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Solution Formula of Korteweg Type by Using Partial Fourier Transform Methods in Half-Space without Surface Tension

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ABSTRACT

Sharp-interface models and diffuse-interface models are the two basic types of models that describe liquid-vapour flow for compressible fluids. Their depictions of the line dividing liquid from vapour are different. The interface is modeled as a hypersurface in sharp-interface models. Sharp-interface models are free-boundary problems from a mathematical perspective since the position of the interface is a priori unknown and therefore a component of the solution to the free-boundary problem. A unique system of partial differential equations describes the motion of the fluid in the liquid and vapour phases, respectively. At the interface, boundary conditions between these systems are connected.. A mathematical model for liquid-vapour flows including phase transition known as the Navier-Stokes-Korteweg system which is the extension of the compressible Navier-Stokes equations. The purpose of Ihis article, we consider the soluton formula of Korteweg fluid model in half-space without surface tension. Since we consider in half-space case, Partial Fourier transform become appropriate method to find the formula of velocity and density for Korteweg type. The solution formula of the model problem for the velocity (u) and the (φ) are obtained by using the invers of partial Fourier transform. It consist multipliers. For the future research, we can investigate the estimation of the multiplier. Furthermore, by using Weis's multiplier theorem we can find not only maximal *Lp-Lq* regularity class, but also we can consider the local well-posedness of the model problem.

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A. INTRODUCTION

In everyday life water occurs in various forms, such as ice, liquid water and water vapour. This different physical states are respectively referred to as the solid, water and vapour (or gas) phase of water. For water vapour one might have in mind the example of boiling water while preparing a cup of tea. However, what we observe coming out of the kettle and filling the kitchen is water steam: a mixture of air, water vapour and fine droplets of water. In the natural sciences, however, water vapour means the gaseous phase of water. Water can be found in many different forms in daily life, including ice, liquid water, and water vapour. One might think of boiling water while brewing a cup of tea as an example of water vapour (Shine et al., 2012).

Instead, we see water steam—a mixture of air, water vapour, and tiny droplets of water—erupting from the kettle and filling the kitchen. Water vapour, on the other hand, refers to the gaseous state of water in the natural sciences. Despite the imprecision in this common example, we nevertheless gain a general understanding of how water vaporizes, or changes phases from the liquid to the vapour phase. On chilly winter days, we can observe the water easily changing back into tiny drops of water by looking out the kitchen window. Condensation, the phase change from the vapour to the liquid phase, is demonstrated here. In contrast to solids, liquids and gases both have the ability to flow. Together they form the class of the fluids. However, they significantly differ in their mass densities (Langer, 2000). In constant temperature, we can use the mass density to distinguish different phases.

Liquids and gases both have the ability to flow, in contrast to solids. Together, they make up the fluids class. Their mass densities, however, greatly differ from one another. This enables us to distinguish between various phases using the mass density, assuming a constant temperature. It makes sense to be interested in the phase boundaries that separate the liquid phase from the vapour phase when thinking about a container filled with a fluid. We anticipate phase borders to be areas where the density function has steep slopes or even leaps, i.e., is discontinuous, as a result of the difference in the mass densities. Due to this, phase borders can be depicted in one of two ways: as narrow regions with steep density gradients, or as infinitesimally tiny regions with density jumps. The terms diffuse-interface models and sharp-interface models, respectively, are used to describe these concepts (Volkov et al., 2015).

Diffuse and sharp-interface models are two separate categories of mathematical models that can be used to describe liquid-vapor fluxes. The interfacial layer where phase changes take place is represented differently in each of them. In sharp-interface models, an infinitesimally thin hypersurface is employed in place of the small, positive thickness that is present in diffuse-interface models. By taking the limit where the interfacial region's thickness goes to zero, the diffuse-interface model can be connected to the related sharp-interface model. This is what we'll refer to as the diffuse-interface model's sharp-interface limit (Magnaudet & Mercier, 2020).

There are many researcher who consider Navier-Stokes equation and others type of fluid flows. Three dimension case of the Stokes equation which known as linearize of Navier-Stokes equation is investigated by (Alif et al., 2021). Meanwhile, reseachers who conducted fluid motion, they considered not only for local well-posedness but also global well-posedness. For instance, Global well-posedness of the Oldroyd-B model fluid flow was studied by (Maryani, 2016a). In the same year, she also investigated same type of fluid flow for free boundary case (Maryani, 2016b). In that article, she considered not only in bounded domain but also in unbounded domain case.

