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The Solution of Generalization of the First and Second Kind of Abel's Integral Equation

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ABSTRACT

Integral equations are equations in which the unknown function is found to be inside the integral sign. N. H. Abel used the integral equation to analyze the relationship between kinetic energy and potential energy in a falling object, expressed by two integral equations. This integral equation is called Abel's integral equation. Furthermore, these equations are developed to produce generalizations and further generalizations for each equation. This study aims to explain generalizations of the first and second kind of Abel's integral equations, and to find solution for each equation. The method used to determine the solution of the equation is an analytical method, which includes Laplace transform, fractional calculus, and manipulation of equation. When the analytical approach cannot solve the equation, the solution will be determined by a numerical method, namely successive approximations. The results showed that the generalization of the first kind of Abel's integral equation solution can be determined using the Laplace transform method, fractional calculus, and manipulation of equation. On the other hand, the generalization of the second kind of Abel's integral equation solution is obtained from the Laplace transform method. Further generalization of the first kind of Abel's integral equation solution can be obtained using manipulation of equation method. Further generalization of the second kind of Abel's integral equation solution cannot be determined by analytical method, so a numerical method (successive approximations) is used.

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A. INTRODUCTION

An integral equation is an equation with an unknown function that appears under the integral sign (Ahmad, 2021). Integral equations can be applied in various fields such as geophysics, engineering, electricity and magnetism, optimal control systems, mathematics, population genetics, and medicine (Ray & Sahu, 2013; Sattaso et al., 2023). In 1823, N. H. Abel pioneered integral equations for mechanical problems. Abel's integral equation is one of the integral equations which is derived directly from certain physics problems without going through the equations (Kumar et al., 2015). Abel's integral equations have two kinds: the first and second. In 1924, the generalized and further generalization of Abel's integral equation on a finite segment was studied (Bairwa et al., 2020). Over time, generalization of Abel's integral equations was applied in several branches of science (Wang et al., 2014; Li et al., 2018). Some of them are modelling plasma (Merk et al., 2013), seismology (Cuha & Peker, 2022), and stereology (Ziada, 2021). The generalization of Abel's integral equation can also be applied to astrophysics (Kumar et al., 2015), optical fibres (Singh et al., 2019), and spectroscopy (Senel et al., 2021).

There are two methods for solving integral equations: the analytical and the numerical methods. In previous research, Abel's integral equations of the first kind $\left(\alpha = \frac{1}{2}\right)$ is solved analytically with Laplace transform (Aggarwal & Sharma, 2019). Meanwhile, generalizations of Abel's integral equations of the first kind ($0 < \alpha < 1$) are solved analytically using fractional calculus (Jahanshahi et al., 2015) and fractional-order Mikusinski operator (M. Li & Zhao, 2013). Solving the Abel's integral equations with numerical methods has been carried out using Touchard and Laguerre polynomials (Abdullah et al., 2021) and the Laguerre wavelet (Mundewadi, 2019). In addition, the generalization of Abel's integral equations has been solved by numerical methods, named the hp-version collocation method (Dehbozorgi & Nedaiasl, 2020).

This study will determine the solutions of generalization of the first and second kinds of Abel's integral equations. The methods that will be used in this study include analytical methods, which include Laplace transforms, fractional calculus, and equation manipulation. In previous study, the Laplace transform method was used by Aggarwal & Sharma (2019) to solve the first kind of Abel's integral equation $\left(\alpha = \frac{1}{2}\right)$. In this paper, the ideas from Aggarwal & Sharma (2019) will be used to determine the generalization solutions of the first and second kinds of Abel's integral equations ($0 < \alpha < 1$). Determination of generalization solutions with fractional calculus using ideas from (Jahanshahi et al., 2015). This study also uses numerical methods when solutions cannot be determined analytically. The numerical method chosen is successive approximations. After obtaining the generalization solution, it is expected to be able to solve various problems (stereology, seismology, geometry, etc.), which are represented in the generalized form of the first or second kind of Abel's integral equations.

B. METHODS

This research studies the generalization of the first and second kinds of Abel's integral equations, and then the solution will be determined. The first step is to learn some of the analytical methods used to determine the solution of the integral equation. Next, select the methods that can be applied to Abel's integral equation, including the Laplace transform, fractional calculus, and equation manipulation. When the solution of the equation cannot be determined by analytical methods, the solution is determined by numerical methods. In this study, the numerical method chosen was successive approximations.

