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Topological Indices of the Relative Coprime Graph of the Dihedral Group

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ABSTRACT

Assuming that G is a finite group and H is a subgroup of G, the graph known as the relative coprime graph of G with respect to H (denoted as $\Gamma_{G,H}$) has vertices corresponding to elements of G. Two distinct vertices x and y are adjacent by an edge if and only if $(|x|,|y|) = 1$ and x or y belongs to H. This paper will focus on finding the general formula for some topological indices of the relative coprime graph of a dihedral group. The study of topological indices in graph theory offers valuable insights into the structural properties of graphs. This study is conducted by reviewing many past literatures and then from there we infer a new result. The obtained outcomes will include measurements of distance, degree of vertex, and various topological indices such as the first Zagreb index, second Zagreb index, Wiener index, and Harary index that are associated with distance and degree of vertex.

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A. INTRODUCTION

Graph theory is a mathematical discipline that concerns the characteristics and traits of graphs, which are visual depictions of entities connected by particular connections or associations. Graph theory has broad implications in multiple areas such as mathematics, computer science, social sciences, network management, transportation, biology, chemistry, and other fields. In chemistry, we utilize graph theory to recognize chemical bonds and analyze the topological indices.

Graph theory is utilized in algebraic structures to illustrate a group or ring as a graph, with its members serving as the vertices. The edges between vertices are determined by the properties of the group or ring. Several studies have utilized graphs as representations of groups, such as commuting graphs, non-commuting graphs, intersection graphs, power graphs, and more. The coprime graph of a group is also an example of a graph representation in algebraic structures. Ma et al. (2014) introduced the coprime graph. Given a finite group G, the coprime graph Γ_G is a graph consisting of vertices corresponding to all elements of G, two distinct vertices are adjacent if their orders are mutually prime. Since then, several studies have examined the properties of coprime graphs, including Hamm & Way (2021) research on the coprime graph's parameters for a group, Dorbidi (2016) research on graph shape, chromatic number, and clique number. Further research has explored the properties of coprime graphs for specific groups such as the integer group modulo Juliana et al. (2020), dihedral groups Syarifudin et al. (2021), and generalized quaternion groups (Nurhabibah et al., 2021). Non-coprime graphs, also known as the complement of coprime graphs, were introduced a few years later by (Mansoori et al., 2016). Since then, several studies have examined non-coprime graphs for various special groups, such as the integer group modulo (Masriani et al., 2020) and the generalized quaternion group (Nurhabibah et al., 2022).

The study of topological indices is an interesting topic that has gained much attention. Numerous studies have focused on various topological indices. For instance, Jahandideh et al. (2015) investigated the topological index of non-commuting graphs of a group, while Alimon et al. (2020) discovered the Szegeb index and Wiener index of the coprime graph of the dihedral group. Zahidah et al. (2021) explored some topological indices of coprime graphs of generalized quaternion groups. Additionally, Husni et al. (2022) discussed the Harmonic index and Gutman index of coprime graphs, as well as the Wiener index, first Zagreb index, and second Zagreb index (Sarmin et al., 2020). Other relevant studies on this topic (Nurhabibah et al., 2021), (Asmarani et al., 2022), and (Syarifudin & Wardhana, 2021).

The author is interested in discussing the coprime relative graph of a group, which was first introduced by Rhani (2018). This graph represents a finite group G and its subgroup H, denoted as $\Gamma_{G,H}$ with vertices representing element of G. Two distinct vertices, x and y, are adjacent if and only if their order $(|x|, |y|) = 1$ and x or y is an element of H. The coprime graph was further developed four years later by Zulkifli & Ali (2021), who investigated various groups for G. The discussion of this graph will focus on the properties of distance and degree of vertices, as well as topological indices such as the first and second Zagreb indices, the Wiener index, and the Harary index.