The Navier-Stokes-Korteweg model's sharp-interface limit, which is an extension of the compressible Navier-Stokes equations. The Dutch mathematician Diederik Johannes Korteweg first presented this diffuse-interface model for liquid-vapour movements in 1901 (Korteweg & De Vries, 1895). In that year, Korteweg developed a constitutive equation of stress tensors with density gradients to explain the effects of fluid capillarity. We are all quite familiar with capillary effects from everyday occurrences like tissue absorbing liquid from a surface, raindrops forming, ink in pens being carried to the tip, and candles burning. In the final illustration, capillary action causes the candle's wick to lift melted wax toward the flame. Once the wax has come into contact with the flame there, it vaporizes and burns (Daube, 2016).

If you place a thin tube into a cup of tea, you will observe the same result as in the tea example earlier: the tea will enter the tube and rise to the top. The tea's top surface concavely shapes a meniscus. The substance determines whether the surface is convex or concave; mercury, for instance, creates convex menisci (an effect known from mercury-in- glass thermometers (Benzoni-Gavage et al., 2005). In the Middle Ages, this liquids in narrow tubes phenomenon was already noted (or even earlier). The Latin word capillus, which means hair, was used to describe it at the time because no other terminology existed (Finn, 2012) and also can be read in (Emmer, 1987). We now understand that this effect is brought on by the tea's own surface tension as well as the interfacial tension between the liquid and the tube's solid surface.

A fluid is said to be Newtonian if it satisfies the Newton's law of viscosity i.e the shear stress is proportional to the rate of shear and the viscosity is the constant of proportionality such as water. Otherwise, depending on the size of the blood vessels and the flow behavior, it is approximated as a Navier-Stokes fluid or as a non-Newtonian fluid. So that the continuum hypotesis holds. The concepts of continuum mechanics developed over the last half-century. From continuum mechanics, we derive fluid mechanics that covers both statics and dynamics of the fluids (Daube, 2016).

The theory of non-Newtonian and viscoelastic fluids flourished in the second half of last century with the developments of (molten and dilute) polymers and the growth of materials science and engineering that generated many new products and applications. The mathematical setting of the constitutive equations required new tools from tensor analysis and algebra. In addition, the non-Newtonian fluids arise in a large variety of industrial applications, such as chemical processes, food industries, construction engineering (Daube, 2016).

Analysis of the behaviour of the fluid motion for non-Newtonian fluids is essentially more complex in comparison with Newtonian fluid motions. It is well known that for Newtonian fluid flows, we can found the analytical solutions, while for non-Newtonian fluid flows are rarely found. Several recent studies investigating this model have been carried out not only on some polymer application but also blood flow using the numerical analysis. Despite this, the mathematical investigation has not been developed yet in the compressible viscous fluid case (Daube, 2016).

Studying about fluid flow is very interesting point in fluid dynamics. Recently, there has been an increasing amount of literature on fluid motion. Many researcher investigated about this subject. However, they conducted in numerical analysis and rarely of them investigated fluid motion in mathematical analysis approach. Therefore, this reason become important motivation for researcher to investigate the fluid flow in the mathematical analysis point of view. The one dimensional of the compressible fluid for Korteweg type with large initial data from vacuum to Cauchy problem has been investigated by Chen et.al (Chen et al., 2015). They use the energy estimate approximation.

This research consider the solution formula of compressible fluid flow of the Korteweg model without surface tension in half-space case with slip boundary condition. The fact that a flowing fluid in touch with a solid body won't have any velocity relative to the body at the contact surface is now beyond dispute. This requirement of not slipping over a solid surface needs to be met by a moving fluid. The no-slip condition is what is meant by this (Inna et al., 2020). In this article, the domain Ω of the problem is the domain of any bounded region of the

N-dimensional Euclidean space $\mathbb{R}^N$ and Γ the boundary of Ω. The equation system of Korteweg with slip boundary condition can be described in the following

