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M/M/1 Non-preemptive Priority Queuing System with Multiple Vacations and Vacation Interruptions

Dillah Rismawati[1], I Wayan Mangku[2], Hadi Sumarno[3]
[1,2,3]Department of Mathematics, IPB University, Indonesia
dillah_rismawati@apps.ipb.ac.id[1], wayanma@apps.ipb.ac.id[2], hadisumarno@apps.ipb.ac.id[3]

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ABSTRACT

Non-preemptive priority queue system is a type of priority queue where customers with higher priorities cannot interrupt low priority one while being served. High priority consumers will still be at the head of the queue. This article discusses the non-preemptive priority queue system with multiple working vacations, where the vacation can be interrupted. Customers are classified into two classes, namely class I (non-preemptive priority customers) and class II, with exponentially distributed service rates. Customers will still receive services at a slower rate than during normal busy periods when they enter the system while it is on vacation. Suppose other customers are waiting in the queue after completing service on vacation. In that case, the vacation will be interrupted, and the service rate will switch to the busy period service rate. The model's performance measurements are obtained using the complementary variable method and analyzing the state change equation following the birth and death processes to find probability generating function for both classes. The results of the numerical solution show that the expected value number of customers and waiting time of customers in the queue for both class customers will be reduced when the vacation times rate (θ) and the vacation service rate (μ_0) increase.

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A. INTRODUCTION

Queue is a phenomenon that frequently occurs in daily life. Queuing happens if there are more customers than service facilities, which makes it unable to serve new customers immediately due to overworked servers. Queuing theory analysis is increasingly applied in reality, such as in the health sector Cho et al. (2017), the banking sector Cowdrey et al. (2018), industrial production optimization Rece et al. (2022), rail traffic transportation Jacyna et al. (2019), and minimizing CO2 emissions at container terminals (Liu & Ge, 2018).

Sometimes servers are unavailable to serve customers within a random period. The vacation period is when the server cannot provide service. On vacation, the server will completely stop service. In contrast to vacation, the server in the working vacation scheme will provide services at a slower rate instead of terminating service. Compared to normal vacations, working vacations can reduce the probability of reneging customers (Dudin et al., 2015). Ibe & Isijola (2014) analyzed multiple vacations wherein two kinds of differentiated vacations are scheduled at the end of a zero-duration busy period (first type) and a nonzero-duration busy period (second type). Ammar (2015) examine an M/M/1 queue with multiple vacations.

Customers may leave the system without getting service first because of the server is on vacation (impatience customers). The server can begin a busy period (non-vacation) even though the working vacation period has not ended if customers wait in the system after finishing services during a working vacation. In that case, the vacation is interrupted. The queuing system with working vacation interruption has been analyzed, such as in Sreenivasan et al. (2013) with MAP/PH/1 queue and Zhang & Liu (2015) with M/G/1 G-queue and server breakdowns. Multiple working vacation and vacation interruption under Bernoulli-scheduled with impatient customers have been examined by (Goswami, 2014). Vijaya Laxmi & Jyothsna (2015) analyzed working vacation and vacation interruption under Bernoulli-scheduled with impatient customers for two working vacation policies, the multiple working vacations and single working vacation.

A priority queuing system classifies customers into two types and gives service priority according to their type. Customers with a higher priority will be served first by the system than those with a lower priority. Based on Kumar (2020), priorities in queueing systems are generally classified as preemptive and non-preemptive. In preemptive priority, the system can interrupt service to every customer in the current service facility immediately when a customer with a higher priority arrives. Customers with higher priorities are always made to move forward in favor of customers with lower priorities. In comparison, non-preemptive systems never interrupt a customer's service once it has begun, although a higher-priority customer enters the system when this service is in progress.

Researchers have widely studied the priority queue model in previous years. Ruth Evangelin & Vidhya (2020) analyzed a non-preemptive M/M/1 priority queue system with server breakdown and repair time. The method used in this research is the complementary variable method to build a Markov process vector. Ajewole et al. (2021) analyzed the preemptive-resume priority queuing system with the Erlang distribution, and the server used was a single server. Ma et al. (2016) analyzed system performance measures and optimized the preemptive-(N,n) queuing system with multiple working vacations using a three-dimensional Markov chain queuing system. Furthermore, Rajadurai et al. (2016) analyzed a performance measure of the repeated preemptive priority queue system with Bernoulli's feedback on a working vacation and vacation interruption. In 2020, Ma et al. (2020) continued their research, namely analyzing system performance evaluation and analyzing discrete queuing systems with non-preemptive priorities and multiple working vacations. Kim et al. (2021) conducted research on the M/M/m non-preemptive priority queue system with server vacation.

This paper investigates an M/M/1 non-preemptive priority queue system with multiple working vacations that can be interrupted. Customers are classified into: class I and class II, with class I customers having a non-preemptive priority over class II customers. The paper aims to determine the performance measures of two classe customers using a complementary variable method and obtain generating function of the length distribution of two classes of customers.

B. METHODS

This research analyzed a non-preemptive priority queueing system with multiple working vacations and vacation interruptions. Class I and class II are the two categories of customer classes, with class I having non-preemptive priority over class II. During busy periods, classes I and II customers enter the system following a Poisson process with an arrival rate of λ_1 and λ_2, respectively. The total system arrival rate is $\lambda = \lambda_1 + \lambda_2$. Customer service time for both classes is exponential with a rate of $\mu_{B_i}, i = 1,2$. When the system has no customers after having finished a service, the server goes on vacation. Customers arriving at λ_i during working vacation will still be served at a lower service rate. The vacation-service rate has an exponential distribution with rates μ_0 and $\mu_0 < \mu_{B_i}$.