B. METHODS

The authors conducted a thorough literature review on relative coprime graphs, dihedral groups and their subgroups, as well as various topological indices including the first Zagreb index, second Zagreb index, Wiener index, and Harary index. First, we study many past papers or books regarding the topic of topological indices. The exploration of these indices will deepen our understanding of complex networks in chemistry or biology. We calculate some topological indices of coprime relative graph of a dihedral group and try to search for a pattern for the formula there. The next step is to determine some properties of coprime relative graph of a dihedral group like distance and degree. Applying this result to the definition of the topological indices, we can deduce a general formula for some topological indices of these graph. This formula becomes our first hypothesis and then we try to prove it directly. If the hypothesis was found to be true, it would be established as a theorem. However, if the hypothesis was found to be false, a new hypothesis would be proposed and tested until a valid hypothesis was discovered.

C. RESULT AND DISCUSSION

The following is one of the definitions of a finite group, namely the dihedral group.

Definition 3.1. (Dummit & Foote, 2004)
Group G is said to be a dihedral group of order $2n$, where $n \in N$ and $n \geq 3$, is a group constructed by two elements $a, b \in G$ with the properties

$$G = \langle a, b | a^n = e = b^2, bab^{-1} = a^{-1} \rangle$$

The dihedral group of order $2n$ is denoted by D_{2n} . The dihedral group D_{2n} can be written in set form as follows:

$$D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$$

Definition 3.2. (Rhani, 2018)
Suppose G is a finite group and H is a subgroup of G. The relative coprime graph of G with respect to H, denoted as $\Gamma_{H,G}$ is a graph whose vertices are members of G, and two distinct vertices x and y are adjacent if only if $(|x|, |y|) = 1$ and x or y are in H.
The following is an example of a relative coprime graph of the dihedral group $\Gamma_{H,D_{2n}}$ with various values for n. Let $n = 3^2$. We investigate two different cases for subgroups of $D_{2\cdot 3^2}$ namely $H_1 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$ and $H_2 = \{e, b\}$. Case 3 is about every subgroup of dihedral group $D_{2.2^2}$.
Given $D_2(3^2) = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b\}$ with each order as shown I Table 1.

Table 1. Order of each element of $D_{2.3^2}$

Element	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
Order	1	9	9	3	9	9	3	9	9
Element	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b	a^8b
Order	2	2	2	2	2	2	2	2	2

Case 1 If $n = 3^2$ with $H_1 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$, then the graph is as shown in Figure 1.

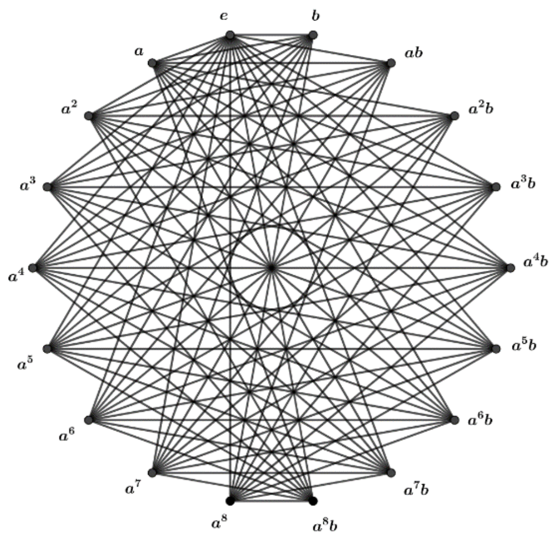

Figure 1. Relative coprime of $D_{2.3^2}$ with H_1

Figure 1 is the form of graph from dihedral group D_{2n} with $n = 3^2$ and H_1 subgroup.

Case 2 If $n = 3^2$ with $H_2 = \{e, b\}$, then the graph is as shown in Figure 2.

