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Characteristic Polynomial and Eigenproblem of Triangular Matrix over Interval Min-Plus Algebra

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ABSTRACT

A min-plus algebra is linear algebra over the semiring $\mathbb{R}_{\varepsilon'}$, equipped with the operations "$\oplus$'=min" and "$\otimes = +$". In min-plus algebra there is the concept of characteristic polynomial obtained from permanent of matrix. Min-plus algebra can be extended to an interval min-plus algebra which is a set $I(\mathbb{R})_{\varepsilon'}$ equipped with the operations $\overline{\oplus}'$ and $\overline{\otimes}$. Matrix over interval min-plus algebra has some special forms, one of which is a triangular matrix. The concept of characteristic polynomial can be applied to triangular matrix. In this research, will be discussed about the characteristic polynomial and eigenproblem of triangular matrix over interval min-plus algebra. From the research result, the permanent formula and characteristic polynomial formula of the triangular matrix are obtained. It is also obtained that the smallest corner of the characteristic polynomial is the principal eigenvalue and the vector corresponding to the principal eigenvalue can be obtained through the matrix A_λ.

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A. INTRODUCTION

In mathematics, the topic of max-plus algebra is often studied. Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ with $\mathbb{R}$ being the set of all real numbers and $\varepsilon = -\infty$ equipped with the operations $\oplus = max$ and $\otimes = plus$ (Gyamerah et al., 2016). Max-plus algebra can be denoted as $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ which is a semiring with neutral element ε and unit element $e = 0$. Can be formed $m \times n$ matrix set whose elements are members of $\mathbb{R}_\varepsilon$ called set of matrices over max-plus algebra. This set of matrices can be denoted as $\mathbb{R}_\varepsilon^{m\times n}$. Max-plus algebra has been applied in several problems by (Haneefa & Siswanto, 2021), (Nurwan & F. Payu, 2022), dan (Prastiwi & Kistiana, 2017). For application in time intervals, max-plus algebra can be extended to interval max-plus algebra. Interval max-plus algebra is the set $I(\mathbb{R})_\varepsilon$ equipped with operations $\overline{\oplus}$ and $\overline{\otimes}$ (Siswanto et al., 2019). A matrix over interval max-plus algebra is a matrix with its entries are members of $I(\mathbb{R})_\varepsilon$. The set of these matrices is denoted by $I(\mathbb{R})_\varepsilon^{m\times n}$.

Besides max-plus algebra, there is another semiring that is also discussed in mathematics, namely min-plus algebra. A min-plus algebra is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$ with $\mathbb{R}$

being the set of all real numbers and $\varepsilon' = +\infty$ equipped with the operations $\oplus'$ and $\otimes$ (Nowak, 2014; Watanabe & Watanabe, 2014). If $p, q \in \mathbb{R}_{\varepsilon'}$ then $p \oplus' q = \min\{p, q\}$ and dan $p \otimes q = p + q$. The set of matrices over min-plus algebra is the set of matrices with its entries are members of $\mathbb{R}_{\varepsilon'}$ and is denoted by $\mathbb{R}_{\varepsilon'}^{m\times n}$. The application of min-plus algebra in several problems has been studied by (Susilowati & Fitriani, 2019) and (Farhi, 2023). Min-plus algebra can also be extended to interval min-plus algebra. An interval min-plus algebra is a set $I(\mathbb{R})_{\varepsilon'}$ equipped with operations $\overline{\oplus'}$ and $\overline{\otimes}$ (Awallia et al., 2020). Matrices with its entries are members of $I(\mathbb{R})_{\varepsilon'}$ are called matrices over interval min-plus algebra. The set of these matrices is denoted by $I(\mathbb{R})_{\varepsilon'}^{m\times n}$.

In conventional algebra, for an $n \times n$ square matrix, its determinant can be found. However, in max-plus algebra the matrix determinant is replaced by the matrix permanent since there is no inverse to the $\otimes$ operation. The permanent of matrix over max plus algebra $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ is $perm(A) = \bigoplus_{\sigma\in P_n} a_{1\sigma(1)} \otimes \ldots \otimes a_{n\sigma(n)}$ with P_n being the permutation group on $\{1, 2, \ldots, n\}$ (Rosenmann et al., 2019). The concept of permanent has also been discussed in other semirings, including interval max-plus algebra by (Siswanto, Kurniawan, et al., 2021) and min-plus algebra by (Siswanto, Gusmizain, et al., 2021). From the permanent of matrix, the characteristic polynomial of a matrix can be obtained.