The time of working vacation is assumed to follow an exponential distribution with a mean of $\frac{1}{\theta}$. If the vacation ends and the queue is empty, the server may go on another vacation until at least one customer is in the queue. This scheme is referred to as a multiple vacation scheme. Suppose the server detects another customer after serving the customer on vacation. In that case, vacation interruption will be activated, and the service rate will switch from μ_0 to μ_{B_i} and begins serving customers class I first. Otherwise, the server can retrieve another vacation. Customers in each class are serviced on a First Come First Served (FCFS). In the FCFS scheme, the customer who arrives first will be served first by the system. The time between arrivals, service time during busy and working vacation periods, and vacation time are independent. Multiple Working Vacation and Vacation Interruptions is illustrated in Figure 1.

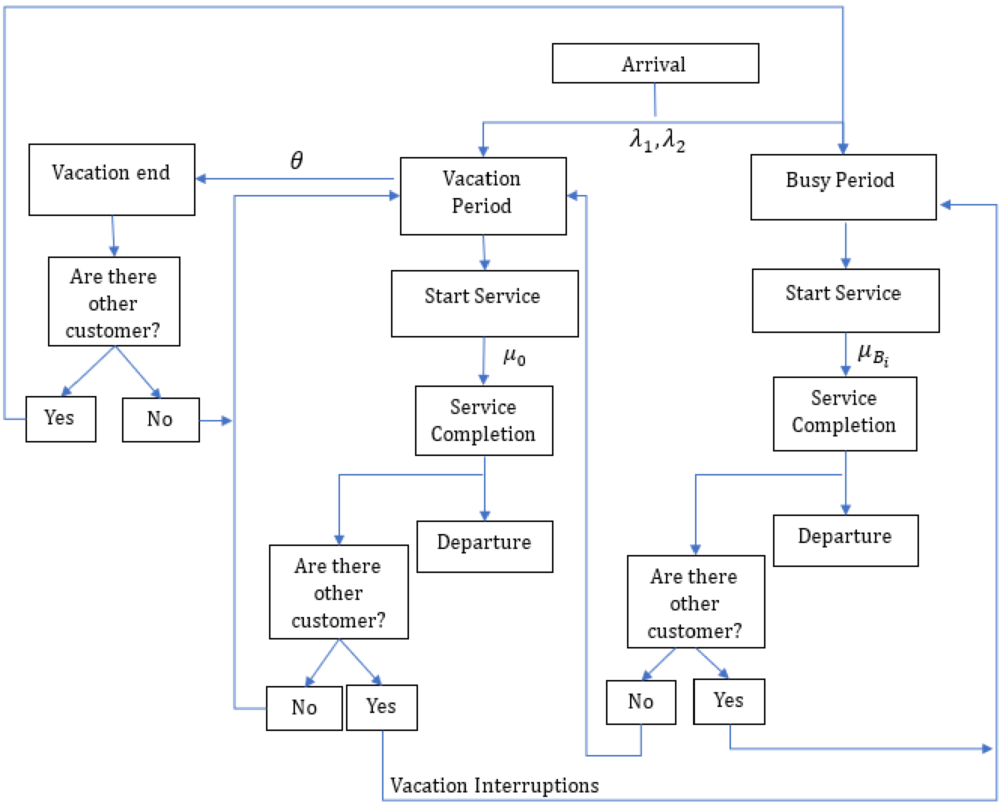

Figure 1. M/M/1 Queueing System with Multiple Working Vacation and Vacation Interruptions

Suppose $N_i(t)$ denotes the number of class-i $(i = I, II)$ customers in the system at time t. The server states denoted by J and defined as follows:

$J = 0 \rightarrow$ Server is in the idle state (working vacation).
$J = 1 \rightarrow$ There is a class I customer in service while the server is on vacation.
$J = 2 \rightarrow$ There is a class II customer in service while the server is on vacation.
$J = 3 \rightarrow$ There is a class I customer in service while the server is in a busy period.
$J = 4 \rightarrow$ There is a class II customer in service while the server is in a busy period.

Thus, the vector process $N(t) = \{N_1(t), N_2(t), J\}, t \geq 0$ is a Markov process having state-space $\{((m, n, j))\ ;\ m, n \geq 0, j = 0,1,2,3,4\}$. The system's busy state is shown by $\rho = \frac{\lambda_1}{\mu_{B_1}} + \frac{\lambda_2}{\mu_{B_2}}$. For stability of the system, it is assumed that $\rho < 1$. Denoting the steady-state probabilities $p_{mnj} = \lim_{t\rightarrow\infty} \Pr\{ N_1(t) = m, N_2(t) = n, J = j\}$. The balanced equation shown below is derived from the birth and death process:

$$(\lambda_1 + \lambda_2)p_{000} = \mu_{B1}p_{103} + \mu_{B2}p_{014} + \mu_0 p_{101} + \mu_0 p_{012}, \tag{1}$$
$$(\lambda_1 + \lambda_2 + \mu_0 + \theta)p_{mn1} = \lambda_2 p_{m,n-1,1}, \tag{2}$$
$$(\lambda_1 + \lambda_2 + \mu_0 + \theta)p_{mn1} = \lambda_1 p_{m-1,n,1} + \lambda_2 p_{m,n-1,1}\ , \tag{3}$$
$$(\lambda_1 + \lambda_2 + \mu_0 + \theta)p_{0n2} = \lambda_2 p_{0,n-1,2}, \tag{4}$$
$$(\lambda_1 + \lambda_2 + \mu_0 + \theta)p_{mn2} = \lambda_1 p_{m-1,n,2}, \tag{5}$$
$$(\lambda_1 + \lambda_2 + \mu_0 + \theta)p_{mn2} = \lambda_1 p_{m-1,n,2} + \lambda_2 p_{m,n-1,2}, \tag{6}$$
$$(\lambda_1 + \lambda_2 + \mu_{B_1})p_{m03} = \lambda_1 p_{m-1,0,3} + \mu_0 p_{m+1,0,3} + \mu_{B_1} p_{m+1,0,3} + \mu_{B_2} p_{m,1,4} + \theta p_{m,0,1} \qquad m \geq 2 \tag{7}$$
$$(\lambda_1 + \lambda_2 + \mu_{B_1})p_{mn3} = \lambda_1 p_{m-1,n,3} + \lambda_2 p_{m,n-1,3} + \mu_0 p_{m+1,n,1} + \mu_0 p_{m,n+1,2} + \mu_{B_1} p_{m+1,n,3} + \mu_{B_2} p_{m,n+1,4} + \theta p_{m,n,1} + \theta p_{m,n,2} \qquad m \geq 2, n \geq 1, \tag{8}$$
$$(\lambda_1 + \lambda_2 + \mu_{B2})p_{0,n,4} = \lambda_2 p_{0,n-1,4} + \mu_0 p_{1,n,1} + \mu_0 p_{0,n+1,2} + \mu_{B_1} p_{1,n,3} + \mu_{B_2} p_{0,n+1,4} + \theta p_{0n2} \tag{9}$$
$$(\lambda_1 + \lambda_2 + \mu_{B2})p_{m,n,4} = \lambda_1 p_{m-1,n,4} + \lambda_2 p_{m,n-1,4} \qquad m \geq 1, n \geq 2. \tag{10}$$