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega_t \\ \rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P(\rho)\mathbf{I}) = \operatorname{Div}(\mathbf{K}(\rho)) & \text{in } \Omega_t \\ \mathbf{n}\cdot\nabla\rho = g & \text{on } \Gamma_t \\ \mathbf{D}(\mathbf{u})\mathbf{n} - \langle \mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n}\rangle \mathbf{n} = \mathbf{h} & \text{on } \Gamma_t \\ \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma_t \end{cases} \tag{1}$$

for $0 < t < T$, the domain Ω is replaced by an unknown domain Ω_t depending on time t with the boundary Γ_t. Let Ω_t and Γ_t the evaluation of reference body Ω and its boundary Γ, respectively. The velocity field $\mathbf{u} = (u_1(x,t), \dots, u_N(x,t))^{\mathrm{T}}$, where $(u_1(x,t), \dots, u_N(x,t))^{\mathrm{T}}$ is the transposed of $(u_1(x,t), \dots, u_N(x,t))$ and $\rho = \rho(x,t)$ is the fluid density as an unknown function.

The material Korteweg of $\mathbf{S}(\mathbf{u})$ and $\mathbf{K}(\rho)$ which introduced by Dunn and Serrin (Dunn & Serrin, 1985) are defined as

$$\mathbf{S}(\mathbf{u}) = \mu\mathbf{D}(\mathbf{u}) + (\nu - \mu)\operatorname{div}\mathbf{u}\,\mathbf{I}, \quad \mathbf{K}(\rho) = \frac{\kappa}{2}\big(\Delta\rho^2 - |\nabla\rho|^2\mathbf{I} - \kappa\nabla\rho\otimes\nabla\rho\big),$$

$\mathbf{D}(\mathbf{u})$ the double deformation tensor whose (i,j) components are $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$, $(\partial_i = \partial/\partial x_j)$, $\mathbf{I}$ the $N \times N$ identity matrix, μ and ν are the first and second viscosity coefficients, respectively, and $\mathbf{n}$ is an outer normal of Γ_t.

The historical of mathematical analysis point of view for compressible fluid model of Korteweg which mean the system of (1) firstly introduced by Kotcshote in 2008 (Kotschote, 2008) for isothermal cases. He proved the existence and uniqueness of strong solution local in time using the contraction mapping principle and result of maximal regularity for $L_p - L_q$, p=q. The similar approach also treated by Kotcshote (Kotschote, 2010), (Freistühler & Kotschote, 2017) and (Kotschote, 2014) in the case of non-isothermal. For the same problem, (Bresch et al., 2003) investigated the equilibria point of stability and the initial densities away from zero. In 2019, (Bresch et al., 2019) investigated Navier-Stokes-Korteweg and Euleur-Korteweg system as application to quantum fluids models. In contrast, (Haspot, 2011) studied the existence of global weak solution.

The equation system of (1) explained a liquid-vapour two-phase flow for compressible fluids with phase transition as a diffuse interface model in addition to modelling capillarity effect, see e.g. (Siddique et al., 2009) and (Li et al., 2020). Beside that in thermodynamic framework for developing boundary condition for Korteweg-type fluids has been considered by (Souček et al., 2020). On the other hand, (Suzuki, 2020) investigated the higher order model and relation to microforces for Korteweg type fluids.

There are two main categories for models that describe liquid-vapour flow for compressible fluids: sharp-interface models and diffuse-interface models. They differ in how they depict the boundary separating liquid from vapour. In sharp-interface models, the interface is represented as a hypersurface. From a mathematical point of view, sharp-interface models are free-boundary problems, since the position of the interface is a priori unknown and thus is part of the solution to the free-boundary problem. A sharp-interface with Lagrangian Eulerian method for rigid-body fluid structure has been studied by Kolahdouz (Kolahdouz et al., 2021). In the liquid and vapour phase, respectively, a distinct system of partial differential equations describes the motion of the fluid. These systems are coupled by boundary conditions at the interface. For a more complete overview and additional references on capillary and interfacial phenomena we refer to the aforementioned references (Benzoni-Gavage et al., 2005) and (Bhatnagar & Finn, 2016) and additionally to (Siddique et al., 2009). For the asymptotic behaviour of solutions to an impermeable wall problem of the compressible fluid model of Korteweg type with density-dependent viscosity and capillarity was investigated by (Chen & Li, 2021).