Characteristic polynomial and eigenproblem of a matrix are interconnected concepts. In max-plus algebra, the characteristic polynomial of matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ is $\chi_A(x) = perm(A \oplus x \otimes I)$ (Hook, 2015; Nishida et al., 2020). The concept of this characteristic polynomial in other semiring, that is, in interval max-plus algebra has been studied by (Siswanto, Kurniawan, et al., 2021) and in min-plus algebra has been studied by (Maghribi et al., 2023). Characteristic polynomial is one of the methods to solve the eigen problem of a matrix. The eigen problem aims to find the eigenvalue and eigenvector corresponding to the eigenvalue (Jiang, 2022). In max-plus algebra, eigenvalues can be obtained from the greatest corner of the characteristic polynomial. This eigenvalue is called the principal eigenvalue. The eigenproblem can be written by the equation $A \otimes x = \lambda \otimes x$. Moreover, λ is called the eigenvalue of matrix A and x is called the eigenvector corresponding to the eigenvalue λ (Siswanto, 2023). The application of eigenvalue problem in max-plus algebra has been studied in (Al Bermanei et al., 2023; De Schutter et al., 2020; Maharani & Suparwanto, 2022; Permana et al., 2020; Subiono et al., 2018).

The concept of characteristic polynomials and eigenproblems can be applied to special matrices, one of which is a triangular matrix. The characteristic polynomial of triangular matrix over interval max-plus algebra has been studied by (Wulandari & Siswanto, 2019). The characteristic polynomial and eigenproblem of triangular matrix over min-plus algebra has been studied by (Maghribi, 2023). In this article, we will discuss the characteristic polynomial and eigenproblem of triangular matrix over interval min-plus algebra.

B. METHODS

The research method used is a literature study. This research begins by explaining the definition of triangular matrix over interval min-plus algebra. Next, determine formulas for the permanent and characteristic polynomials of a triangular matrix. After obtaining the characteristic polynomial formula, the eigenvalue can be determined which in turn can be obtained the eigenvector corresponding to the eigenvalue. First, some definitions and theorems used for the discussion in this article are given.

Definition 1. (Maghribi, 2023) A matrix is called a triangular matrix over min-plus algebra if all entries of above or below the diagonal of the matrix is equal to ε'.

Theorem 1. (Maghribi, 2023) If $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$, then the permanent of triangular matrix A is

$$perm(A) = a_{11} \otimes a_{22} \otimes \dots \otimes a_{nn}.$$

Definition 2. (Rahayu et al., 2021) Given $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. If $\lambda(A) \in \mathbb{R}_{\varepsilon'}$ and $v \in \mathbb{R}_{\varepsilon'}$ such that v has at least one finite entry and

$$A \otimes v = \lambda \otimes v$$

then λ is an eigenvalue of A and v is the corresponding eigenvector.

Definition 3. (Rahayu et al., 2021) Given $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and λ defined as in Definition 2, the matrix A_λ is defined as

$$A_\lambda = a_{ij} - \lambda,$$

matrix A_λ^+ is defined as

$$A_\lambda^+ = A_\lambda \oplus' A_\lambda^{\otimes 2} \oplus' \dots \oplus' A_\lambda^{\otimes n},$$

A_λ^* is defined as

$$A_\lambda^* = I \oplus' A_\lambda^+.$$

Theorem 2. (Maghribi et al., 2023) If $(A_\lambda^+)_{vv} = 0$ then the v-th columns in matrix A_λ^+ are eigenvectors corresponding to $\lambda(A)$.