1. Gamma and Beta Functions

According to Goyal et al. (2021) the Gamma function is defined as:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt, \qquad x \geq 0 \tag{1}$$

and the Beta function is defined as:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, y > 0. \tag{2}$$

According to Salah (2015) and Al-Gonah et al. (2018), the Gamma and Beta functions have the following properties:

a. $\Gamma(1) = 1$

b. $\Gamma(x+1) = x\Gamma(x), (x > 0)$,

c. $\Gamma(n+1) = n!$ where n is a non-negative integer

d. $\Gamma(x) = 2\int_0^\infty e^{-t^2}\, t^{2x-1} dt$

e. $\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\ d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$

f. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

g. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

h. $B(x,y) = B(y,x)$

i. $B(x+1,y) = \frac{x}{x+y}B(x,y)$ and $B(x,y+1) = \frac{y}{x+y}B(x,y)$.

j. $\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$

k. $\Gamma(x)\ \Gamma(1-x) = \frac{\pi}{\sin \pi x}$.

2. Laplace Transform

According to Saif et al. (2020) the Laplace transform of the function $f(t)$ is defined as follows:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s), \tag{3}$$

with $f(t) = 0$ for $t \leq 0$, and s is a complex variable. The inverse Laplace transform, denoted as

$$\mathcal{L}^{-1}[F(s)] = f(t). \tag{4}$$

Suppose $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$. According to Debnath (2016) and Wang & Chen (2019) the properties of the Laplace transform are:

$$\mathcal{L}[f(t) + g(t)] = F(s) + G(s)$$

a. $\mathcal{L}[af(t)] = aF(s); a \in \mathbb{R}$

b. $\mathcal{L}\left[f'(t)\right] = sF(s) - f(0)$

c. $\mathcal{L}\left[f''(t)\right] = s^2F(s) - f'(0) - sf(0)$

d. $\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$, where $p > -1$.

Theorem 1

Suppose f_1 and f_2 are continuous functions defined at $(0,\infty)$. The Laplace transform of $f_1(x)$ and $f_2(x)$ is

$$\mathcal{L}\{f_1(x)\} = F_1(s),\ \ \mathcal{L}\{f_2(x)\} = F_2(s) \tag{5}$$

The Laplace convolution product $\left((f * g)(x)\right)$ is defined by

$$\mathcal{L}\left\{\int_0^t f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s) \tag{6}$$

(Yang, 2014).

3. Fractional Calculus

Fractional calculus is a branch of classical mathematics which deals with the generalization of operations of differentiation and integration to fractional order (Delkhosh, 2013). The n-derivative of a function $f(x)$ is $\frac{d^n f(x)}{dx^n}$. In general, the value of n is a positive integer, n can be a rational, irrational, or complex number.

Definition of fractional integral by Riemann–Liouville:

$$I_x^{1-\alpha}u(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{1}{(x-t)^\alpha}u(t)\,dt, \tag{7}$$

and the Riemann–Liouville fractional derivative is defined as:

$$D_x^{1-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^x \frac{1}{(x-t)^{1-\alpha}}f(t)\,dt. \tag{8}$$

C. RESULT AND DISCUSSION

1. Generalization of Abel's Integral Equations of the First Kind

Let f be any known function, and u be the function to be determined. The Abel's integral equation of the first kind is

$$f(x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}u(t)dt, \qquad x \geq 0. \tag{9}$$

Equation (9) can be generalized as follows:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}u(t)dt, \quad 0 < \alpha < 1, \qquad x \geq 0, \tag{10}$$

where $K(x,t) = \frac{1}{(x-t)^\alpha}$ is called an Abel's kernel. In this case, equation (9) is a special case of generalization of the first kind of Abel's integral equation with a value of $\alpha = \frac{1}{2}$.

A further generalization of first kind of Abel's integral equations considers the following Abel's kernel:

$$K(x,t) = \frac{1}{[g(x)-g(t)]^\alpha}, \qquad 0 < \alpha < 1,$$

equation (10) becomes

$$f(x) = \int_0^x \frac{1}{[g(x)-g(t)]^\alpha}u(t)dt\,, \qquad 0 < \alpha < 1, \qquad x \geq 0, \tag{11}$$

where $g(t)$ is a a monotonic function of increasing and decreasing in the interval $0 < t < b$ and $g'(t) \neq 0$ for every t in the interval.