C. RESULT AND DISCUSSION

1. Stationery-Equation Solution

The probability generating function can be used to determine the mean and variance of a probability distribution. In this paper, the probability generating function will be used to determine the average number of customers and customer waiting time in the system. Let X be a random variable, then $G_m(z_2) = \mathrm{E}[z_2^X] = \sum_{n=0}^{\infty} P_{mnj} z_2^n$ with $\sum_{n=0}^{\infty} P_{mnj} = 1$ is the probability generating function of the random variable X (Shortle et al., 2018) and $G(z_1, z_2) = \sum_{m=1}^{\infty} G_m(z_2) z_1^m$ is join probability generating function. Based on the stationary equation above, define the probability generator function as follows:

$$G_m^a(z_2) = \sum_{n=0}^{\infty} P_{mn1} z_2^n, \quad m \geq 1, \qquad G^a(z_1, z_2) = \sum_{m=1}^{\infty} G_m^a(z_2) z_1^m,$$
$$G_m^b(z_2) = \sum_{n=1}^{\infty} P_{mn2} z_2^n, \quad m \geq 0, \qquad G^b(z_1, z_2) = \sum_{m=0}^{\infty} G_m^b(z_2) z_1^m,$$
$$G_m^c(z_2) = \sum_{n=0}^{\infty} P_{mn3} z_2^n, \quad m \geq 1, \qquad G^c(z_1, z_2) = \sum_{m=1}^{\infty} G_m^c(z_2) z_1^m,$$
$$G_m^d(z_2) = \sum_{n=1}^{\infty} P_{mn4} z_2^n, \quad m \geq 0, \qquad G^d(z_1, z_2) = \sum_{m=0}^{\infty} G_m^d(z_2) z_1^m.$$

By using the probability generating function above, multiply equation (3) with z_2^n, summing the number of possible n, it is obtained as follows:

$$(\lambda_1 + (1 - z_2)\lambda_2 + \mu_0 + \theta)[G_m^a] = [G_{m-1}^a]\lambda_1. \tag{11}$$

Similarly, from equations (4) and (9) respectively,

$$G_0^b = \frac{\lambda_2 z_2}{\lambda_1 + \lambda_2(1 - z_2) + \mu_0 + \theta} \tag{12}$$
$$\left(\lambda_1 + (1 - z_2)\lambda_2 + \left(1 - \frac{1}{z_2}\right)\mu_{B_2}\right) G_0^d = G_1^a \mu_0 + G_1^c \mu_{B_1} + \left(\frac{\mu_0}{z_2} + \theta\right) G_0^b - (\lambda_1 + \lambda_2). \tag{13}$$

Furthermore, using equations (6), (8) and (10), the corresponding results were:

$$(\lambda_1 + (1 - z_2)\lambda_2 + \mu_0 + \theta)[G_m^b] = [G_{m-1}^b]\lambda_1, \tag{14}$$
$$(\lambda_1 + (-z_2 + 1)\lambda_2 + \mu_{B1}) G_m^c(z_2) = G_{m-1}^c(z_2)\lambda_1 + G_{m+1}^a(z_2)\mu_0 + \left(\frac{\mu_0}{z_2} + \theta\right) G_m^b(z_2) + G_{m+1}^c(z_2)\mu_{B1} + \frac{\mu_{B2}}{z_2} G_m^d(z_2) + \theta G_m^a(z_2), \tag{15}$$
$$\left(\lambda_1 + (1 - z_2)\lambda_2 + \mu_{B_2}\right)[G^d(z_1, z_2) - G_0^d] = \lambda_1 z_1 G^d(z_1, z_2). \tag{16}$$

Next, multiply equation (11) by z_1^m and sum according to the number of possible m, it is obtained the following:

$$G^a(z_1, z_2) = \frac{z_1 \lambda_1 p_{000}}{((1 - z_1)\lambda_1 + (1 - z_2)\lambda_2 + \mu_0 + \theta)}. \tag{17}$$

Similarly, from equations (14)-(16), it is obtained respectively,

$$G^b(z_1, z_2) = \frac{(\lambda_1 + (1 - z_2)\lambda_2 + \mu_0 + \theta) G_0^b}{((1 - z_1)\lambda_1 + (1 - z_2)\lambda_2 + \mu_0 + \theta)}, \tag{18}$$
$$\left((1 - z_1)\lambda_1 + (1 - z_2)\lambda_2 + \left(1 - \frac{1}{z_1}\right)\mu_{B_1}\right) G^c(z_1, z_2) = \left(\frac{\mu_0}{z_1} + \theta\right) G^a(z_1, z_2) + \left(\frac{\mu_0}{z_2} + \theta\right) G^b(z_1, z_2) - \mu_0 G_1^a(z_2) - \mu_{B_1} G_1^c(z_2) - \left(\frac{\mu_0}{z_2} + \theta\right) G_0^b + \mu_{B2} z_2^{-1} \left(G^d(z_1, z_2) - G_0^d(z_2)\right), \tag{19}$$
$$G^d(z_1, z_2) = \frac{(\lambda_1 + (1 - z_2)\lambda_2 + \mu_{B_2}) G_0^d}{((1 - z_1)\lambda_1 + (1 - z_2)\lambda_2 + \mu_{B_2})}. \tag{20}$$