Theorem 3. (Maghribi, 2023) The min-plus characteristic polynomial of a triangular matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is

$$\chi_A(x) = (x \oplus' a_{11}) \otimes (x \oplus' a_{22}) \otimes \dots \otimes (x \oplus' a_{nn}).$$

Theorem 4. (Maghribi, 2023) The smallest corner of the characteristic polynomial of the triangular matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ which is also the principal eigenvalue is

$$\lambda(A) = \oplus'_{i\in\mathbb{N}} \ a_{ii}.$$

Theorem 5. (Maghribi, 2023) Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ be a triangular matrix that satisfies the following 2 conditions

1. there is more than one diagonal element that is greater than or equal to λ(A) and
2. all entries above or below the diagonal are greater than or equal to λ(A).

The columns of matrix A_λ containing diagonal element 0 are the eigenvectors of A corresponding to λ(A).

Definition 5. (Awallia et al., 2020) Interval min-plus algebra is the set of $I(\mathbb{R})_{\varepsilon'}$ defined as

$$I(\mathbb{R})_{\varepsilon'} = \{x = [\underline{x}, \overline{x}] | \underline{x}, \overline{x} \in \mathbb{R}_{\varepsilon'}, \underline{x} \leq \overline{x} < \varepsilon'\} \cup \{[\varepsilon', \varepsilon']\}, \varepsilon' = \infty.$$

The interval min-plus algebra is equipped with the operations $\oplus'$ and $\otimes$ that for all $x, y \in I(\mathbb{R})_{\varepsilon'}$ holds

1. $x \overline{\oplus}' y = [\underline{x} \oplus' \underline{y}, \overline{x} \oplus' \overline{y}]$ and
2. $x \overline{\otimes} y = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$.

Definition 6. (Awallia et al., 2020) Matrices over interval min-plus algebra is defined as the set of matrices with its entries are members of $I(\mathbb{R})_{\varepsilon'}$ denoted by $I(\mathbb{R})_{\varepsilon'}^{m\times n}$ that is

$$I(\mathbb{R})_{\varepsilon'}^{m\times n} = \{A = [A_{ij}] | A_{ij} \in I(\mathbb{R})_{\varepsilon'}, i = 1, 2, \dots, m, j = 1, 2, \dots, n\}.$$

Definition 7. (Awallia et al., 2020) Matrices over interval min-plus algebra have binary operations $\oplus'$ and $\otimes$ defined that for all $A, B \in I(\mathbb{R})_{\varepsilon'}^{m\times n}$ and $k \in I(\mathbb{R})_{\varepsilon'}$ holds

$$[A \overline{\oplus}' B]_{ij} = A_{ij} \overline{\oplus}' B_{ij} \text{ dan } [k \overline{\otimes} A]_{ij} = k \overline{\otimes} A_{ij}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Definition 8. (Awallia et al., 2020) For every $A \in I(\mathbb{R})_{\varepsilon'}^{m\times p}$ and $B \in I(\mathbb{R})_{\varepsilon'}^{p\times n}$ is defined

$$[A \overline{\otimes} B]_{ij} = \overline{\oplus}'^{p}_{k=1} A_{ik} \otimes B_{kj}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Definition 9. (Awallia et al., 2020) Given an interval matrix $A \in I(\mathbb{R})_{\varepsilon'}^{m\times n}$ with $\underline{A}$ and $\overline{A}$ being its lower bound matrix and upper bound matrix, respectively. Define the matrix interval of A as

$$[\underline{A}, \overline{A}] = \{A \in I(\mathbb{R})_{\varepsilon'}^{m\times n} | \underline{A} \leq A \leq \overline{A}\}$$

and the set of matrix interval of A is

$$I(\mathbb{R}_{\varepsilon'}^{m\times n})_b = \{A = [\underline{A}, \overline{A}] | A \in I(\mathbb{R})_{\varepsilon'}^{m\times n}\}.$$

Definition 10. (Awallia, 2020) Given $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$. An interval scalar $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is called interval min-plus eigenvalue of matrix A if there exists an interval vector $\lambda \in I(\mathbb{R})_{\varepsilon'}^{n}$ with $v \neq \overline{\varepsilon'}_{n\times 1}$ such that $A \overline{\otimes} v = \lambda \overline{\otimes} v$. The vector v is called the interval min-plus eigenvector of matrix A corresponding to λ.

