JTAM (Jurnal Teori dan Aplikasi Matematika)
http://journal.ummat.ac.id/index.php/jtam
p-ISSN 2597-7512 | e-ISSN 2614-1175
Vol. 7, No. 3, July 2023, pp. 864-875

Analysis Dynamics Two Prey of a Predator-Prey Model with Crowley–Martin Response Function

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Article History:
Received : 08-05-2023
Revised : 06-07-2023
Accepted : 08-07-2023
Online : 18-07-2023

Keywords:
Dynamics;
Predator-Prey;
Crowley–Martin;
Population Modeling.

ABSTRACT

The predator-prey model has been extensively developed in recent research. This research is an applied literature study with a proposed dynamics model using the Crowley–Martin response function, namely the development of the Beddington-DeAngelis response function. The aim of this research is to construct a mathematical model of the predator-prey model, equilibrium analysis and population trajectories analysis. The results showed that the predator-prey model produced seven non-negative equilibrium points, but only one equilibrium point was tested for stability. Stability analysis produces negative eigenvalues indicating fulfillment of the Routh-Hurwitz criteria so that the equilibrium point is locally asymtotically stable. Analysis of the stability of the equilibrium point indicates stable population growth over a long period of time. Numerical simulation is also given to see the trajectories of the population growth movement. The population growth of first prey and second prey is not much different, significant growth occurs at the beginning of the growth period, while after reaching the peak the trajectory growth slopes towards a stable point. Different growth is shown by the predator population, which grows linearly with time. The growth of predators is very significant because predators have the freedom to eat resources. Various types of trajectory patterns in ecological parameters show good results for population growth with the given assumptions.

Crossref
https://doi.org/10.31764/jtam.v7i3.14506

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A. INTRODUCTION

The past few decades there have been many major events that have made humanity suffer due to natural disasters such as environmental damage, prolonged air pollution, species extinction, virus outbreaks, etc. It is necessary to understand what is happening in the environment (May, 1973). As humans, our job is to provide strategies to reduce pressure from the environment (Batabyal et al., 2020). The two researchers focused on discussing the effects of balance in ecosystems and their effects on predator-prey populations. Mathematical modeling is one of the most concerns about seeing and uncovering trends in environmental change, because many biologists, ecologists, and mathematicians are committed to studying the balance of environmental events.

Population mathematical modeling that focuses on the growth of the number of species is inseparable from the shape of the ecological system (Pratama et al., 2022). The dynamic relationship of the predator-prey interaction is characterized by the interaction of different species. Modeling plays an important role in every species interaction event (Pratama, 2022) (Yulida & Karim, 2019). The two research results emphasize that interactions in predator-prey populations play a very important role in a mathematical model. Knowledge of population modeling is useful for gathering information about interactions that occur in predator-prey systems. Many ecological perspectives have been studied in depth by Lotka and Volterra, regarding effects, model forms, and function responses (Ghanbari et al., 2020). The Lotka-Volterra predator-prey model is used to describe the dynamic relationship between the two species. The development of ecological mathematical modeling science does not only answer the need for a quantitative understanding of ecological systems, but can improve mathematical understanding.

The last few decades, there have been many developments in the science of predator-prey mathematical modeling, for example in empirical and theoretical studies of population mathematical models. Comparisons and assumption tests are often carried out to describe in detail close to the actual conditions in ecosystem life. The response function plays an important role in every predator-prey interaction. In population modeling, the response function is the center of research attention. The response function is a function that describes the number of prey, which is preyed upon by each predator in a unit of time. The response function determines the stability, oscillation and dynamics of the predator-prey model system (Pratama, 2022) (Puspitasari et al., 2021). Many predator-prey models focus specifically on the development of response function forms. The interaction of predator-prey species always has its own characteristics, which are inherited by the species. The novelty of this study is the development of an increased number of species, namely two populations of prey species (Mortoja et al., 2019). This consideration refers to the basic ecological assumption that in a carrying capacity ecosystem there must be more than one species inhabiting the living area of the ecosystem.