Substitute equations (12), (13), (17), (18) and (20) into equation (19), and obtain

$$G^c(z_1,z_2)=\frac{(A(z_1,z_2)-1)(\lambda_1+(1-z_2)\lambda_2+\mu_{B_2})G_0^d(z_2)-[\lambda_1+\lambda_2-B(z_1,z_2)-C(z_1,z_2)]p_{000}}{(1-z_1)\lambda_1+(1-z_2)\lambda_2+\mu_{B1}\left(1-\frac{1}{z_1}\right)}, \tag{21}$$

where:

$$A(z_1,z_2)=\frac{\mu_{B2}}{z_2\left((1-z_1)\lambda_1+(1-z_2)\lambda_2+\mu_{B_2}\right)},$$

$$B(z_1,z_2)=\left(\frac{\mu_0}{z_1}+\theta\right)\frac{z_1\lambda_1}{\left((1-z_1)\lambda_1+(1-z_2)\lambda_2+\mu_0+\theta\right)},$$

$$C(z_1,z_2)=\left(\frac{\mu_0}{z_2}+\theta\right)\frac{z_2\lambda_2}{\left((1-z_1)\lambda_1+(1-z_2)\lambda_2+\mu_0+\theta\right)}.$$

$G^c(z_1,z_2)$ and $G_0^d(z_2)$ are unknown probabilities in equation (21). Using the Kernel method, the numerator of equation (21) must be zero if the denominator is.

$$(1-z_1)\lambda_1+(1-z_2)\lambda_2+\left(1-\frac{1}{z_1}\right)\mu_{B1}=0,$$
$$\lambda_1 z_1^2-(\lambda_1+\lambda_2(1-z_2)+\mu_{B1})z_1+\mu_{B1}=0. \tag{22}$$

When $|z_1|=1, |z_2|<1$,

$$|\lambda_1 z_1^2+\mu_{B1}|\leq \lambda_1+\mu_{B1}<|\lambda_1+(1-z_2)\lambda_2+\mu_{B1}|=|(\lambda_1+(1-z_2)\lambda_2+\mu_{B1})z_1|.$$

Rouche's theorem states that equation (22) has unique roots in the region $|z_2|<1$. Assume that the unique root is $f(z_2)$. For any fixed value z_2, the value $z_1=f(z_2)$ causes the denominator of equation (21) to vanish. In this case, to obtain the correct G_0^d value, the numerator in equation (21) also vanishes when $z_1=f(z_2)$. The numerator of the equation must be made equal to zero, so we get

$$(A_1(z_2)-1)(\lambda_1+(1-z_2)\lambda_2+\mu_{B_2})G_0^d(z_2)-[\lambda_1+\lambda_2-B_1(z_2)-C_1(z_2)]p_{000}=0,$$
$$G_0^d(z_2)=\frac{[\lambda_1+\lambda_2-B_1(z_2)-C_1(z_2)]p_{000}}{(\lambda_1+(1-z_2)\lambda_2+\mu_{B_2})(A_1(z_2)-1)}, \tag{23}$$
$$\begin{aligned}G_0^{d'}(z_2)=p_{000}(\lambda_2(-1+A_1(z_2))(\lambda_1+\lambda_2-B_1(z_2)-C_1(z_2))-A_1'(z_2)(\lambda_1+\lambda_2-z_2\lambda_2\\+\mu_{B_2})(\lambda_1+\lambda_2-B_1(z_2)-C_1(z_2))-(\lambda_1+\lambda_2-z_2\lambda_2+\mu_2)(B_1'(z_2)\\+C_1'(z_2))(-1+A_1(z_2)))/(\lambda_1+\lambda_2-z_2\lambda_2+\mu_{B_2})^2(-1+A_1(z_2))^2\end{aligned}$$

where:

$$A_1(z_2)=\frac{\mu_{B2}}{z_2\left((1-f(z_2))\lambda_1+(1-z_2)\lambda_2+\mu_{B_2}\right)},$$

$$B_1(z_2)=\left(\frac{\mu_0}{f(z_2)}+\theta\right)\frac{f(z_2)\lambda_1}{\left((1-f(z_2))\lambda_1+(1-z_2)\lambda_2+\mu_0+\theta\right)},$$

$$C_1(z_2)=\left(\frac{\mu_0}{z_2}+\theta\right)\frac{z_2\lambda_2}{\left((1-f(z_2))\lambda_1+(1-z_2)\lambda_2+\mu_0+\theta\right)},$$

$$A_1'(z_2) = -\frac{\mu_{B_2}(\lambda_1+\lambda_2-2\,z_2\lambda_2+\mu_{B_2}-\lambda_1(f(z_2)+z_2f'(z_2)))}{z_2^2(\lambda_1+\lambda_2-z_2\lambda_2+\mu_{B_2}-\lambda_1 f(z2))^2},$$

$$B_1'(z_2) = \frac{\lambda_1(\lambda_2\mu_0+\theta\,\lambda_2 f(z_2)+(\theta(\theta+\lambda_1+\lambda_2-z_2\lambda_2)+(\theta+\lambda_1)\mu_0)f'(z2))}{(\theta+\lambda_1+\lambda_2-z_2\lambda_2+\mu_0-\lambda_1 f(z_2))^2},$$

$$C_1'(z_2) = \frac{\lambda_2(\theta(\theta+\lambda_1+\lambda_2)+(\theta+\lambda_2)\mu_0-\theta\lambda_1 f(z_2)+\lambda_1(z_2\theta+\mu_0)f'(z_2))}{(\theta+\lambda_1+\lambda_2-z_2\lambda_2+\mu_0-\lambda_1 f(z_2))^2}.$$