Theorem 2

Let f be any known function, with

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}u(t)dt,\ x \geq 0$$

then

$$u(x)=\frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{1}{(x-t)^{1-\alpha}}f(t)\,dt.$$

Proof:

There are three methods to prove Theorem 2.

a. Laplace Transform

The proof of the Laplace transformation uses ideas from Aggarwal & Sharma (2019). The first step is to use the definition of convolution for equation (10) so that

$$f(x)=\int_0^x \frac{1}{(x-t)^{\alpha}}u(t)\,dt=\frac{1}{x^{\alpha}}*u(x). \tag{12}$$

Then a Laplace transform is performed on both sides of equation (12)

$$\mathcal{L}\{f(x)\}=\mathcal{L}\{x^{-\alpha}\}\,\mathcal{L}\{u(x)\}. \tag{13}$$

Suppose $F(s)=\mathcal{L}\{f(x)\}$ and $U(s)=\mathcal{L}\{u(x)\}$, so

$$F(s)=\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}U(s) \tag{14}$$

$$U(s)=\frac{s^{1-\alpha}}{\Gamma(1-\alpha)}F(s), \tag{15}$$

where Γ is the Gamma function. To obtain the inverse Laplace transform, equation (15) can be expressed by

$$\mathcal{L}\{u(x)\}=\frac{s}{\Gamma(1-\alpha)\Gamma(\alpha)}\mathcal{L}\{y(x)\} \tag{16}$$

where

$$y(x)=\int_0^x \frac{1}{(x-t)^{1-\alpha}}f(t)\,dt. \tag{17}$$

From the Laplace transform properties are obtained

$$\mathcal{L}\{y'(x)\}=s\mathcal{L}\{y(x)\}-y(0). \tag{18}$$

Equation (16) becomes the following equation:

$$\mathcal{L}\{u(x)\}=\frac{\sin \pi\alpha}{\pi}\mathcal{L}\{y'(x)\}. \tag{19}$$

By applying the inverse of the Laplace transform ($\mathcal{L}^{-1}$) to equation (19), we get

$$u(x)=\frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{1}{(x-t)^{1-\alpha}}f(t)\,dt.$$

b. Equation Manipulation

Both sides of equation (10) are multiplied by $\frac{1}{(s-x)^{1-\alpha}}$ and integrated at the lower bound 0 and upper bound s over x, so

$$\begin{aligned}\int_0^s \frac{f(x)}{(s-x)^{1-\alpha}}dx &= \int_0^s \frac{1}{(s-x)^{1-\alpha}}\left(\int_0^x \frac{u(t)}{(x-t)^{\alpha}}dt\right)dx\\ &= \int_0^s u(t)\left(\int_t^s \frac{1}{(s-x)^{1-\alpha}(x-t)^{\alpha}}dx\right)dt,\end{aligned} \tag{20}$$

by substituting $y=\frac{s-x}{s-t}$, we get $dx=-(s-t)dy$. This resulted

$$\int_t^s \frac{1}{(s-x)^{1-\alpha}(x-t)^{\alpha}}dx = \int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\,dy = B(\alpha,\,1-\alpha), \tag{21}$$

where B is the Beta function. Equation (20) can be expressed by

$$\int_0^s \frac{f(x)}{(s-x)^{1-\alpha}}dx = \int_0^s u(t)\,B(\alpha,\,1-\alpha)\,dt. \tag{22}$$

From the properties of the Gamma and Beta functions, we get

$$\frac{\sin\pi\alpha}{\pi}\int_0^s \frac{f(x)}{(s-x)^{1-\alpha}}dx = \int_0^s u(t)\;dt. \tag{23}$$

Furthermore, both sides are derived from the variable s so that

$$u(s) = \frac{\sin\pi\alpha}{\pi}\frac{d}{ds}\int_0^s \frac{f(x)}{(s-x)^{1-\alpha}}dx. \tag{24}$$

