graph $T_{2,2}$

Now, for even $n \ge 4$ we have $\left\lceil \frac{n^2+1}{2} \right\rceil = \frac{n^2}{2} + 1$, define $f: V(T_{n,2}) \to \left\{1, 2, \dots, \frac{n^2}{2} + 1\right\}$ by the following

$$f(c_i) = \begin{cases} \frac{n^2}{2} + 1 & \text{for } i = 0, n, \\ 1 & \text{for } i = 1, \\ \frac{n}{2} i + 1 & \text{for } i = 2, 3, \dots, n - 1. \end{cases} \qquad (3)$$

For $j = 1, 2, \dots, n$

$$f(v_i^j) = \begin{cases} j & \text{for } i = 1, 2, \\ \frac{n}{2}i - n + j & \text{for } i = 3, 4, \dots, n-1, \\ \frac{n^2}{2} - n + j & \text{for } i = n. \end{cases} \quad (4)$$

As an illustration of the formulation of vertex labelling in (3) and (4), consider the Figure 5.

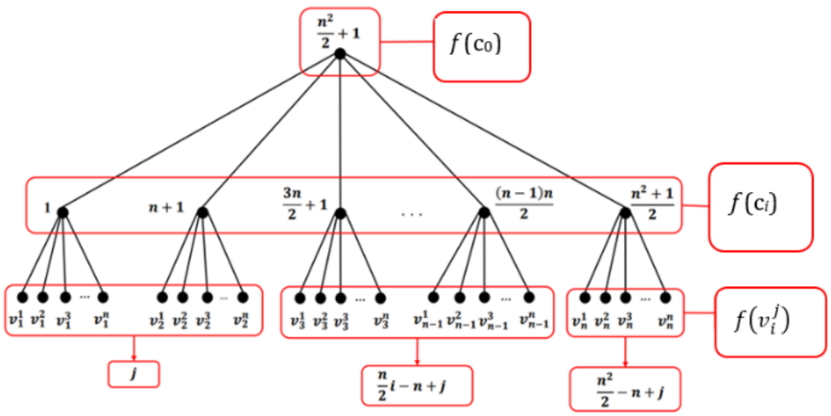

Figure 6. The vertex labelling of $T_{n,2}$ by Equation (3) and (4)

According to the vertex labelling in (3) and (4), we can count the vertex weight, as follow.

$$wt(c_i) = \begin{cases} \frac{n^3+3n^2+2n+4}{4}, & \text{for } i = 0, \\ \frac{2n^2+n+4}{2}, & \text{for } i = 1, \\ \frac{(n^2+n)i+n+4}{2}, & \text{for } i = 2, 3, \dots, n-1, \\ \frac{n^3+n^2+n+4}{2}, & \text{for } i = n. \end{cases}$$

$$wt(v_i^j) = \begin{cases} j+1 & \text{for } i = 1, \text{and } j = 1, 2, \dots, n, \\ ni - n + j + 1 & \text{for } i = 2, 3, \dots, n, \text{and } j = 1, 2, \dots, n. \end{cases}$$

For $i = 2, 3, \dots, n-1$ we obtain $\frac{2n^2+3n+4}{2} \le wt(c_i) \le \frac{n^3+4}{2}$. Since $wt(c_1) = \frac{2n^2+n+4}{2} < \frac{2n^2+3n+4}{2}$ and $\frac{n^3+4}{2} < \left(\frac{n^3+4}{2}\right) + \frac{n^2+n}{2} = wt(c_n)$, then $wt(c_1) < wt(c_i) < wt(c_n)$. We can check easily that $wt(c_1) < wt(c_0) < wt(c_n)$. Next, we assume that $wt(c_0) = wt(c_i)$ for $i = 2, 3, \dots, n-1$, then we obtain

$$\frac{n^3 + n^2 + 2n + 4}{4} = \frac{(n^2+n)i + n + 4}{2}$$

$$\frac{n^3 + n^2 + 2n + 4}{4} - \left(\frac{n+4}{2}\right) = \frac{(n^2+n)i}{2}$$

$$\frac{2(n^3+n^2-4)}{4} = (n^2+n)i$$
$$i = \frac{n^3+n^2-4}{2n^2+2n}.$$

So, i is not integer, a contradiction. Therefore $wt(c_0) \neq wt(c_i)$. Next, we have $2 \leq wt(v_i^j) \leq n^2+1 < wt(c_1)$, since $wt(c_1) = \frac{2n^2+n+4}{2} = (n^2+1) + \frac{n+2}{2}$. Therefore $wt(v_i^j) < wt(c_1) < wt(c_i) < wt(c_n)$ and off course $wt(v_i^j) < wt(c_0)$. So it is completely proof. ■