When z_2 converges to 1, substitute $z_1 = f(z_2)$ into equation (22),

$\lambda_1(f(1))^2 - (\lambda_1+\mu_{B1})f(1)+\mu_{B1} = 0.$

Solve the equation above, $f(1) = 1$ and $f(1) = \frac{\mu_{B_1}}{\lambda_1} > 1$. Considering the circumstances $\rho < 1$, according to Rouche's theorem, only one real root in the region $|z_2| < 1$ of equation (22), i.e., $f(1) = 1$. $f(z_2)$ is the root of a quadratic equation with z_1, it must be derivative. When z_2 converges to 1, rederivative (22) to obtain $f'(1) = \frac{-\lambda_2}{\lambda_1-\mu_{B1}} = \frac{\lambda_2}{\mu_{B1}-\lambda_1}$. By using the L'Hospital rule,

$$G_0^d(1) = \lim_{z_2\to 1} G_0^d(z_2) = \frac{-B_1'-C_1'}{-\lambda_2(-1+A_1)+(\lambda_1+\mu_{B_2})A_1'}p_{000} \tag{24}$$

$$G_0^{d'}(1) = \lim_{z_2\to 1} G_0^{d'}(z_2) = \Big[p_{000}\lambda_2(\theta+\lambda_1+\lambda_2)\big(-\lambda_1\lambda_2^2(\theta+\mu_0)(\lambda_1-\mu_{B_1})\mu_{B_1}^2 + D_2 + (-\lambda_1^3\lambda_2(\theta+\mu_0)+C_2-B_2+E_2)\mu_{B_2}^2+\mu_{B_1}(A_2)\mu_{B_2}^3\big)\Big] / \Big[(\theta+\mu_0)^2(\lambda_1-\mu_{B_1})^2(\lambda_2-\mu_{B_2})(\lambda_1+\mu_{B_2})^2\big(\lambda_2\mu_{B_1}+(\lambda_1-\mu_{B_1})\mu_{B_2}\big)\Big] \tag{25}$$

where:

$A_1 = \lim_{z_2\to 1} A_1(z_2) = 1,\ B_1 = \lim_{z_2\to 1} B_1(z_2) = \lambda_1,\ C_1 = \lim_{z_2\to 1} C_1(z_2) = \lambda_2,$

$A_2 = \big(\lambda_1(\theta(\lambda_1+\lambda_2)+\lambda_2\mu_0+\lambda_1(\lambda_2+\mu_0)) - 2\lambda_1(\theta+\lambda_2+\mu_0)\mu_{B_1}+(\theta+\lambda_2+\mu_0)\mu_{B_1}^2\big),$

$B_2 = \lambda_1(2\theta(\lambda_1+\lambda_2)-\lambda_2(\lambda_2-2\mu_0)+2\lambda_1(\lambda_2+\mu_0))\mu_{B_1}^2,$

$C_2 = \lambda_1^2(\theta(\lambda_1+4\lambda_2)+4\lambda_2\mu_0+\lambda_1(\lambda_2+\mu_0))\mu_{B_1},$

$D_2 = \lambda_1\lambda_2(\lambda_1-\mu_{B_1})\mu_{B_1}(-\lambda_2(\theta+\mu_0)+(\theta+\lambda_2+\mu_0)\mu_{B_1})\mu_{B_2},$

$E_2 = (\theta\lambda_1-\lambda_2^2+\lambda_1(\lambda_2+\mu_0))\mu_{B_1}^3,$

$$A_1' = \lim_{z_2\to 1} A_1'(z_2) = \frac{-\lambda_2\mu_{B_1}-\lambda_1\mu_{B_2}+\mu_{B_1}\mu_{B_2}}{(\lambda_1-\mu_{B_1})\mu_{B_2}},$$

$$B_1' = \lim_{z_2\to 1} B_1'(z_2) = -\frac{\lambda_1\lambda_2(\theta+\mu_{B_1})}{(\theta+\mu_0)(\lambda_1-\mu_{B_1})},$$

$$C_1' = \lim_{z_2\to 1} C_1'(z_2) = \frac{\theta\,\lambda_1\lambda_2-\lambda_2(\theta+\lambda_2)\mu_{B_1}}{(\theta+\mu_0)(\lambda_1-\mu_{B_1})},$$

Finally, to get p_{000}, based on the normalization condition where:

$$G^a(1,1) = \frac{p_{000}\lambda_1}{\theta+\mu_0},$$

$$G^b(1,1) = \frac{p_{000}\lambda_2}{\theta+\mu_0},$$

$$G^c(1,1) = -\frac{p_{000}\lambda_1(\theta+\lambda_1+\lambda_2)\mu_{B_2}}{(\theta+\mu_0)\left(\lambda_2\mu_{B_1}+\left(\lambda_1-\mu_{B_1}\right)\mu_{B_2}\right)},$$
$$G^d(1,1) = -\frac{p_{000}\lambda_2(\theta+\lambda_1+\lambda_2)\mu_{B_1}}{(\theta+\mu_0)\left(\lambda_2\mu_{B_1}+\left(\lambda_1-\mu_{B_1}\right)\mu_{B_2}\right)},$$
Thus, $p_{000} = -\frac{(\theta+\mu_0)(\lambda_2\mu_{B_1}+(\lambda_1-\mu_{B_1})\mu_{B_2})}{-\lambda_2\mu_0\mu_{B_1}-\lambda_1\mu_0\mu_{B_2}+(\theta+\lambda_1+\lambda_2+\mu_0)\mu_{B_1}\mu_{B_2}}$. (26)

2. Performance Measure

The value of the performance measurements changes with time since a queueing model represents a dynamic system. When all transient behaviour has finished, the system has stabilized, and the values of a model effectiveness measurement are independent of time, the system is said to be in a steady state (Bolch et al., 2006).