The variable x, s is a dummy variable, then

$$u(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt.$$

c. Fractional Calculus

The proof of the fractional calculus uses ideas from Jahanshahi et al. (2015). Based on the definition of fractional integral by Riemann–Liouville, equation (10) can be written as

$$f(x) = I_x^{1-\alpha}u(x)\Gamma(1-\alpha). \tag{25}$$

Then both sides are derived by definition of the Riemann–Liouville fractional derivative so that

$$\frac{1}{\Gamma(1-\alpha)}D_x^{1-\alpha}f(x) = D_x^{1-\alpha}I_x^{1-\alpha}u(x). \tag{26}$$

Based on the properties of calculus fractions are obtained:

$$\frac{1}{\Gamma\,(1-\alpha)}\frac{1}{\Gamma\,(\alpha)}\frac{d}{dx}\int_0^x \frac{1}{(x-t)^{1-\alpha}}f(t)\,dt = u(x). \tag{27}$$

So, the solution is

$$u(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{1}{(x-t)^{1-\alpha}}f(t)\,dt.$$

Theorem 3

Let f be any known function, and $g(t)$ be a monotone increasing and decreasing function on the interval $0 < t < b$ and $g'(t) \neq 0$, with

$$f(x) = \int_0^x \frac{1}{[g(x)-g(t)]^{\alpha}}u(t)dt, \qquad 0<\alpha<1, \qquad x \geq 0,$$

then

$$u(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{g'(t)\,f(t)}{\big(g(x)-g(t)\big)^{1-\alpha}}dt.$$

Proof:

The method used to prove Theorem 3 is equation manipulation. The steps are to multiply both sides of equation (11) by $\frac{g'(x)}{(g(s)-g(x))^{1-\alpha}}$, then integrate over x from the lower bound 0 and upper bound s so that

$$\int_0^s \frac{g'(x)\,f(x)}{\big(g(s)-g(x)\big)^{1-\alpha}}\,dx = \int_0^s u(t)\left(\int_t^s \frac{g'(x)}{\big(g(s)-g(x)\big)^{1-\alpha}\big(g(x)-g(t)\big)^{\alpha}}\,dx\right)dt. \quad (28)$$

By substituting $y=\frac{g(s)-g(x)}{g(s)-g(t)}$ then $g'(x)dx=-\big(g(s)-g(t)\big)dy$ which resulted

$$\begin{aligned}\int_t^s \frac{g'(x)}{\big(g(s)-g(x)\big)^{1-\alpha}\big(g(x)-g(t)\big)^{\alpha}}\,dx &= \int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\,dy \\ &= B(\alpha,1-\alpha) \\ &= \frac{\pi}{\sin\pi\alpha}. \end{aligned} \quad (29)$$

From equation (28), it is obtained that

$$\frac{\sin\pi\alpha}{\pi}\int_0^s \frac{g'(x)\,f(x)}{\big(g(s)-g(x)\big)^{1-\alpha}}\,dx = \int_0^s u(t)\,dt. \quad (30)$$

Then both sides are derived to s, then

$$u(s) = \frac{\sin\pi\alpha}{\pi}\frac{d}{ds}\int_0^s \frac{g'(x)\,f(x)}{\big(g(s)-g(x)\big)^{1-\alpha}}\,dx. \quad (31)$$

The variable x, s is a dummy variable, so it applies

$$u(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{g'(t)\,f(t)}{\big(g(x)-g(t)\big)^{1-\alpha}}\,dt.$$

2. Generalization of Abel's Integral Equations of the Second Kind

Let f be any known function, and u be the function to be determined. The Abel's integral equation of the second kind is

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}u(t)dt, \qquad x \geq 0. \quad (32)$$

Equation (32) can be generalized as follows:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^{\alpha}}u(t)dt, \qquad 0<\alpha<1, \qquad x\geq 0. \quad (33)$$

where $\frac{1}{(x-t)^{\alpha}} = K(x,t)$ is called an Abel's kernel. Equation (32) is a special case of generalization of the second kind of Abel's integral equation with a value of $\alpha=\frac{1}{2}$.

A further generalization of the second kind of Abel's integral equations considers the kernel

$$K(x,t) = \frac{1}{[g(x)-g(t)]^{\alpha}}, \qquad 0<\alpha<1,$$

so that

$$u(x) = f(x) + \int_0^x \frac{1}{[g(x)-g(t)]^{\alpha}}u(t)dt, \qquad 0<\alpha<1, \qquad x\geq 0, \quad (34)$$

where $g(t)$ is a monotonic function of increasing and decreasing in the interval $0<t<b$ and $g'(t) \neq 0$ for each t in the interval.