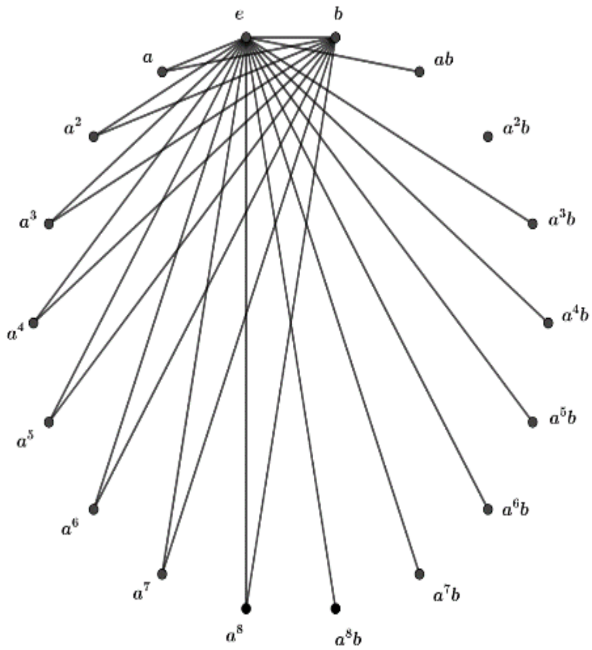

Figure 2. Relative coprime of $D_{2.3^2}$ with H_2

Figure 2 is the form of graph from dihedral group D_{2n} with $n = 3^2$ and H_2 subgroup.
Given $D_{2.2^2} = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ with each order as shown in Table 2.

Table 2. Order of each element of $D_{2.2^2}$

Element	e	a	a^2	a^3	b	ab	a^2b	a^3b
Order	1	4	2	4	2	2	2	2

Case 3. If $n = 2^2$ with H is any subgroup of dihedral group D_{2n}, then the graph is as shown in Figure 3.

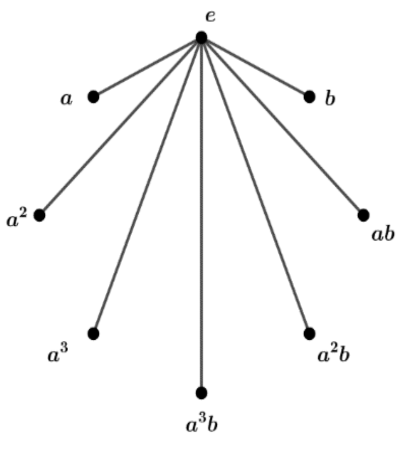

Figure 3. Relative coprime of $D_{2.2^2}$ with H

Figure 3 is the form of graph from dihedral group D_{2n} with $n = 2^2$ and H is any subgroup.

1. Distances ad Degrees of Vertex

First, we talk about the distance. In graph theory, the distance between two vertices is defined as the length of the shortest path between them. The shortest path is the sequence of edges that connects the two vertices, and the length of the path is the sum of the weights of the edges in the path. The relative coprime graph is unweighted, so then the length of the path is simply the number of edges in the path. The distance between two vertices is sometimes referred to as the geodesic distance or the shortest path distance. The distance between vertices can also be used to define other important graph properties, such as diameter, eccentricity, and centrality. The following theorem provides a result on the distance of the relative coprime graph of any dihedral group.

Theorem 3.1. The distance on the relative coprime graph of a dihedral group with any subgroup less than or equal to it is at most 2.

Proof. Note that every subgroup of a dihedral group always contains the element e and notice that $|e| = 1$. As a result, every vertex in the relative coprime graph of a dihedral group with any subgroup is connected to e. Therefore, the distance on this graph is at most 2. ■
The degree of a vertex is defined as the number of edges that are incident to the vertex. That is, the degree of a vertex is equal to the number of edges that are connected to it. The degree of a vertex is a fundamental property of a graph and is used to characterize the graph's structure. The degree of vertices in the relative coprime graph of a dihedral group can be obtained, which is now given in the following three theorems.

Theorem 3.2. Let D_{2n} be a dihedral group with $n = p^k$, where p is an odd prime and $k \in N$, and let H be a subgroup where $H = \langle a \rangle$. Then, the degree of the vertices in $\Gamma_{H,G}$ is given by:

$$\deg(x) = \begin{cases} 2n-1, & x = e \\ n+1, & x \in \{a, a^2, \dots, a^{n-1}\} \\ n, & x \in \{b, ab, a^2b, \dots, a^{n-1}b\} \end{cases}$$