For example, an inclusive distance vertex irregular labelling of $T_{4,2}$ is given in Figure 4.

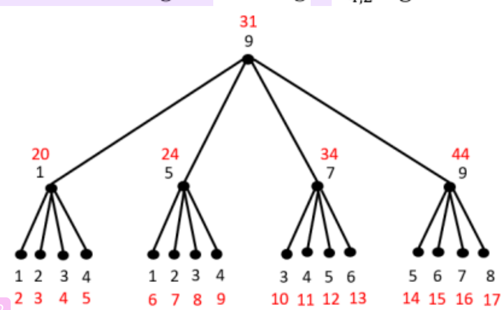

Figure 7. An inclusive distance vertex irregular labelling of graph $T_{4,2}$

By Lemmas 1 and 2, we have the exact value of distance vertex irregularity strength of the complete *n-ary* tree to level 2, $T_{n,2}$ by the following theorem.

Theorem 3 Let $n \geq 2$ be an integer and $T_{n,2}$ be an *n-ary* tree to level 2. The inclusive distance irregularity strength of $T_{n,2}$ is

$$\widehat{dis}(T_{n,2}) = \left\lceil \frac{n^2+1}{2} \right\rceil.$$

Proof. According to (1) we have

$$\widehat{dis}(T_{n,2}) \geq \max_{\delta \leq r \leq \Delta} \left\{ \left\lceil \frac{\delta + \sum_{j=\delta}^{r} n_j}{r+1} \right\rceil \right\} = \max_{1 \leq r \leq n+1} \left\{ \left\lceil \frac{1 + \sum_{j=1}^{r} n_j}{r+1} \right\rceil \right\} = \max_{1 \leq i \leq n+1} \left\{ \left\lceil \frac{n^2+1}{2} \right\rceil, \left\lceil \frac{n^2+2}{n+1} \right\rceil, \left\lceil \frac{n^2+n+2}{n+2} \right\rceil \right\}$$
$$= \left\lceil \frac{n^2+1}{2} \right\rceil,$$

since

$$\frac{n^2+1}{2} = \frac{\frac{n^3}{2}+\frac{n^2}{2}+\frac{n}{2}+\frac{1}{2}}{n+1} \geq \left\lceil \frac{\frac{n^3}{2}+\frac{n^2}{2}+\frac{n}{2}+\frac{1}{2}-\left(\frac{n^3}{2}-\frac{n^2}{2}+\frac{n}{2}-\frac{3}{2}\right)}{n+1} \right\rceil = \left\lceil \frac{n^2+2}{n+1} \right\rceil \text{ and}$$

$$\frac{n^2+1}{2} = \frac{\frac{n^3}{2}+n^2+\frac{n}{2}+1}{n+2} \geq \left\lceil \frac{\frac{n^3}{2}+n^2+\frac{n}{2}+1-\left(\frac{n^3}{2}-\frac{n}{2}-1\right)}{n+1} \right\rceil = \left\lceil \frac{n^2+n+2}{n+2} \right\rceil.$$

Next, the upper bound of the inclusive distance irregularity strength of $T_{n,2}$ has been proved by Lemmas 1 and 2. Therefore, we obtain $\widehat{dis}(T_{n,2}) = \left\lceil \frac{n^2+1}{2} \right\rceil$. ■

D. CONCLUSION AND SUGGESTIONS

The inclusive distance irregularity strength of n-ary tree at level two is obtained. A challenging problem that remains open is get the inclusive distance irregularity strength of the complete n-ary tree at level greather than two, $T_{n,k}$, for $k > 2$.

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