a. Expected Number of Customers in the Queue

Suppose L_V and L_B represent the expected number of customers in the system during the working vacation and busy periods, respectively. The expected number of class I customers in the system during working vacation (L_{V_1}) and busy periods $\left(L_{B_1}\right)$ are as follows:

$$L_{V_1} = \left.\frac{\partial\left(G^a(z_1,1)+G^b(z_1,1)\right)}{\partial z_1}\right|_{Z_1=1} = \frac{p_{000}\lambda_1(\theta+\lambda_1+\lambda_2+\mu_0)}{(\theta+\mu_0)^2}$$

$$L_{B_1} = \left.\frac{\partial\left(G^c(z_1,1)+G^d(z_1,1)\right)}{\partial z_1}\right|_{Z_1=1}$$
$$= \frac{\lambda_1}{(\lambda 1-\mu_{B_1})^2}\left(\frac{p_{000}(\theta+\lambda_1+\lambda_2)\left(-\lambda_1^2+(\theta+\lambda_1+\mu_0)\mu_{B_1}\right)}{(\theta+\mu_0)^2} + \frac{G_0^d(1)\mu_{B_1}\left(\lambda_1+\mu_{B_2}\right)\left(-\lambda_1+\mu_{B_1}+\mu_{B_2}\right)}{\mu_{B_2}^2}\right).$$

Thus, the following is the expected number class I customers in the system:

$$L_1 = L_{V_1} + L_{B_1}$$
$$= \frac{\lambda 1}{(\lambda_1-\mu_{B_1})^2}\left(\frac{p_{000}\left[\lambda_1^2\mu_0+(\theta-\lambda_1)(\theta+\lambda_1+\lambda_2)\mu_{B_1}+(\theta-\lambda_1+\lambda_2)\mu_0\mu_{B_1}+(\theta+\lambda_1+\lambda_2+\mu_0)\mu_{B_1}^2\right]}{(\theta+\mu_0)^2} + \frac{G_0^d(1)\mu_{B_1}(\lambda_1+\mu_{B_2})(-\lambda_1+\mu_{B_1}+\mu_{B_2})}{\mu_{B_2}^2}\right) \quad (27)$$

Similarly, the expected number of class II customers in the system during working vacation (L_{V_2}) and busy periods $\left(L_{B_2}\right)$ are:

$$L_{V\,2}=\left.\frac{\partial\left(G^{a}(1,z_2)+G^{b}(1,z_2)\right)}{\partial z_2}\right|_{z_2=1}=\frac{p_{000}\lambda_2(\theta+\lambda_1+\lambda_2+\mu_0)}{(\theta+\mu_0)^2},$$

$$L_{B\,2}=\left.\frac{\partial\left(G^{c}(1,z_2)+G^{d}(1,z_2)\right)}{\partial z_2}\right|_{z_2=1}$$

$$=-\frac{p_{000}\lambda_2(\theta+\lambda_1+\lambda_2)}{(\theta+\mu_0)^2}+\frac{\left(\lambda_1\left(\lambda_2-\mu_{B_2}\right)-\mu_{B_2}^2\right)G_0^d(1)+\mu_{B_2}(\lambda_1+\mu_{B_2})G_0^{d'}(1)}{\lambda_2\mu_{B_2}}.$$

Therefore, the expected number of class II customers in the system is:

$$L_2=L_{V\,2}+L_{B\,2}=\frac{p_{000}\lambda_2\mu_0}{(\theta+\mu_0)^2}+\frac{\left(\lambda_1(\lambda_2-\mu_{B_2})-\mu_{B_2}^2\right)G_0^d(1)+\mu_{B_2}(\lambda_1+\mu_{B_2})G_0^{d'}(1)}{\lambda_2\mu_{B_2}}, \tag{28}$$

$G_0^d(1)$ and $G_0^{d'}(1)$ are in equations (24) and (25), respectively.

The expected number of customers in the queue is the difference between the number of customers in the system and in service. Suppose L_Q is the number of customers in the queue. Probability of class I and class II customers using the server is defined in the following expression:

$$P_{class-I}=G^{a}(1,1)+G^{c}(1,1)=\frac{\lambda_1\lambda_2\mu_{B_1}-\lambda_1\left(\theta+\lambda_2+\mu_{B_1}\right)\mu_{B_2}}{\lambda_2\mu_0\mu_{B_1}+\lambda_1\mu_0\mu_{B_2}-(\theta+\lambda_1+\lambda_2+\mu_0)\mu_{B_1}\mu_{B_2}},$$

$$P_{class-II}=G^{b}(1,1)+G^{d}(1,1)=-\frac{\lambda_2\left((\theta+\lambda_1)\mu_{B_1}+\left(-\lambda_1+\mu_{B_1}\right)\mu_{B_2}\right)}{\lambda_2\mu_0\mu_{B_1}+\lambda_1\mu_0\mu_{B_2}-(\theta+\lambda_1+\lambda_2+\mu_0)\mu_{B_1}\mu_{B_2}}.$$

So, mean queue length for class I $\left(L_{Q_1}\right)$ and class II $\left(L_{Q_2}\right)$ customers are:

$$L_{Q_1}=L_1-\frac{\lambda_1\lambda_2\mu_{B_1}-\lambda_1\left(\theta+\lambda_2+\mu_{B_1}\right)\mu_{B_2}}{\lambda_2\mu_0\mu_{B_1}+\lambda_1\mu_0\mu_{B_2}-(\theta+\lambda_1+\lambda_2+\mu_0)\mu_{B_1}\mu_{B_2}}, \tag{29}$$

$$L_{Q_2}=L_2-\left(-\frac{\lambda_2\left((\theta+\lambda_1)\,\mu_{B_1}+\left(-\lambda_1+\mu_{B_1}\right)\mu_{B_2}\right)}{\lambda_2\mu_0\mu_{B_1}+\lambda_1\mu_0\mu_{B_2}-(\theta+\lambda_1+\lambda_2+\mu_0)\mu_{B_1}\mu_{B_2}}\right), \tag{30}$$