Proof. If $x = e$, then x is adjacent to all other vertices, so $\deg(x) = 2n - 1$. Next, consider the set $\{a, a^2, \dots, a^{n-1}\}$. Since $n = p^k$, each element in this set has an order of p^l with $1 \le l \le k$. Therefore, there are no two adjacent vertices in the set $\{a, a^2, \dots, a^{n-1}\}$. Also, note that $|a^m b| = 2$ for $m = 0, \dots, n-1$, so the vertices $a^m b$ are not adjacent to each other.
Now, let $x \in \{a, a^2, \dots, a^{n-1}\}$ and $y \in \{b, ab, a^2b, \dots, a^{n-1}b\}$. Clearly, these two vertices are adjacent because $(|x|, |y|) = (p^l, 2) = 1$, and $x \in H = \langle a \rangle$. Therefore, if $x \in \{a, a^2, \dots, a^{n-1}\}$, then x is adjacent to $\{e, a, a^2, \dots, a^{n-1}\}$, so $deg\ (x) = n + 1$. If $x \in \{b, ab, a^2b, \dots, a^{n-1}b\}$, then x is adjacent to $\{e, a, a^2, \dots, a^{n-1}\}$, so $\deg(x) = n$. ■

Theorem 3.3. Let D_{2n} be a dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and let H be a subgroup where $H = \{e, a^i b\}$ for some $i = 0,1, \dots, n-1$. Then, the degree of vertex $\Gamma_{H,G}$ is given by:

$$\deg(x) = \begin{cases} 2n-1, & x = e \\ 2, & x \in \{a, a^2, \dots, a^{n-1}\} \\ n, & x = a^i b, i = 0,2, \dots, n-2 \\ 1, & x \in \{b, ab, a^2 b, \dots, a^{n-1} b\} \backslash \{a^i\} \end{cases}$$

Proof. If $x = e$, then it is clear that x is adjacent to all other vertices, so $\deg(x) = 2n - 1$. Next, consider the set $\{a, a^2, \dots, a^{n-1}\}$. Since $n = p^k$, each element in this set will have an order of p^l with $1 \leq l \leq k$. Therefore, there are no two adjacent vertices in the set $\{a, a^2, \dots, a^{n-1}\}$. It is known that $|a^i b| = 2$, so every $x \in \{a, a^2, \dots, a^{n-1}\}$ is adjacent to $a^i b$. Hence, if $x \in \{a, a^2, \dots, a^{n-1}\}$, then $\deg(x) = 2$.
Let $x = a^i b$ for some i. This vertex is not adjacent to $a^j b$ for $j \neq i$ because $(|a^i b|, |a^j b|) = 2$. Thus, this vertex is adjacent to $\{e, a, a^2, \dots, a^{n-1}\}$, and so $deg\ deg\ (x) = n$. Furthermore, if $x \in \{b, ab, a^2 b, \dots, a^{n-1} b\} \setminus \{a^i\}$, then x is only adjacent to $\{e\}$, so $deg\ (x) = 1$.■

Theorem 3.4. Let D_{2n} is a dihedral group with $n = 2^k, k \geq 2$, and H is a subgroup where $H = \{e, a^i b\}$ for some $i = 0,1, \dots, n-1$ or $H = \langle a \rangle$. Then the degree of vertex $\Gamma_{H,G}$ is as follows.

$$\deg(x) = \begin{cases} 2n-1, & x = e \\ 1, & \text{other} \end{cases}$$

Proof. If $x = e$, then it is obvious that x is adjacent to all other vertices, so $deg\ (x) = 2n - 1$. Since $n = 2^k$, for every $i = 1, \dots, n-1$ then $|a^i| = 2^j$ for some $j \in \{1, \dots, k\}$. Thus, if $x \in \{a, a^2, \dots, a^{n-1}\}$ and $y \in \{b, ab, a^2 b, \dots, a^{n-1} b\}$, then $(|x|, |y|) = 2$, so there are no two vertices from these two sets are adjacent. It has been previously shown that if $x, y \in \{a, a^2, \dots, a^{n-1}\}$ then x and y are not adjacent, and similarly for $x, y \in \{b, ab, a^2 b, \dots, a^{n-1} b\}$. Therefore, a vertex x that is not e will only be adjacent to e, so $\deg(x) = 1$. ■