Mathematically stability analysis has been carried out by many previous studies. The stability that occurs in a population model is very dependent on the form of the response function (Maiti et al., 2019). The response function Bazykin is one example, which depends on the species of prey under certain parametric conditions (Ghanbari et al., 2020). We observed that the population predator-prey interaction system model depends on the prey and ratio depends on it being irrational to consider when competition between predators is encountered.

The response function in the predator-prey system involves dependence on predators, namely, Hassel-Varley, Beddington-DeAngelis, and Crowley-Martin. The three response functions provide a better picture of the form of predator predation on the predator-prey population density (Meng et al., 2014). The Crowley-Martin response function has the same properties or characteristics as the Beddington-DeAngelis response function, which is a predation characteristic that explains the mutual interference of predators even at very high prey densities (Hossain et al., 2021). Predation disturbances during the predation process are ignored at very high prey densities, this is because predators have no attachment to handling prey. While the Crowley–Martin considers the competitive interference factor in each species when preying on or looking for prey. Effect such a characteristic differentiates it from Beddington-Deangelis or from other response functions (Shang et al., 2021). The types of species in the functional response with these characteristics are *Heterospilus Prosopidi* in the type of parasite larvae, Coccinellidae Predaceous in the type of aphids and *Callosobruchus Chinensis* in the type of azuki bean weevil (Tripathi et al., 2020)(Tripathi et al., 2016). These species function as a biocontrol in the ecosystem. Referring to previous research, the novelty of this research is developing the model formulated takes into account the Crowley-Martin function response scheme which can lead to better dynamics than the previously formulated model.

B. METHODS

The research conducted is a type of literature study. The mathematical form of population predator-prey modeling with the characteristics of the reciprocal relationship between predator responses, in general, was introduced by DeAngelis (Hossain et al., 2021). Here we describe some of the response function forms $f(x,y)$ that are often encountered in previous studies. There are several interesting forms of the response function in population dynamics, including, the

1. $f(x,y)=f(x)$, function response only depends on x

a. Holling Type I

$$f(x)=mx,$$

b. Holling Type II

$$f(x)=\frac{mx}{x+a},$$

c. Holling Type III

$$f(x)=\frac{mx^2}{x^2+a},$$

d. Holling Type IV

$$f(x)=\frac{mx}{x^2+bx+a}.$$

The following shows the trajectory of population movement with the characteristics of each response function, as shown in Figure 1.

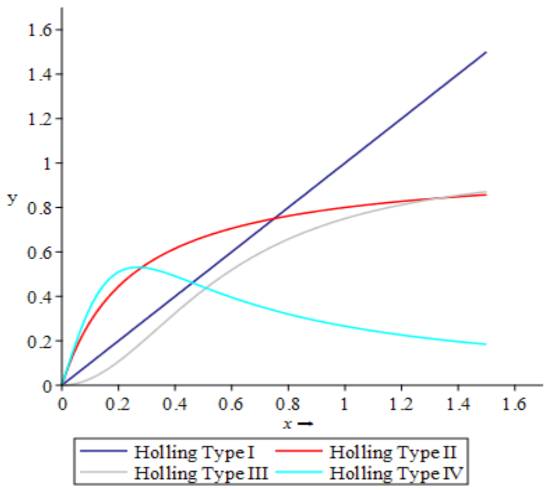

Figure 1. Trajectories Function Responses Holling I, Holling II, Holling III and Holling IV