Theorem 6. (Awallia, 2020) Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$ and $\lambda(A)$ is an eigenvalue of matrix A. If $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})] < [\varepsilon', \varepsilon']$ then the columns of A_λ^+ with the lower bound of its diagonal elements as the eigenvectors of matrix A and the upper bound of its diagonal elements 0, are the eigenvectors of matrix A corresponding to the eigenvalues of $\lambda(A)$.

C. RESULT AND DISCUSSION

1. Characteristic polynomial of triangular matrix

This section discusses the characteristic polynomial of triangular matrix over interval min-plus algebra. First defined the triangular matrix over interval min-plus algebra then determine the permanent formula of the triangular matrix which is then used to determine the characteristic polynomial formula of the triangular matrix. The following is given definition of a triangular matrix over interval min-plus algebra, theorem about the permanent formula of the triangular matrix, and theorem about the characteristic polynomial formula of the triangular matrix.

Definition 11. A matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is called a triangular matrix if all entries of above or below the diagonal is $[\varepsilon', \varepsilon']$.

Theorem 7. The permanent of a triangular matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is

$$perm(A) = [\underline{a}_{11}, \overline{a}_{11}] \otimes [\underline{a}_{22}, \overline{a}_{22}] \otimes \ldots \otimes [\underline{a}_{nn}, \overline{a}_{nn}].$$

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be an upper triangular matrix with its diagonal elements are $[\underline{a}_{11}, \overline{a}_{11}], [\underline{a}_{22}, \overline{a}_{22}], \ldots, [\underline{a}_{nn}, \overline{a}_{nn}]$. Matrix A can be written as

$$A = \begin{bmatrix} [\underline{a}_{11}, \overline{a}_{11}] & [\underline{a}_{12}, \overline{a}_{12}] & \cdots & [\underline{a}_{1n}, \overline{a}_{1n}] \\ [\varepsilon', \varepsilon'] & [\underline{a}_{22}, \overline{a}_{22}] & \cdots & [\underline{a}_{2n}, \overline{a}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\varepsilon', \varepsilon'] & [\varepsilon', \varepsilon'] & \cdots & [\underline{a}_{nn}, \overline{a}_{nn}] \end{bmatrix}$$

so that the lower bound matrix of matrix A is obtained

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \varepsilon' & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \underline{a}_{nn} \end{bmatrix}$$

and the upper bound matrix of matrix A is obtained

$$\overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \varepsilon' & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \overline{a}_{nn} \end{bmatrix}$$

such that $A \approx [\underline{A}, \overline{A}] \in I\left(\mathbb{R}_{\varepsilon'}^{n\times n}\right)_b$. Based on Theorem 1, it can be obtained that

$$perm(\underline{A}) = perm \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \varepsilon' & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \underline{a}_{nn} \end{bmatrix} = \underline{a}_{11} \otimes \underline{a}_{22} \otimes \ldots \otimes \underline{a}_{nn}$$

and

$$perm(\overline{A}) = perm \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \varepsilon' & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \overline{a}_{nn} \end{bmatrix} = \overline{a}_{11} \otimes \overline{a}_{22} \otimes \ldots \otimes \overline{a}_{nn}$$

so that

$$perm(A) = [perm(\underline{A}), perm(\overline{A})]$$
$$= [\underline{a}_{11} \otimes \underline{a}_{22} \otimes \dots \otimes \underline{a}_{nn}, \overline{a}_{11} \otimes \overline{a}_{22} \otimes \dots \otimes \overline{a}_{nn}]$$
$$= [\underline{a}_{11}, \overline{a}_{11}] \overline{\otimes} [\underline{a}_{22}, \overline{a}_{22}] \overline{\otimes} \dots \overline{\otimes} [\underline{a}_{nn}, \overline{a}_{nn}].$$

It can be proved for the lower triangular matrix in the same way.

■

Theorem 8. The interval min-plus characteristic polynomial of a triangular matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is

$$\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$$

with

$$\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn})$$

and

$$\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn}).$$

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be a triangular matrix with $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$. Based on Theorem 3, it is obtained that the characteristic polynomial of $\underline{A}$ is

$$\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn})$$

and the characteristic polynomial of $\overline{A}$ is

$$\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn}).$$

Since $A \approx [\underline{A}, \overline{A}]$, then can obtained the characteristic polynomial of A is $\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$ with $\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn})$ and $\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn})$.