2. Topology Indices

The topological indices discussed in this study are related to the distance and degree of vertices in graphs, namely the first Zagreb index, second Zagreb index, Wiener index, and Harary index. Many of this index are related to each other and have been studied extensively for its use in other field such as chemistry. Now, we will use the result from the previous section to help determine all these topological indices.

a. The First Zagreb Index
   The first Zagreb index is graph invariant that is defined as the sum of the squares of the degrees of all vertices in a given graph. It was introduced by Mihalic and Gutman in 2000 and is a measure of the complexity or topological richness of a graph. Formally, it can be defined as follows.

Definition 3.5. (Sarmin et al., 2020)
Let a connected graph G, $t$he first Zagreb index of G is the sum of squares of the degree of each vertex in G, written as:

$$M_1(G) = \sum_{x \in G} (\deg(x))^2$$

The following are some theorems regarding the first Zagreb index based on each case.

Theorem 3.5. Let D_{2n} is a dihedral group with $n = p^k$, p is an odd prime number and $k \in N$ and H is the subgroup where $H = \langle a \rangle$. Then, the first Zagreb index $\Gamma_{H,D_{2n}}$ is:

$$M_1(\Gamma_{H,D_{2n}}) = 2(p^{3k}) + 5(p^{2k}) - 5(p^k)$$

Proof. Based on **Theorem 3.2,** then obtained

$$\sum_{x \in V(\Gamma_{H,D_{2n}})} (\deg(x))^2 = (\deg(e))^2 + \sum_{x \in \{a,a^2,\dots,a^{n-1}\}} (\deg(x))^2 + \sum_{x \in \{b,ab,\dots,a^{n-1}b\}} (\deg(x))^2$$
$$= (2n-1)^2 + \textstyle\sum_{x \in \{a,a^2,\dots,a^{n-1}\}} (n+1)^2 + \sum_{x \in \{b,ab,\dots,a^{n-1}b\}} (n)^2$$
$$= (2n-1)^2 + (n-1)(n+1)^2 + n(n)^2$$
$$= (4n^2 - 4n + 1) + (n-1)(n^2 + 2n + 1) + n^3$$
$$= 2n^3 + 5n^2 - 5n$$
$$= 2(p^{3k}) + 5(p^{2k}) - 5(p^k) \blacksquare$$

Theorem 3.6. Let $\Gamma_{H,D_{2n}}$ denotes the graph associated with the subgroup H of the dihedral group D_{2n}, where $n = p^k$ for some odd prime p and $k \in N$, and $H = \{e, a^i b\}$ for every $i = 0,1, \dots, n-1$. Then the first Zagreb index of $\Gamma_{H,D_{2n}}$ is given by

$$M_1(\Gamma_{H,D_{2n}}) = 5(p^{2k}) + p^k - 4$$

Proof. Based on **Theorem 3.3**, then obtained

$$\textstyle\sum_{x \in V(\Gamma_{H,D_{2n}})} (\deg(x))^2 = (\deg(e))^2 + \sum_{x \in \{a,a^2,\dots,a^{n-1}\}} (\deg(x))^2 + \sum_{x = a^i} (\deg(x))^2 +$$
$$\textstyle\sum_{x \in \{b,ab,\dots,a^{n-1}b\} \backslash \{a^i\}} (\deg(x))^2$$
$$= (2n-1)^2 + \textstyle\sum_{x \in \{a,a^2,\dots,a^{n-1}\}} (2)^2 + (n)^2 + \sum_{x \in \{b,ab,\dots,a^{n-1}b\}} (1)^2$$
$$= (2n-1)^2 + (n-1)2^2 + (n)^2 + (n-1)(1)^2$$
$$= (4n^2 - 4n + 1) + (4n - 4) + n^2 + (n-1)$$
$$= 5n^2 + n - 4 \blacksquare$$