Recently, many researchers studied about Korteweg type. In 2022 (Kobayashi et al., 2022) investigated the estimation of resolvent problem for a compressible fluid model of Korteweg type and their application. Two year before, (Inna et al., 2020) and (Saito, 2020) studied R-boundedness solution operator of Korteweg model fluid flow in half-space case and the regularity of $L_p - L_q$ framework for a compressible fluid model of Korteweg in general domains, respectively. In 2021, Saito investigated the R-boundedness of the solution operator for compressible fluid flow of of the Korteweg in half-space not only for large resolvent (λ) parameter but also for small resolvent (λ) (Saito, 2021). Meanwhile, (Lauro, 2014) studied the linear stability for Korteweg fluid. In (Inna et al., 2020), they considered the estimating for multiplier which appear in the solution formula of the Korteweg type in half-space by using Weis's multiplier theorem. Then, they found a positive constant which known as R-boundedness of the solution formula for Korteweg model fluid flow. Different from that article, in this article we consider the solution formula of the model problem (2) by using partial Fourier transform.

Notation $\mathbb{N}$ denotes the sets of natural numbers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{C}$ and $\mathbb{R}$ denote the sets of complex numbers and real numbers, respectively. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial_x^\kappa = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$ with $x = (x_1, \dots, x_N)$. For scalar function f and N-vector of function $\mathbf{g}$, we get

$$\nabla f = (\partial_1 f, \dots, \partial_N f), \nabla \mathbf{g} = \{\, \partial_i g_j \mid i, j = 1, \dots, N\},$$
$$\nabla^2 f = \{\partial_i \partial_j f \mid i, j = 1, \dots, N\},\ \ \nabla^2 \mathbf{g} = \{\partial_i \partial_j g_k \mid i, j, k = 1, \dots, N\},$$
$$W_q^{m,\ell}(\Omega) := \{\, (\mathbf{f}, \mathbf{g}) \mid \mathbf{f} \in W_q^m(\Omega),\ \mathbf{g} \in W_q^\ell(\Omega)\,\}.$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and Fourier inverse transform, respectively, which are defined by

$$\mathcal{F}_x[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx,\ \ \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\, d\xi. \tag{1a}$$

In other hand, the partial Fourier transform with respect to $x^{\prime} = (x_1, \dots, x_{N-1})$ and its inverse transform are defined as

$$\mathcal{F}_{x^{\prime}}\left[u(x^{\prime}, x_N)\right](\xi) = \hat{u}(\xi^{\prime}, x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix^{\prime}\cdot\xi^{\prime}} u\left(x^{\prime}, x_N\right) dx^{\prime}, \tag{1b}$$

$$\mathcal{F}_{\xi^{\prime}}^{-1}\left[u(\xi^{\prime}, x_N)\right]\left(x^{\prime}\right) = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix^{\prime}\cdot\xi^{\prime}} u\left(\xi^{\prime}, x_N\right) d\xi^{\prime}, \tag{1c}$$

where $\xi^{\prime} = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$.

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform, respectively, which are defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\, dt, \quad \mathcal{L}^{-1}[g](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\tau)\, d\tau,$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. For $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$, we set $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{N} x_j y_j$. For scalar functions f, g and N-vectors of function , $\mathbf{g}$, we get $(\mathbf{k}, \mathbf{g})_D = \int_D \mathbf{k} \cdot \mathbf{g}\, dx$, $(k, g)_{\Gamma} = \int_{\Gamma} \mathrm{kg}\, d\sigma$, $(\mathbf{k}, \mathbf{g})_{\Gamma} = \int_{\Gamma} \mathbf{k} \cdot \mathbf{g}\, d\sigma$, where σ is the surface element of Γ. For $N \times N$ matrices of function $\mathbf{F} = (F_{ij})$ and $\mathbf{G} = (G_{ij})$, we get $(\mathbf{F}, \mathbf{G})_D = \int_D \mathbf{F} : \mathbf{G}\, dx$, $(\mathbf{F}, \mathbf{G})_{\Gamma} = \int_{\Gamma} \mathbf{F} : \mathbf{G}\, d\sigma$, where $\mathbf{F} : \mathbf{G} \equiv \sum_{i,j=1}^{N} F_{ij} G_{ij}$. The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on $a, b, \dots$. The values of constants C and $C_{a,b,\dots}$ denote a positive constant which maybe different even in a single chain of inequalities. We use small boldface letter, e.g. $\mathbf{u}$ to denote vector-valued functions and capital boldface letters, e.g. $\mathbf{H}$ to denote matrix-valued functions, respectively. But, we also use the Greek letters, e.g $\sigma, \rho, \theta, \tau, \omega$ such as mass densities.