■

2. Eigenproblem of Triangular Matrix

This section discusses the eigenproblem of triangular matrix over interval min plus algebra. The eigen problem aims to find the eigenvalue and eigenvector corresponding to the eigenvalue. The following theorems are given about the smallest corner of the characteristic polynomial of the triangular matrix, the theorem about the eigenvalue of the triangular matrix, and the theorem about the eigenvector corresponding to the eigenvalue.

Theorem 9. The smallest corner of characteristic polynomial of a triangular matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is

$$\delta_1 = [\underline{\delta}_1, \overline{\delta}_1]$$

with $\underline{\delta}_1 = \bigoplus'^{n}_{i=1} \underline{a}_{ii}$ and $\overline{\delta}_1 = \bigoplus'^{n}_{i=1} \overline{a}_{ii}$.

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be a triangular matrix with $A \approx [\underline{A}, \overline{A}]$ where $[\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$. Based on Theorem 8, it is obtained that the characteristic polynomial of A is

$$\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$$

with

$$\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn})$$

and

$$\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \ \dots \otimes (\overline{x} \oplus' \overline{a}_{nn}).$$

Furthermore, by Theorem 4, can obtained that the smallest corner of $\chi_{\underline{A}}(\underline{x})$ is $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ and the smallest corner of $\chi_{\overline{A}}(\overline{x})$ is $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$. Therefore, the smallest corner of the characteristic polynomial $\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$ is $\delta_1 = [\underline{\delta}_1, \overline{\delta}_1]$ with $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ and $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$.

■

Theorem 10. If $A \in I(\mathbb{R})^{n\times n}_{\varepsilon'}$ is a triangular matrix where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}^{n\times n}_{\varepsilon'})_b$ then the smallest corner of $\chi_A(x)$ is $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$.

Proof. Let $A \in I(\mathbb{R})^{n\times n}_{\varepsilon'}$ be a triangular matrix with $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}^{n\times n}_{\varepsilon'})_b$. Based on Theorem 9, the smallest corner characteristic polynomial of A is $\delta_1 = [\underline{\delta}_1, \overline{\delta}_1]$ with $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ dan $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$. Furthermore, based on Theorem 4, it is obtained that the smallest corner of $\chi_{\underline{A}}(\underline{x})$ is $\underline{\lambda}(\underline{A})$ and the smallest corner of adalah $\overline{\lambda}(\overline{A})$. Since $\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$ then the smallest corner of $\chi_A(x)$ is $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$.

■

Theorem 11. If $A \in I(\mathbb{R})^{n\times n}_{\varepsilon'}$ a triangular matrix where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}^{n\times n}_{\varepsilon'})_b$ and $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$ is the principal eigenvalues of matrix A then the eigenvectors of matrix A corresponding to $\lambda(A)$ are the columns of matrix A_λ with the lower bound of its diagonal elements as the eigenvectors of matrix A and the upper bound of its diagonal elements 0.

Proof. Let $A \in I(\mathbb{R})^{n\times n}_{\varepsilon'}$ be a triangular matrix where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}^{n\times n}_{\varepsilon'})_b$ and $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$ is the principal eigenvalues of matrix A with $\underline{\lambda}(\underline{A})$ being the eigenvalue of matrix $\underline{A}$ and $\overline{\lambda}(\overline{A})$ being the eigenvalue of matrix $\overline{A}$. Can be determined the matrix $\underline{A}_{\underline{\lambda}}$ with $\underline{A}_{\underline{\lambda}} = \underline{a}_{ij} - \underline{\lambda}(\underline{A})$ and the matrix $\overline{A}_{\overline{\lambda}}$ with $\overline{A}_{\overline{\lambda}} = \overline{a}_{ij} - \overline{\lambda}(\overline{A})$. Let $\underline{g}_k$ and $\overline{g}_k$, $k = 1,2,\dots,n$ be the columns of matrices $\underline{A}_{\underline{\lambda}}$ and $\overline{A}_{\overline{\lambda}}$ respectively. Next, the matrix A_λ is formed with its columns determined as follows.