Theorem 3.7 Let D_{2n} be a dihedral group with $n = 2^k$, p an odd prime number, and $k \geq 2$, and let H be a subgroup where $H = \{e, a^i b\}$ for each $i = 0,1, \dots, n-1$ and $H = \langle a \rangle$. Then the first Zagreb index $\Gamma_{H,D_{2n}}$ is given by:

$$M_1(\Gamma_{H,D_{2n}}) = 2^{2(k+1)} - 2^{k+1}$$

Proof. Based on **Theorem 3.4,** then obtained

$$
\begin{aligned}
\textstyle\sum_{x\in V(\Gamma_{H,D_{2n}})}(\deg(x))^2 &= (\deg(e))^2 + \textstyle\sum_{x\in V(\Gamma_{H,D_{2n}})\setminus\{e\}}(\deg(x))^2 \\
&= (2n-1)^2 + (2n-1)(1)^2 \\
&= 2n(2n-1) \\
&= 4n^2 - 2n \\
&= 4(2^{2k}) - 2(2^k) \\
&= 2^{2(k+1)} - 2^{k+1} \blacksquare
\end{aligned}
$$

b. The Second Zagreb Index
The second Zagreb index is a graph invariant that is defined as the sum of the products of the degrees of all pairs of adjacent vertices in the given graph. The second Zagreb index was also introduced by Mihalic and Gutman in 2000 and is used to quantify the degree of branching and connectivity in a graph. Formally, it can be defined as follows.

Definition 3.6. (Sarmin et al., 2020)
Let a connected graph G, the second Zagreb index of G is the sum of the product of degrees of each pair of adjacent vertices in G, and it can be written as follows.

$$M_2(G) = \sum_{(x,y)\in E(G)} \deg(x)\deg(y)$$

Below are several theorems about second Zagreb Index based on each case.

Theorem 3.8. Let D_{2n} be the dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and let H be a subgroup where $H = \langle a \rangle$. Then the second Zagreb index $\Gamma_{H,D_{2n}}$ is given by:

$$M_2\left(\Gamma_{H,D_{2n}}\right) = p^{4k} + 4p^{3k} - 3p^{2k} - 2p^k + 1$$

Proof. Based on **Theorem 3.2,** then obtained

$$
\begin{aligned}
\textstyle\sum_{(x,y)\in E(\Gamma_{H,D_{2n}})} \deg(x)\deg(y) &= \textstyle\sum_{y\in\{a,a^2,\dots,a^{n-1}\}} \deg(e)\deg(y) + \\
&\quad \textstyle\sum_{y\in\{b,ab,\dots,a^{n-1}b\}} \deg(e)\deg(y) + \\
&\quad \sum_{\substack{x\in\{a,a^2,\dots,a^{n-1}\} \\ y\in\{b,ab,\dots,a^{n-1}b\}}} \deg(x)\deg(y) \\
&= (n-1)(2n-1)(n+1) + (n)(2n-1)(n) + \\
&\quad (n-1)(n)(n+1) \\
&= (n^2-1)(2n-1) + n^2(2n-1) + n^2(n^2-1) \\
&= n^4 + 4n^3 - 3n^2 - 2n + 1 \\
&= p^{4k} + 4p^{3k} - 3p^{2k} - 2p^k + 1 \blacksquare
\end{aligned}
$$

Theorem 3.9. Let D_{2n} be the dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and let H be a subgroup where $H = \{e, a^i b\}$ for every $i = 0,1, \dots, n-1$. Then, the second Zagreb index $\Gamma_{H,D_{2n}}$ is given by:

$$M_2(\Gamma_{H,D_{2n}}) = 10p^{2k} - 12p^k + 3$$

Proof. Based on **Theorem 3.3,** then obtained

$$\begin{aligned}\sum_{(x,y)\in E(\Gamma_{H,D_{2n}})} \deg(x)\deg(y) &= \sum_{y\in\{a,a^2,\dots,a^{n-1}\}} \deg(e)\deg(y) + \\ &\quad \sum_{y\in\{b,ab,\dots,a^{n-1}b\}\setminus\{a^ib\}} \deg(e)\deg(y) + \deg(e)\deg(a^ib) \\ &\quad + \sum_{y\in\{a,a^2,\dots,a^{n-1}\}} \deg(a^ib)\deg(y) \\ &= (n-1)(2n-1)(2) + (n-1)(2n-1)(1) + \\ &\quad (2n-1)(n) + (n-1)(n)(2) \\ &= (2n^2-3n+1)(2) + (2n^2-3n+1) + (2n^2-n) + \\ &\quad (2n^2-2n) \\ &= 10n^2 - 12n + 3 \\ &= 10p^{2k} - 12p^k + 3 \ \blacksquare\end{aligned}$$