Theorem 4

Let f be any known function, with

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \qquad x \geq 0,$$

then

$$u(x) = \mathcal{L}^{-1}\left\{\frac{s^{1-\alpha}}{s^{1-\alpha} - \Gamma(1-\alpha)} \mathcal{L}\{f(x)\}\right\}.$$

Proof:

The Laplace transform method is used to prove Theorem 4. By carrying out the Laplace transformation on both sides of equation (33), it is obtained that

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{f(x)\} + \mathcal{L}\left\{\int_0^x \frac{1}{(x-t)^\alpha} u(t)\, dt\right\} \tag{35}$$

$$U(s) = \frac{s^{1-\alpha}}{s^{1-\alpha} - \Gamma(1-\alpha)} F(s), \tag{36}$$

where Γ is a gamma function, $U(s) = \mathcal{L}\{u(x)\}$, and $F(s) = \mathcal{L}\{f(x)\}$. The inverse of the Laplace transforms ($\mathcal{L}^{-1}$) for both sides is

$$u(x) = \mathcal{L}^{-1}\left\{\frac{s^{1-\alpha}}{s^{1-\alpha} - \Gamma(1-\alpha)} \mathcal{L}\{f(x)\}\right\}.$$

The solution $u(x)$ in Theorem 4 is still in the inverse Laplace form, so it is difficult for some functions f. Therefore, the equation (33) solution can also be approached using numerical methods, namely successive approximations. This method begins by determining the initial guess. Then the initial guess is an approximation of the next function (Kanwal, 2013). The recurrence relation of equation (33) for this method is

$$u_{n+1}(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u_n(t)\, dt, \qquad n \geq 0. \tag{37}$$

When the initial guess $u_0(x) = 0$, some successive approximations u_k with $k \geq 1$ are

$$\begin{aligned} u_1(x) &= f(x) \\ u_2(x) &= f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u_1(t)\, dt \\ u_3(x) &= f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u_2(t)\, dt \\ &\vdots \end{aligned}$$

So, $u(x) = \lim_{n\to\infty} u_n(x)$.

Further generalization solutions of the second kind of Abel's integral equations can be approached using successive approximations. When the initial guess is $u_0(x) = 0$, then

$$\begin{aligned} u_1(x) &= f(x) \\ u_2(x) &= f(x) + \int_0^x \frac{1}{\big(g(x) - g(t)\big)^\alpha} u_1(t)\, dt \end{aligned}$$

$$u_3(x) = f(x) + \int_0^x \frac{1}{\big(g(x)-g(t)\big)^\alpha} u_2(t)\,dt$$
$$\vdots$$

So, $u(x) = \lim_{n\to\infty} u_n(x)$.

3. Illustration of Abel's Integral Equation

The following are some illustrations in solving generalizations of the first and second kinds of Abel's integral equations:

a. Suppose $f(x) = \frac{16}{5}x^{\frac{5}{4}} - x - 4x^{\frac{1}{4}} + 1$, with $\alpha = \frac{3}{4}$, then equation (33) becomes

$$u(x) = \frac{16}{5}x^{\frac{5}{4}} - x - 4x^{\frac{1}{4}} + 1 + \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}} u(t)\,dt.$$

In this case, the Laplace transform method is used so that the formulation $u(x)$ in Theorem 4 becomes

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\left\{\frac{s^{\frac{1}{4}}}{s^{\frac{1}{4}} - \Gamma\left(\frac{1}{4}\right)}\,\mathcal{L}\left\{\frac{16}{5}x^{\frac{5}{4}} - x - 4x^{\frac{1}{4}} + 1\right\}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s^{\frac{1}{4}}}{s^{\frac{1}{4}} - \Gamma\left(\frac{1}{4}\right)}\left(\frac{16}{5}\frac{\Gamma\left(\frac{5}{4}+1\right)}{s^{\frac{5}{4}+1}} - \frac{1}{s^2} - 4\frac{\Gamma\left(\frac{1}{4}+1\right)}{s^{\frac{1}{4}+1}} + \frac{1}{s}\right)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2}\right\}, \end{aligned}$$