1. If for k in pairs of $\underline{g}_k$ and $\overline{g}_k$ holds $\underline{g}_k \leq \overline{g}_k$, then define the k-column of A_λ as the interval vector $g_k \approx [\underline{g}_k, \overline{g}_k]$.
2. If for k in pairs of $\underline{g}_k$ and $\overline{g}_k$ holds $\underline{g}_k \nleq \overline{g}_k$, then define $\underline{g}^*_k = -\delta \otimes \underline{g}_k$ with $\delta = maks_i\left(\left(\underline{g}_k\right)_i - \left(\overline{g}_k\right)_i\right), i = 1,2,..,n$ and the k-column of A_λ as the interval vector $g_k \approx [\underline{g}^*_k, \overline{g}_k]$.

Based on Theorem 2, it is obtained that the eigenvector of the matrix $\underline{A}$ corresponding to $\underline{\lambda}(\underline{A})$ is any column of the matrix $\underline{A}^+_{\underline{\lambda}}$ with its diagonal element is 0. By Definition 3, can be obtained $\underline{A}^+_{\underline{\lambda}} = \underline{A}_{\underline{\lambda}} \oplus' \underline{A}^{\otimes 2}_{\underline{\lambda}} \oplus' \dots \oplus' \underline{A}^{\otimes n}_{\underline{\lambda}}$. Since $\underline{\lambda}(\underline{A}) = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$, then elements of $\underline{a}_{ii} - \underline{\lambda}(\underline{A})$ in the matrix $\underline{A}_{\underline{\lambda}}$ are nonnegative for $\underline{a}_{ii} \neq \lambda(A)$ and 0 for $\underline{a}_{ii} = \lambda(A)$. Furthermore, the elements of $\underline{A}_\lambda$ that are nonnegative will always be nonnegative for the elements of $\underline{A}_\lambda$ that are adjacent, and the elements of $\underline{A}_\lambda$ will be less than or equal to the elements of $\underline{A}_\lambda^{\otimes n}$. On the other side, the diagonal elements of $\underline{A}_\lambda$ that equal to 0 are also always equal to 0 for the corresponding diagonal elements of $\underline{A}_\lambda^{\otimes n}$. Therefore, for a triangular matrix, $\underline{A}_\lambda^{+}$ is $\underline{A}_\lambda$. That is, the eigenvector of the matrix $\underline{A}$ corresponding to $\underline{A}_\lambda$ is any column of the matrix $\underline{A}_\lambda$ with its diagonal element is 0. In the same way, it is obtained that $\overline{A}_{\overline{\lambda}}^{+}$ is $\overline{A}_{\overline{\lambda}}$ and the eigenvector of the matrix $\overline{A}$ corresponding to $\overline{A}_{\overline{\lambda}}$ is any column of the matrix $\overline{A}_{\overline{\lambda}}$ with its diagonal element is 0. Since $\underline{A}_\lambda^{+}$ is $\underline{A}_\lambda$ and $\overline{A}_{\overline{\lambda}}^{+}$ is $\overline{A}_{\overline{\lambda}}$ then A_λ^{+} is A_λ. Furthermore, based on Theorem 6, the eigenvectors of matrix A corresponding to $\lambda(A)$ are the columns of matrix A_λ and the lower bound of its diagonal elements as the eigenvectors of matrix $\underline{A}$ and the upper bound of its diagonal elements 0.

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D. CONCLUSION AND SUGGESTIONS

From the results and discussion, it is obtained that the permanent of the triangular matrix is the multiplication of its diagonal elements. Besides that, it is obtained that the characteristic polynomial formula of the triangular matrix with its smallest corner is the main eigenvalue and the eigenvector corresponding to the main eigenvalue are the columns of the matrix A_λ with the lower bound of its diagonal elements as the eigenvector of matrix A and the upper bound of its diagonal elements 0. For readers who are interested in this topic, further research can be done on the generalized eigenproblem of interval min-plus algebra.

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