B. METHODS

This article's research approach makes use of a literature review of linked topics, particularly the article of (Inna et al., 2020). With their establishment of the velocity formula, we defined the solution of velocity differently in this article. In the steps that follow to determine the model problem's formula, we first use the Laplace transform to transform model problem (1), then we get the resolvent issue, which is expressed in (2). Furthermore, by using partial Fourier transform and inverse partial Fourier transform of the equation system of (2), we have the solution formula of equation (2). Therefore, we obtain the answer formula or velocity $\mathbf{u}$ and density ρ in half space case. Studying the partial Fourier transform is thus the first and most crucial step towards proving Theorem 1. The research procedure can be described in the following flowchart, as shown in Figure 1.

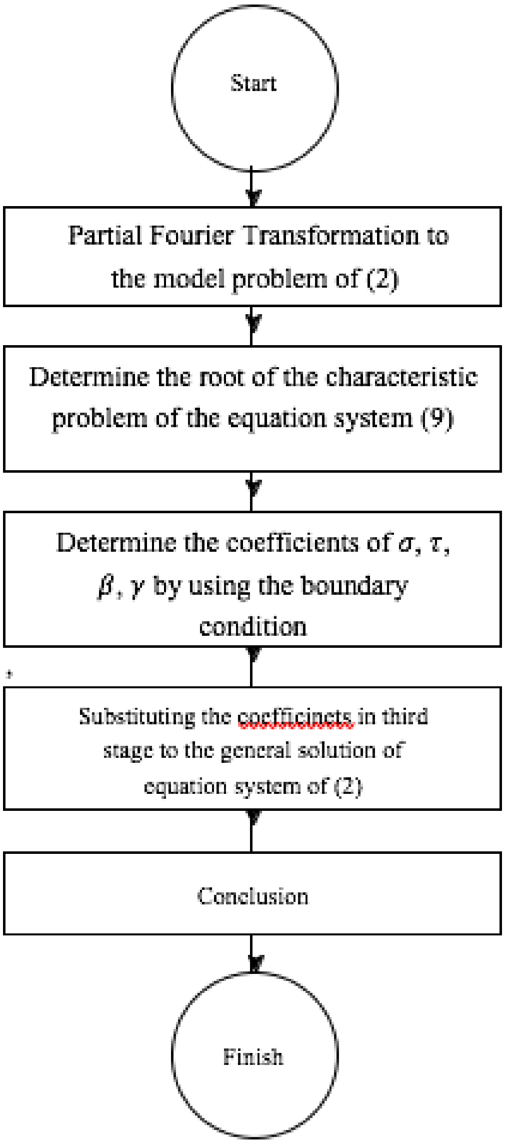

Figure 1. Reserach Procedure

C. RESULT AND DISCUSSION

In everyday life, we are very familiar with capillary effects, such as tissue absorbs liquid from a surface and a raindrop forms. These effects make the surface concave or convex. We know that this is happening depends on the material. The phase transitions of the material can be described in the mathematical model. In this section, we consider the two phenomena of phase change. This phenomena known as Korteweg type. Some researcher consider the solution formula of the model problem in the numerical point of view. However, in this section, we consider the solution formula of the model problem for velocity $\mathbf{u}(\mathrm{x},\mathrm{t})$ and density $\rho(x,t)$ in the mathematical point of view. In the following, we state the main theorem from the problem of Korteweg type without surface tension. The method of to find the solution formula of the model problem (2) for velocity and density in half-space case is followed (Kobayashi et al., 2022).