2. $f(x,y)$**, response function that depends on** x **and** y

a. Ration-Dependent

$$f(x,y)=\frac{mx}{x+ay},$$

b. Beddington-DeAngelis

$$f(x,y)=\frac{mx}{ax+by+c},$$

c. Hassel-Varley

$$f(x,y)=\frac{mx}{ax+by^{\gamma}+c},\ \gamma=\frac{1}{2},\frac{1}{3}.$$

The assumption of the functional response parameter a,b,c and m is a positive constant that has a specific biological assumption in each function response. The response function will be the most important part of the predator-prey model system. A brief description of the predator-prey system that has been studied previously becomes a novelty reference for the current study. Lotka-Volterra type predator-prey system known as the Wangersky–Cunningham population model,

$$\begin{aligned}\frac{dx}{dt}&=rx\left(1-\frac{x}{k}\right)-mxy,\\ \frac{dy}{dt}&=emxy-\delta y.\end{aligned}\tag{1}$$

where, $x(t)$ and $y(t)$ is the population density of predator and prey species, at time t. The predator-prey model largely adopts the predator-prey model of form (1), which is a function of the density of prey. Model (1) is very popular and is widely referred to as a prey-dependence model, because the predator-prey system only depends on the prey. The system in the model (1) is developed by involving a more complex algebraic response function equation, namely,

$$\begin{aligned}\frac{dx}{dt}&=rx\left(1-\frac{x}{k}\right)-\frac{mxy}{ax+c},\\ \frac{dy}{dt}&=\frac{emxy}{ax+c}-\delta y.\end{aligned}\tag{2}$$

Changes in the response function that involves a highly competitive search process are very realistic to consider both biologically and physiologically. Model (2) is modified to form,

$$\begin{aligned}\frac{dx}{dt}&=rx\left(1-\frac{x}{k}\right)-\frac{mxy}{ax+by},\\ \frac{dy}{dt}&=\frac{emxy}{ax+by}-\delta y.\end{aligned}\tag{3}$$

The development of science convincingly Beddington, revealed the interaction of predator-prey in parasites, through the following mathematical equations;

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{ax+by+c}, \\ \frac{dy}{dt} &= \frac{emxy}{ax+by}-\delta y-hy^2. \end{aligned} \tag{4}$$

where, x and y represent population density of predator and prey in time t respectively. This research will focus on a simple basic model of form (4), which does not involve the intrinsic competition of predators. The condition is mathematically namely $h=0$. In addition, the novelty of this research involves two prey species, namely x_1 and x_2, which have identical characteristics or are still in the same species family. The two species still live in a good ecosystem environment area. This assumption is realistic to consider because in an ecological system a family species is a form of balance in an ecological system. The mathematical equation formed next is,

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1\left(1-\frac{x_1}{k}\right)-\frac{\sigma_1 x_1 y}{ax_1+by+c}, \\ \frac{dx_2}{dt} &= sx_2\left(1-\frac{x_2}{k}\right)-\frac{\sigma_2 x_2 y}{ax_2+by+c} \\ \frac{dy}{dt} &= \frac{m\sigma_1 x_1 y}{ax_1+by+c}+\frac{p\sigma_2 x_2 y}{ax_2+by+c}-\delta y. \end{aligned} \tag{5}$$

where all parameters in model (5) are considered with dimensional parameters, presented in Table 1.

Table 1. List of Parameters, Biological Meanings and Unit

Parameters	Meaning	Unit
x_1	First prey population (time dependent),	$[N]$
x_2	Second prey population (time dependent),	$[N]$
y	Predator population (time dependent),	$[N]$
r	First Prey's intrinsic growth rate,	$[T]^{-1}$
s	Second prey's intrinsic growth rate,	$[T]^{-1}$
k	Prey's environmental carrying capacity,	$[N]$
a	Prey saturation constant,	$[T]^{-1}$
b	Predator interference,	$[T]^{-1}$
c	Another saturation constant,	$[N][T]^{-1}$
σ_1	Consumption rate first prey,	$[T]^{-1}[T]^{-1}$
σ_2	Consumption rate second prey,	$[T]^{-1}[T]^{-1}$

m	Conversion factor of biomass due to change of food level first prey,	-
p	Conversion factor of biomass due to change of food level second prey,	-
δ	Predators death rate.	$[T]^{-1}$