L_1 and L_2 are in equations (27) and (28), respectively.

b. Expected Waiting Time of customers in the Queue

Suppose W_Q denotes the expected waiting time of customers in the queue. Based on Little's law, the expected number of customers in the queue $\left(L_Q\right)$ equals the arrival rate (λ) multiplied by the mean waiting time of customers in the queue $\left(W_Q\right)$. In other words, $L_Q=\lambda W_Q$. As a result,

$$W_Q=\frac{L_Q}{\lambda}.$$

Thus, the expected waiting time of customers in the queue for class I (W_{Q_1}) and class II (W_{Q_2}) are:

$$W_{Q_1} = \frac{L_{Q_1}}{\lambda_1}, \quad (31)$$
$$W_{Q_2} = \frac{L_{Q_2}}{\lambda_2}, \quad (32)$$

L_{Q_1} and L_{Q_2} are in equations (29) and (30), respectively.

3. Simulation of the Effect of Vacation-Service Rate (μ_0) on Mean Queue Length (L_Q)

Assume that the parameters $\lambda_1 = 1.22, \lambda_2 = 0.7$, and $\mu_1 = \mu_2 = 2$ have been set with different values of θ. Equation (29) and (30) calculates the expected number of customers in the queue for class I $\left(L_{Q_1}\right)$ and class II $\left(L_{Q_2}\right)$, respectively, as shown in Figure 2 and Figure 3.

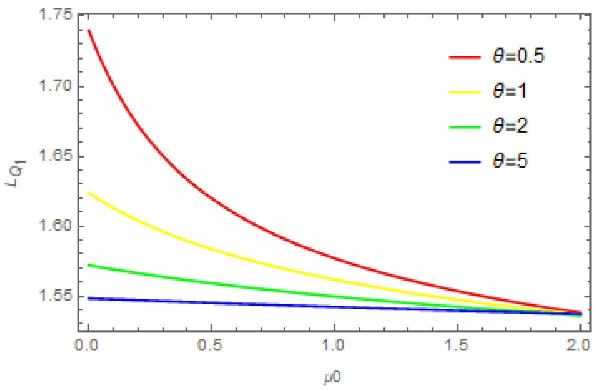

Figure 2. L_{Q_1} vs μ_0

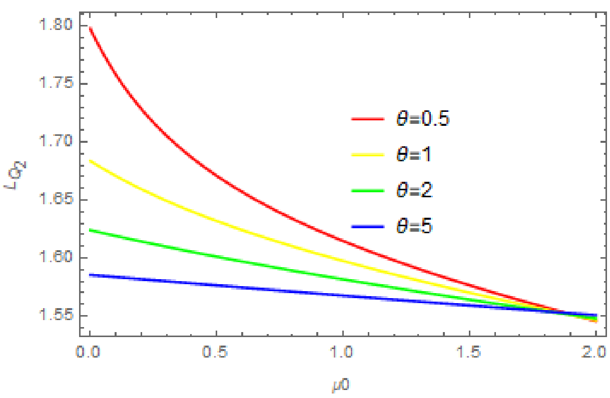

Figure 3. L_{Q_2} vs μ_0

Figures 2 and Figure 3 show that L_{Q_1} and L_{Q_2} decrease as the vacation-service rate (μ_0) increases. When μ_0 approaches 2, L_{Q_1} and L_{Q_2} approach a constant value, corresponding to a queueing model without vacations. The queue length for the model with classical vacation ($\mu_0 = 0$) is longer than for the model without vacation ($\mu_0 = 2$). Moreover, as vacation time (θ) increases, L_{Q_1} and L_{Q_2} will decrease. When the rate of vacation time is small ($\theta \leq 0.5$), the effect of μ_0 to mean queue length is more significant. Besides that, based on Figure 1 and Figure 2, the expected queue length for class I is lower than for class II. For example, for $\theta = 0.5$, the queue length for class I is about 5% lower than class II's.

4. Simulation of the Effect of Vacation-Service Rate (μ_0) on Mean Waiting time Customers in Queue $\left(W_Q\right)$ and Heterogeneous Service Rate on Mean Waiting Time Customers in Queue $\left(W_Q\right)$

Assume that the parameters $\lambda_1 = 1.22, \lambda_2 = 0.7$, and $\mu_1 = \mu_2 = 2$ have been set with varying θ values. Equation (31) and (32) calculates the expected waiting time for customers in the queue (queue length) for class I (W_{Q_1}) and class II (W_{Q_2}), respectively, as shown in Figure 4, Figure 5, and Figure 6.