1. Main Theorem

Before we state the main result for the linear problem of equation (1), we Introduce the definition of Sobolev space $W_q^{m,n}(\Omega)$ and technical lemma in the following

Definition 1. (Adams & Fournier, 2003)
Let $k \in \mathbb{N} \cup \mathbb{N}_0$ and $p \in [1, \infty)$ then the Sobolev Space $W_q^m(\Omega)$ is defined by

$$W_q^m(\Omega) := \{\mathbf{u} \in L_q(\Omega) \mid D^\alpha \mathbf{u} \in L_q(\Omega), \forall \alpha \text{ with } |\alpha| \leq m\}$$

Following theorem is the main result of this article

Theorem 1. Let $\rho(x,t)$ be a density and $\mathbf{u}(x,t)$ velocity in N-dimensional Euclidean space $\mathbb{R}^N$, $N \geq 2$ and set $\boldsymbol{x}' = (x_1, \dots, x_{N-1})$ and $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$ then for $\eta_*^\omega > 0$ the equation system of (2) has a unique solution formula of $(\rho, \mathbf{u}) \in W_q^{3,2}(\mathbb{R}_+^N)$ with

$$\rho = \mathcal{F}_{\xi'}^{-1}\left[\frac{s_1\lambda}{t_1}\beta_N e^{-t_1 x_N} + \frac{s_2\lambda}{t_2}\gamma_N e^{-t_2 x_N}\right](\boldsymbol{x}', x_N)$$

and

$$u_j = \mathcal{F}_{\xi'}^{-1}\left[\left(-\frac{1}{\omega_\lambda}\left(\hat{h}_j(0) + \frac{i\xi_j}{t_1}(\omega_\lambda - t_1)\beta_N + \frac{i\xi_j}{t_2}(\omega_\lambda - t_2)\gamma_N\right)\right)e^{-\omega_\lambda x_N} + \left(-\frac{i\xi_j}{t_1}\beta_N\right)(e^{-t_1 x_N} - e^{-\omega_\lambda x_N}) + \left(-\frac{i\xi_j}{t_2}\gamma_N\right)(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}).\right](\boldsymbol{x}', x_N), \quad j = 1, \dots, N-1$$

and also

$$u_N = \mathcal{F}_{\xi'}^{-1}[\beta_N(e^{-t_1 x_N} - e^{-\omega_\lambda x_N}) + \gamma_N(e^{-t_2 x_N} - e^{-\omega_\lambda x_N})](\boldsymbol{x}', x_N).$$

where

$$\beta_N = \frac{1}{s_2 - s_1}\left(-\hat{g}(0) + s_2\mu\lambda^{-1} i\xi'\hat{h}'(0)\right),$$

$$\gamma_N = \frac{\hat{g}(0) - s_1\beta_N}{s_2}, \quad t_{1,2} = \pm\sqrt{\left|\xi'\right|^2 + s_k\lambda},$$

and

$$s_\pm = \frac{(\mu + \nu)}{2\kappa} \pm \sqrt{\eta_*^\omega}\,.$$

2. Proof of the main theorem

a. Resolvent problem

In this part, we consider the resolvent problem of the linearized of equation (1) in half-space by using Laplace transformation, we have

$$\begin{cases} \lambda\rho + \operatorname{div}\mathbf{u} & = & 0 & \text{in } \mathbb{R}_+^N \\ \lambda\mathbf{u} - \mu\Delta\mathbf{u} - \nu\nabla\operatorname{div}\mathbf{u} - \kappa\nabla\Delta\rho & = & 0 & \text{in } \mathbb{R}_+^N \\ \mathbf{n}\cdot\nabla\rho & = & g & \text{on } \mathbb{R}_0^N \\ \partial_N u_j + \partial_j u_N & = & h_j & \text{on } \mathbb{R}_0^N \\ u_N & = & 0 & \text{on } \mathbb{R}_0^N \end{cases} \qquad (2)$$

for $j = 1, \dots N-1$ and $\mathbb{R}_+^N$, $\mathbb{R}_0^N$, $N \geq 2$, are the upper half-space and its boundary, respectively, which defined as

$$\mathbb{R}_+^N = \{x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N \mid x_N > 0\},$$
$$\mathbb{R}_0^N = \{x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

and also $\mathbf{n} = (0, \dots 0, -1)^{\mathrm{T}}$ is the outward unit normal vector on $\mathbb{R}_0^N$.

b. Partial Fourier Transform

Before we elaborate the part of transformation process to the equation system of (2), first of all the definition of partial Fourier transform and inverse partial Fourier transform as in (1b) and (1c), respectively.