C. RESULT AND DISCUSSION

1. Equilibrium analysis

The predator-prey model that was carried out in this study was a predator-prey model with two prey populations and a single predator population. The mathematical model is assumed to be suitable or closest to the real conditions in the ecosystem environment where the species live, the form of the formulation is model (5). In order to get the non-negative equilibrum points of model (5), the model is equalized to zero and then solved in x_1, x_2, and y :

$$\begin{aligned} rx_1\left(1-\frac{x_1}{k}\right)-\frac{\sigma_1 x_1 y}{ax_1+by+c}&=0, \\ sx_2\left(1-\frac{x_2}{k}\right)-\frac{\sigma_2 x_2 y}{ax_2+by+c}&=0, \\ \frac{m\sigma_1 x_1 y}{ax_1+by+c}+\frac{p\sigma_2 x_2 y}{ax_2+by+c}-\delta y&=0. \end{aligned} \tag{6}$$

there are seven non-negative equilibrium points obtained from model (5), with details as follows, $N_0(0,0,0)$, $N_1(x_1,0,0)$, $N_2(0,x_2,0)$, $N_3(x_1,x_2,0)$, $N_4(0,x_2,y)$, $N_5(x_1,0,y)$ and $N_6(x_1,x_2,y)$. The most realistic equilibrium to consider is $N_6(x_1,x_2,y)$ to proceed with the Routh-Hurwitz criterion test. The equations associated with the equilibrium N_6,

$$Ax^6-Bx^5+Cx^4+Dx^3-Ex^2+Fx+G=0.$$

Equation (6) is the final equation with differential operation. The polynomial function (6) is a type of sextic equation polynomial form. It is clear that will have six solution roots. The sextic polynomial has even degrees, the sextic function is similar to the quartic function. The derived form of the sextic function is the quintic function. If the leading coefficient of equation (5) A is positive, then the function increases to infinity positive on both sides, so that the function has a global minimum. If the coefficient of equation (5) B is negative, then the sextic function decreases to negative infinity and has a global maximum. From the analysis of the form equation (6), six roots of the solution are obtained. The solving roots consist of four positive real roots and two negative real roots (Kulkarni, 2008). It is difficult to show analytically the roots of the solution, because the form of the sextic function is a special polynomial. The roots of the sextic function will be shown in the numerical simulation section with the help of the Mathematics computer program.

In modeling form (5) the overall equilibrium is fourteen equilibrium points, but only seven equilibrium points fulfill the non-negative elements. Of the seven equilibrium points, there is only one equilibrium point that meets the stability selection criteria (Yin et al., 2014). Once $N_6(x_1,x_2,y)$ obtained, the next step is testing the stability of the equilibrium point. Ecological considerations are also a determining factor for the equilibrium point to be taken. It is hoped that the sustainability of all living things in the ecosystem will be the final decision in research to become a recommendation for policymakers either in companies or the government.

2. Numerical Simulation

The numerical simulation section describes applying parameter values to legally and reliably growing models (5). The parameter values used in the simulation steps come from references and assumptions. In the first part, simulation model (5) will be shown to obtain the equilibrium point, to form local stability. The next stage is to show the movement of population growth based on trajectories (Pratama et al., 2019). In numerical simulations, parameter values are given which are taken from research assumptions and previous relevant studies. Parameter values are given mathematically as follows, $r = 1.32$, $s = 1.50$, $k = 100$, $a = 0.2$, $b = 0.5$, $\delta = 0.03$, $\sigma_1 = 0.05$, $\sigma_2 = 0.5$, $m = 0.07$ and $p = 0.6$. Parameter substitution performed in the model (5) results in the form,