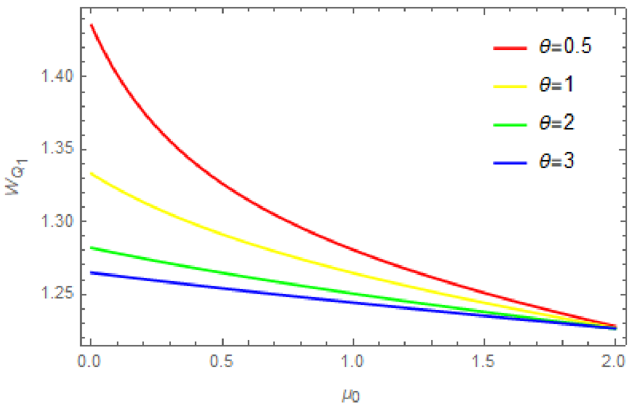

Figure 4. W_{Q_1} vs μ_0

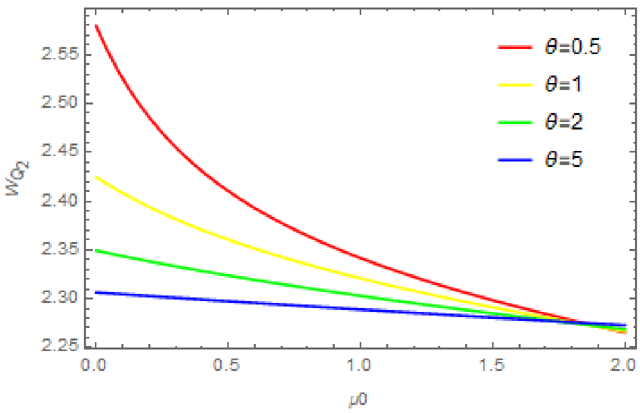

Figure 5. W_{Q_2} vs μ_0

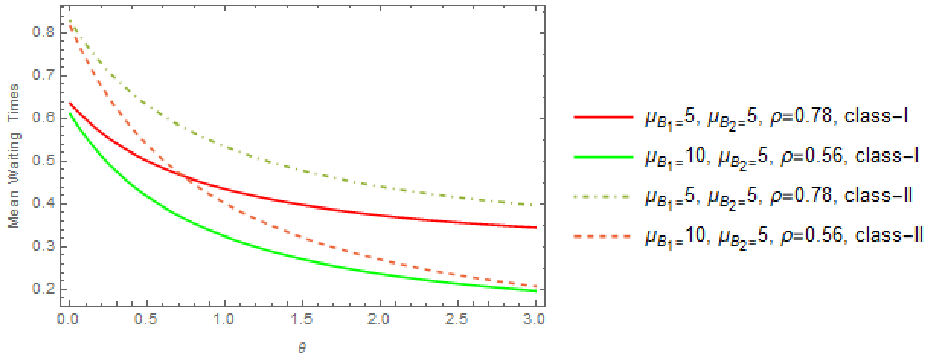

Figure 6. Mean Waiting Times with Heterogeneous Service Rates and traffic intensity($\boldsymbol{\rho}$)

Based on Figure 4 and Figure 5, the expected waiting time in queue decreases as the vacation-service rate (μ_0) increases. The queue with working vacation will reduce to a normal vacation model if the vacation-service rate degenerates to zero ($\mu_0 = 0$). The decrease in the mean waiting time is more significant in the system with a more extended vacation ($\theta \leq 0.5$). It is also obvious that W_{Q_1} is shorter than that of W_{Q_2}. For example, for $\theta = 0.5$, W_{Q_1} is about 45% lower than that for W_{Q_2}. This condition is relevant to the model where class I has non-preemptive priority over class II, so class I customers should not be required to wait too long in the queue.

The effect of the heterogeneous service rates and traffic intensity (ρ) is illustrated in Figure 6. When the service rate of class I is twice as fast as that of class II, the expected waiting time for the two classes will be shorter than when the rates of the two classes are homogeneous. The graph shows a downward trend initially and stabilizes for large values of θ. Meanwhile, customers' queue waiting time will also increase with a high traffic load. This is reasonable for practice. Systems with shorter vacation times significantly reduce customer waiting time when the traffic load is high.

5. Simulation of Comparisons between MWV+VI and MV Models

The following Comparisons between MWV+VI and MV models, as shown in Figure 7.

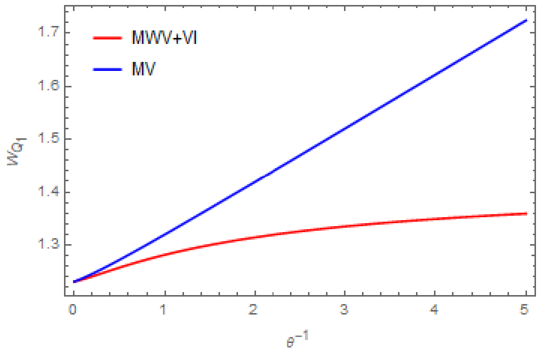

Figure 7. Comparisons between MWV+VI and MV Models

Figure 7 compares the multiple working vacations and vacation interruptions (MWV+VI) and multiple classical vacations (MV) models. As the mean vacation rate (θ^{-1}) increases, the customer waiting time in class I will increase. Based on Figure 6, the MWV+VI model performs better than MV because it has a mean waiting time that tends to be stable. The system will be more optimal because the server can be used effectively.

Similar to (Guo-xi & Qi-Zhou, 2009) research regarding priority non-preemptive queue systems (without vacations), the expected value of waiting time for class I and class II customers will increase as the arrival rate of each class increases. However, the presence of vacation in this paper causes the expected waiting time of customers to be higher than the non-preemptive priority model without vacation. This is because, during vacations, customers will be served at a lower rate than during normal busy periods. In (Majid & Manoharan, 2019), it can be seen that the expected number of customers in the queue and the expected waiting time of customers also decrease as the value of θ increases. In this paper, the multiple working vacations (MWV) model without vacation interruption has a higher waiting time expectation value than the model with vacation interruptions (MWV+VI).

D. CONCLUSION AND SUGGESTIONS

A non-preemptive priority queue system with multiple working vacations has been analyzed in this paper, where the vacation can be interrupted if there are other customers after completing services during a working vacation. Class I and class II are the two categories of customer classes, with class I having non-preemptive priority over class II. A probability generating function for the distribution of the length of the queue system for two types of customers was obtained by using the complementary variable method to analyze state-change equations based on the birth-death process and building a vector Markov process. The resulting measure of system performance is the expected number of customers and the waiting time of customers in the system and the queue.

Based on the numerical simulation result, the expected value of the queue length and the expected waiting time in the queue decreases for both classes when the vacation service rate (μ_0) increases. The system with a more extended vacation $(\theta \leq 0.5)$ significantly affects mean queue length and waiting time. This is because when vacation is longer, after completing one service during a working vacation, the vacation is interrupted and normal busy period begins. Moreover, if the service rate of class I is twice as fast as that of class II, the expected waiting time for the two classes will be shorter than when the rates of the two classes are homogeneous. For future research, one can consider this model with multiple servers and more than two types of customer priority. In addition, one can add cost analysis to find the minimum cost per unit of time to optimize the service rate.

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