Applying partial Fourier transform to equation (2), and let $\varphi = \operatorname{div}\mathbf{u}$, we have

$$\begin{aligned} \lambda\hat{\rho} + \hat{\varphi} &= 0, & x_N > 0 \\ \lambda\hat{u}_j - \mu\left(\partial_N^2 - \left|\xi'\right|^2\right)\hat{u}_j - \nu i\xi_j\hat{\varphi} - \kappa i\xi_j(\partial_N^2 - |\xi'|^2)\hat{\rho} &= 0, & x_N > 0 \\ (\lambda\hat{u}_N - \mu\left(\partial_N^2 - \left|\xi'\right|^2\right)\hat{u}_N - \nu\partial_N\hat{\varphi} - \kappa\partial_N(\partial_N^2 - |\xi'|^2)\hat{\rho} &= 0, & x_N > 0 \end{aligned} \qquad (3)$$

and boundary condition when $x_N = 0$

$$\begin{aligned} \partial_N\hat{\rho}(0) &= -\hat{g}(0), & \\ \partial_N\hat{u}_j(0) + i\xi_j\hat{u}_N(0) &= \hat{h}_j(0), & j = 1, \dots, N-1 \\ \hat{u}_N(0) &= 0. & \end{aligned} \qquad (4)$$

Substituting first equation to the second equation of (3) then multiplying by $i\xi_j$, we have

$$\lambda^2\hat{u}_j - \lambda\mu\left(\partial_N^2 - \left|\xi'\right|^2\right)\hat{u}_j - \lambda\nu i\xi_j\hat{\varphi} - \lambda\kappa i\xi_j(\partial_N^2 - |\xi'|^2)\hat{\varphi} = 0. \qquad (5)$$

$$\lambda^2 i\xi_j\hat{u}_j - \lambda\mu i\xi_j\left(\partial_N^2 - \left|\xi'\right|^2\right)\hat{u}_j - \lambda\nu\left|\xi'\right|^2\hat{\varphi} - \lambda\kappa\left|\xi'\right|^2(\partial_N^2 - |\xi'|^2)\hat{\varphi} = 0. \qquad (6)$$

Applying same technique to third equation of (3) then differentiate with respect to x_N, we have

$$\lambda^2\partial_N\hat{u}_N - \lambda\mu\partial_N\left(\partial_N^2 - \left|\xi'\right|^2\right)\hat{u}_N - \lambda\nu\partial_N^2\hat{\varphi} + \kappa\partial_N^2(\partial_N^2 - |\xi'|^2)\hat{\varphi} \quad = 0. \tag{7}$$

By using $i\xi_j\hat{u}_j + \partial_N\hat{u}_N = \hat{\varphi}$, adding equation (6) and (7), we have

$$P_\lambda(\partial_N)\hat{\varphi} \quad = 0, \tag{8}$$

where

$$P_\lambda(\partial_N) \quad = \quad \Big(\lambda^2 - \lambda(\mu+\nu)\left(\partial_N^2 - \left|\xi'\right|^2\right) + \kappa(\partial_N^2 - |\xi'|^2)^2\Big). \tag{9}$$

Furthermore, we have the formula of $P_\lambda(t)$ i.e

$$P_\lambda(t) \quad = \quad \Big(\lambda^2 - \lambda(\mu+\nu)\left(t^2 - \left|\xi'\right|^2\right) + \kappa(t^2 - |\xi'|^2)^2\Big). \tag{10}$$

Multiplying equation (5) by $P_\lambda(\partial_N)$ we have

$$\left(\lambda^2 - \lambda\mu\left(\partial_N^2 - \left|\xi'\right|^2\right)\right)P_\lambda(\partial_N)\hat{u}_j - i\xi_j\big(\lambda\nu - \kappa(\partial_N^2 - |\xi'|^2)\big)P_\lambda(\partial_N)\hat{\varphi} \quad = 0. \tag{11}$$