$$1.32x_1\left(1-\frac{x_1}{100}\right)-\frac{0.05x_1y}{0.2x_1+0.5y+0.4}=0,$$

$$1.50x_2\left(1-\frac{x_2}{100}\right)-\frac{0.5x_2y}{0.2x_2+0.5y+0.4}=0,$$

$$\frac{0.0035x_1y}{0.2x_2+0.5y+0.4}+\frac{0.30x_2y}{0.2x_2+0.5y+0.4}-0.03y=0.$$

From equation, seven non-negative equilibrium points are obtained including $N_0(0,0,0)$, $N_1(100,0,0)$, $N_2(0,100,0)$, $N_3(100,100,0)$, $N_4(0,33.47463290,6680.736726)$, $N_5(93.75064916,0,180.4512550)$ and $N_6(92.46553379,33.47023650,6894.877715)$. All equilibrium points can be considered for holistic stability testing. The most realistic equilibrium to choose is $N_6(92.46553379,33.47023650,6894.877715)$ with ecological considerations that want all species populations to survive for a very long time. The population continues to grow stably, exploitation or harvesting can also be carried out or at least the sustainability of the species can be seen by future generations. To test the stability of the equilibrium point is to use the Routh-Hurwithz criteria. The characteristic equation that emerges is as follows,

$$\lambda^3+1.723127738\lambda^2+0.6153085991\lambda+0.001833785888=0.$$

Resulting in the appearance of the eigenvalues associated with equation (9), namely $\lambda_1 = -1.22001440563688$, $\lambda_2 = -0.00300552282810151$, and $\lambda_3 = -0.500107809946015$. Local asymptotic stability is due to the associated eigenvalues having the values $\lambda_1 < 0$, $\lambda_2 < 0$, and $\lambda_3 < 0$. The stability at other equilibrium points was not tested, because the authors assume that the point $N_6(92.46553379,33.47023650,6894.877715)$ is the most ideal for the continuity of the dynamic system in the species. In terms of the significant value of population growth, it can be seen that for prey species the growth is threefold. The growth of first prey species is high when compared to first prey species. Things like this can happen to ecosystems, prey species can dominate the prey density. While in the predator population the growth is very significant, this can happen because of the abundant food resources, namely first prey and second prey. Predators can choose food according to taste, choosing to prey on first prey or second prey, or both at the same time. We all know that abundant food resources can support nutrition, fertilization, mating, reproduction, and food for the next generation of predator species populations. Trajectories analysis is also given to model (5), looking at the consistency of growth for the survival of all species. Trajectories are given by taking initial values for all populations namely $x_1(0) = 5$, $x_2(0) = 3$, and $y(0) = 10$. Taking the initial value is adjusted to the equilibrium point taken, namely N_6. The trajectories of the stability model of a model (5) are as shown in Figure 2.

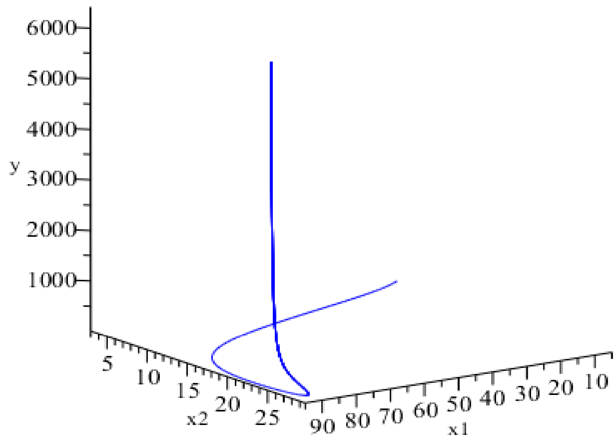

Figure 2. Local stable asymtotic equilibrium N_6

In Figure 2 it is clear that model (5) acquires the form of local asymptotic stability. Associated N_6 with equilibrium and fulfilling the Routh-Hurwitz criteria, the equilibrium point is at positive points. This condition is very possible for predator species to prey on first prey or second prey in a balanced way. For example, when predation occurs more on first prey, it can be ascertained that the predator species needs first prey for its needs. The same thing can also happen to the second prey species. The growth of each population over time will also be shown in Figure 3, Figure 4 and Figure 5.