Therefore, $u(x) = 1 - x$.

b. Suppose $f(x) = x^n$ and $\alpha = \frac{1}{2}$, then equation (10) becomes

$$x^n = \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}} u(t)dt, \qquad n > -1, \qquad x \geq 0.$$

with the equation manipulation method, it is obtained that

$$\begin{aligned} u(x) &= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{t^n}{(x-t)^{\frac{1}{2}}}\,dt \\ &= \frac{\left(n+\frac{1}{2}\right)\ \Gamma(n+1)}{\sqrt{\pi}\ \Gamma\left(n+\frac{3}{2}\right)} x^{n-\frac{1}{2}}. \end{aligned}$$

c. Suppose $f(x) = x^2, g(x) = x^2$ and $\alpha = \frac{1}{2}$, then equation (34) becomes

$$u(x) = x^2 + \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{2}}} u(t)\,dt.$$

When the initial guess is $u_0\ (x) = 0$, then

$u_1(x) = x^2$

$$
\begin{aligned}
u_2(x) &= x^2 + \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{2}}} t^2\,dt \\
&= x^2 + \frac{\pi}{4}x^2 \\
&= \left(1+\frac{\pi}{4}\right)x^2 \\
u_3(x) &= x^2 + \int_0^x \frac{1}{(x^2-t^2)^{\frac{1}{2}}} \left(1+\frac{\pi}{4}\right) t^2\,dt \\
&= x^2 + \left(1+\frac{\pi}{4}\right)\frac{\pi}{4}x^2 \\
&= \left(1+\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2\right)x^2 \\
u_n(x) &= \left(1+\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2+\cdots+\left(\frac{\pi}{4}\right)^{n-1}\right)x^2.
\end{aligned}
$$

So, $u(x) = \left(1+\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2+\cdots+\left(\frac{\pi}{4}\right)^n+\cdots\right)x^2$

$$
\begin{aligned}
&= x^2\sum_{n=0}^{\infty}\left(\frac{\pi}{4}\right)^n \\
&= \frac{4}{4-\pi}x^2.
\end{aligned}
$$

d. According to Thórisdóttir & Kiderlen (2013), the first kind of Abel's integral equation is applied in stereology, namely in the random ball model, with the equation is

$$t(x_2) = \frac{2}{r}\int_{x_2}^{M} \frac{D(x_1)}{(x_1^2-x_2^2)^{\frac{1}{2}}}\,dx_1, \qquad 0 \le x_2 \le x_1 \le M < \infty,$$

where

$r = \int_0^M D(x_2)dx_2 = \frac{\pi}{2H}$ and $H = \int_0^M \frac{t(x_2)}{x_2}dx_2$,

with

$D(x_1)$: spherical particle size distribution
$t(x_2)$: circular section size distribution of the random plane section particle
M : upper limit on the maximum size of spherical particles
r : average radius of the sphere.

D. CONCLUSION AND SUGGESTIONS

Suppose f is any known function, and g is a known increasing and decreasing monotone function, and u is the function to be determined, then: The generalization of the first kind of Abel's integral equation solution using the Laplace transform method, fractional calculus, and equation manipulation is

$$u(x) = \frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{1}{(x-t)^{1-\alpha}} f(t)\,dt.$$

The further generalization of the first kind of Abel's integral equation solution is

$$u(x) = \frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{g'(t)\,f(t)}{\left(g(x)-g(t)\right)^{1-\alpha}}\,dt.$$

The generalization of the second kind of Abel's integral equation solution using the Laplace transform method is

$$u(x) = \mathcal{L}^{-1}\left\{\frac{s^{1-\alpha}}{s^{1-\alpha} - \Gamma(1-\alpha)}\mathcal{L}\{f(x)\}\right\}.$$

The three analytical methods cannot determine further generalization of the second kind of Abel's integral equations solution, so determined by numerical methods (successive approximations). The solution is

$$u(x) = \lim_{n\to\infty} u_{n+1}(x).$$

In this study, the further generalization solution of the second kind of Abel's integral equation is determined using a numerical method, namely successive approximations. Future research is expected to determine the advanced generalization solution of the second kind of Abel's integral equation using analytical methods.

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