By equation (8), we can write the equation (10) to be

$$\left(\lambda^2 - \lambda\mu\left(\partial_N^2 - \left|\xi'\right|^2\right)\right)P_\lambda(\partial_N)\hat{u}_j \quad = 0. \tag{12}$$

Since $\hat{u}_j \neq 0$, then $P_\lambda(\partial_N) = 0$ or $\left(\lambda^2 - \lambda\mu\left(\partial_N^2 - \left|\xi'\right|^2\right)\right) = 0$. Moreover, roots of the equation (10) i.e

$$P_\lambda(t) \quad = \quad \lambda^2\kappa A(s),$$

with

$$A(s) \quad = \quad \left(\frac{1}{\kappa} - \frac{(\mu+\nu)}{\kappa}s + s^2\right),$$

then, for $\eta_*^\omega = \left(\frac{\mu+\nu}{2\kappa}\right)^2 - \frac{1}{\kappa} \neq 0$ we have

$$s_\pm \quad = \quad \begin{cases} \dfrac{(\mu+\nu)}{2\kappa} \pm \sqrt{\eta_*^\omega} & \eta_*^\omega > 0 \\ \dfrac{(\mu+\nu)}{2\kappa} \pm i\sqrt{\eta_*^\omega} & \eta_*^\omega < 0 \end{cases}.$$

and

$$t_{1,2} \quad = \quad \pm\sqrt{\left|\xi'\right|^2 + s_k\lambda}.$$

In this article we consider only for $\eta_*^\omega > 0$ and $t_1 \neq t_2$, $\omega_\lambda = t_1$ and $\omega_\lambda = t_2$. Therefore,

$$\hat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \beta_J(e^{-t_1 x_N} - e^{-\omega_\lambda x_N}) + \gamma_J(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}), \quad j = 1, \dots, N \quad (13)$$

Multiplying equation (13) by $i\xi_j$ and differentiate equation (13) respect to x_N variable, summing up then substituting to formula $i\xi_j\hat{u}_j + \partial_N\hat{u}_N = \hat{\varphi}$, we can find the formula of $\hat{\varphi}$ that is

$$\hat{\varphi} = \sigma e^{-t_1 x_N} + \tau e^{-t_2 x_N},$$

with

$$\sigma = i\frac{\left(|\xi'|^2 - t_1^2\right)}{t_1}\beta_N, \quad \tau = \frac{(|\xi'| - t_2)}{t_2}\gamma_N, \;\; \beta_j = -\frac{i\xi_j}{t_1}\beta_N, \;\; \gamma_j = -\frac{i\xi_j}{t_2}\gamma_N$$

$$\alpha_j = -\frac{1}{\omega_\lambda}\left(\hat{h}_j(0) + \frac{i\xi_j}{t_1}(\omega_\lambda - t_1)\beta_N + \frac{i\xi_j}{t_2}(\omega_\lambda - t_2)\gamma_N\right), j = 1, \dots, N-1$$

$$\beta_N = \frac{1}{s_2 - s_1}\left(-\hat{g}(0) + s_2\mu\lambda^{-1}i\xi'\hat{h}'(0)\right).$$

Furthermore, the formula of $\hat{\rho}$ is writing in the following

$$\hat{\rho} = -\frac{\hat{\varphi}}{\lambda} = -\frac{\sigma e^{-t_1 x_N} + \tau e^{-t_2 x_N}}{\lambda}.$$

By using the same technique in (Kobayashi et al., 2022) and applying inverse partial Fourier transform , we complete the proof of Theorem 1.■

D. CONCLUSION AND SUGGESTIONS

The velocity and density of the model Korteweg in equation (2) by using the partial Fourier transform and inverse partial Fourier transform contains multipliers. For the future research, we can consider the estimation of these multiplier by using Weis's Multipliers Theorem for the same problem with surface tension. Multipliers are used to build the solution formula for the model problem velocity and density. Solution operator families are the name given to these multipliers. The boundedness of the model problem's solution operator families. The boundedness of the solutions operator with surface tension will be a topic for future research, as was indicated at the conclusion of this paper. Be aware that the proofs and the entire result can be applied to fluid dynamics research in the future. The regularity of the model problem's solution is a crucial consideration from a purely mathematical standpoint. Additionally, this discovery serves as a crucial first step in demonstrating boundlessness in bent-half space and the general domain.

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