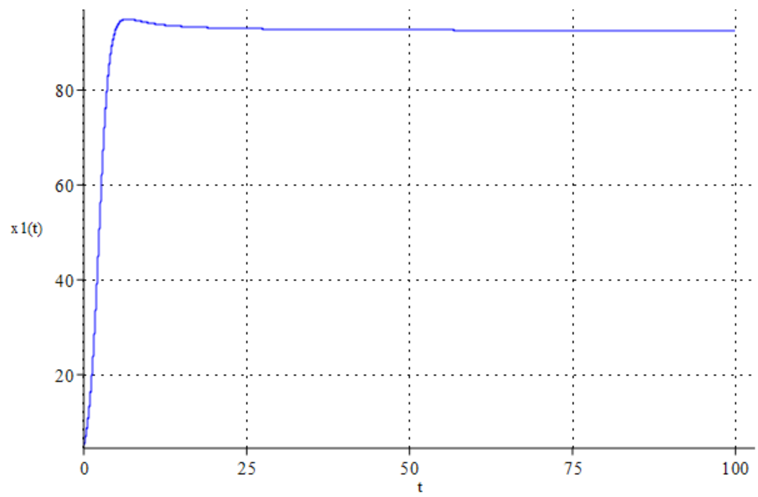

Figure 3. Trajectories Population First prey

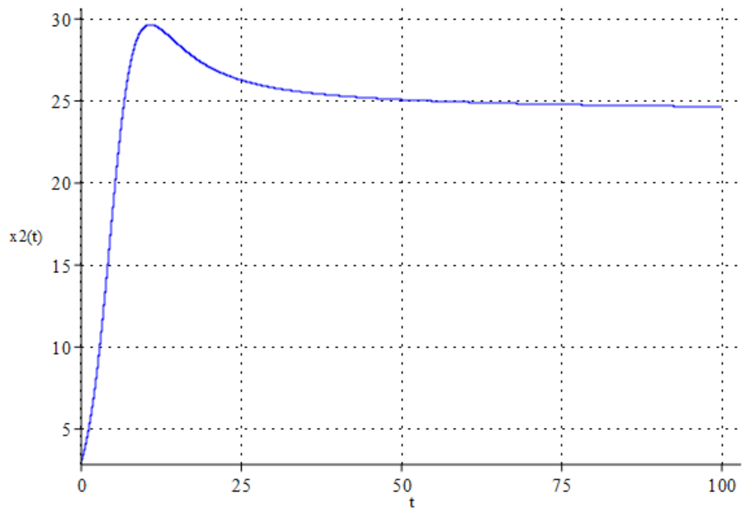

Figure 4. Trajectories Population Second prey

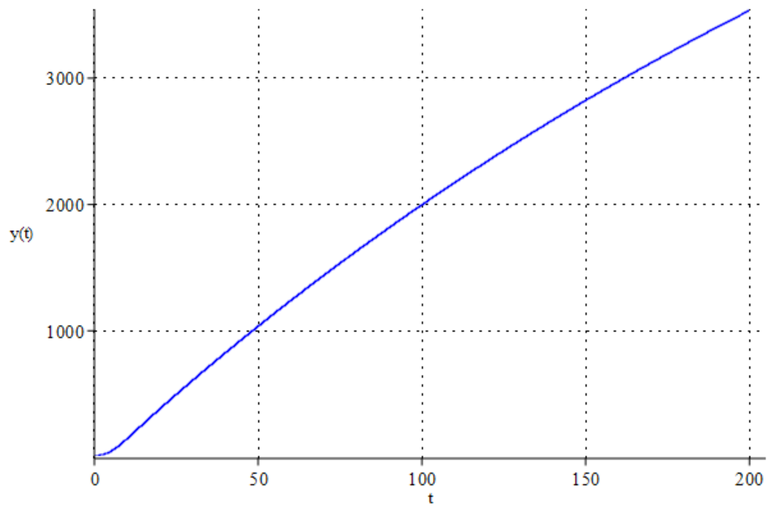

Figure 5. Trajectories Population Predator

Trajectories in each species differ significantly, even in the same ecosystem. Refers to the trajectories population in figure 3, figure 4 and figure 5 are the most stable trajectory movements over a long time. For example, in the population of first prey, the characteristic that emerges is significant growth in the first few months. Significant growth at the beginning was actually able to form a slight growth peak before sloping to reach full stability (Parshad et al., 2017). Significant increase in trajectories early in growth such as this is usually due to slow predation by predator populations (Yang, 2017). Of course, predator species must first recognize the characteristics of the first prey that will be preyed on. It is in this process that the prey develops significantly. After passing the peak growth rate in a relatively short time, population growth has slowed down towards a stable growth point, namely $x_1 = 92.46553379$.

In population second prey, the movements formed in the curve are not much different. Significant growth was also shown significantly by the population of second prey. Even the peak of population growth is significant when compared to the movement of the population first prey growth curve. The growth curve for second prey species drops when it has passed the peak population growth point, and towards a stable growth point for that species, namely, $x_2 = 33.47023650$.

Based on figure 5 shows that the growth is very significant in the predator population. As previously explained, predator species play a very large role in the system dynamic cycle. Population growth with a very large number and is consistent over a relatively long time. Mathematically, it is obtained that the ratio of details of growth to time is about 1: 20. This ratio is of course very large for species that live within a carrying capacity of 100. At 1 unit of time, the growth of predators is as many as 20 species. As explained earlier, predatory species find comfort in the presence of abundant resources. The growth of very significant predator species continued to grow without experiencing a decline until it reached a stability point according to model (5), namely $y = 6894.877715$.

D. CONCLUSION AND SUGGESTIONS

The predator-prey model always involves many interactions in ecosystem life. The aim of the research is to obtain a stable predator-prey model represented by model (5). Model (5), which consists of first prey, two prey and predator species, provides a more complex population dynamics model. Model (5) is analyzed by linearizing differential equations, so that seven non-negative equilibrium points are obtained, namely $N_0(0,0,0)$, $N_1(x_1,0,0)$, $N_2(0,x_2,0)$, $N_3(x_1,x_2,0)$, $N_4(0,x_2,y)$, $N_5(x_1,0,y)$ and $N_6(x_1,x_2,y)$. Of the seven equilibriums, only one equilibrium point is taken which will be tested using the Routh-Hurwitz stability criterion. Tests are carried out to show locally asymptotically stable equilibrium points. The stability of the model (5) is shown in the eigenvalues that meet that is $\lambda_1 = -1.22001440563688$, $\lambda_2 = -0.00300552282810151$, and $\lambda_3 = -0.500107809946015$

Trajectories analysis illustrates that each population has its characteristics. The characteristic that appears in the first prey population is the significant growth in the first few months. Significant increase in trajectories early in growth such as this is usually due to slow predation by predator populations. Mathematically, it is obtained that the ratio of details of growth to time is about 1:20. This ratio is of course very large for species that live within a carrying capacity of 100. At 1 unit of time, the growth of predators is as many as 20 species. The continuation of this research can provide interventions for exploitation efforts or consider the nature of competition between predator species. Very significant amount of predator growth can provide an economic benefit, consumption, medical or scientific experimentation. The nature of competition between predators is also worth considering because the rapid growth of predators allows for interactions between predators to occur or better known as intraspecific interactions. These factors deserve to be included as a dynamic predator-prey system intervention variable.

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