Theorem 3.10. Let D_{2n} be a dihedral group with $n = 2^k$, p an odd prime number, and $k \geq 2$, and let H be a subgroup where $H = \{e, a^i b\}$ for every $i = 0,1, \dots, n-1$ and $H = \langle a \rangle$. Then, the second Zagreb index order $\Gamma_{H,D_{2n}}$ is given by:

$$M_2(\Gamma_{H,D_{2n}}) = 2^{2k+2} - 2^{k+2} + 1$$

Proof. Based on **Theorem 3.4**, then obtained

$$\begin{aligned}\sum_{(x,y)\in E(\Gamma_{H,D_{2n}})} \deg(x)\deg(y) &= \sum_{y\in V(\Gamma_{H,D_{2n}})\setminus\{e\}} \deg(e)\deg(y) \\ &= (2n-1)(2n-1)(1) \\ &= 4n^2 - 4n + 1 \\ &= 4(2^k)^2 - 4(2^k) + 1 \\ &= 2^{2k+2} - 2^{k+2} + 1\end{aligned}$$

c. The Wiener Index

The Wiener index is a graph invariant that is defined as the sum of the distances between all pairs of vertices in a given graph. It is used to quantify the complexity or topological richness of a graph. More formally, it can be defined as follows.

Definition 3.7. (Alimon et al., 2020)
Let a connected graph G, the Wiener index of G is the sum of the unordered pairs of vertices in G, written as

$$W(G) = \sum_{\{x,y\} \in V(G)} d(x,y)$$

The following are some theorems about the Wiener index based on each case.

Theorem 3.11. Let D_{2n} be a dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and let $H = \langle a \rangle$ be a subgroup. Then, the Wiener index $\Gamma_{H,D_{2n}}$ is

$$W(\Gamma_{H,D_{2n}}) = 3p^{2k} - 2p^k + 1$$

Proof. Based on Theorem 3.1 then obtained

$$\begin{aligned}
\textstyle\sum_{x,y \in V(\Gamma_{H.D_{2n}})} d(x,y) &= \textstyle\sum_{y \in V(\Gamma_{H,D_{2n}}) \setminus \{e\}} d(e,y) + \sum_{x,y \in \{a,a^2,\ldots,a^{n-1}\}} d(x,y) + \\
&\quad \textstyle\sum_{x,y \in \{b,ab,\ldots,a^{n-1}b\}} d(x,y) + \sum_{\substack{x \in \{a,a^2,\ldots,a^{n-1}\} \\ y \in \{b,ab,\ldots,a^{n-1}b\}}} d(x,y) \\
&= (2n-1)(1) + \binom{n-1}{2}(2) + \binom{n}{2}(2) + n(n-1)(1) \\
&= 2n - 1 + (n^2 - 2n + 2) + 2(n^2 - n) \\
&= 3n^2 - 2n + 1 \\
&= 3p^{2k} - 2p^k + 1 \blacksquare
\end{aligned}$$

Theorem 3.12. Let D_{2n} be a dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and let H be a subgroup such that $H = \{e, a^i b\}$ for every $i = 0,1,\ldots,n-1$. Then, the Wiener index $\Gamma_{H,D_{2n}}$ is

$$W(\Gamma_{H,D_{2n}}) = 4p^{2k} - 4p^k + 2$$

Proof. Based on **Theorem 3.1,** then obtained

$$\begin{aligned}
\textstyle\sum_{x,y \in V(\Gamma_{H,D_{2n}})} d(x,y) &= \textstyle\sum_{y \in V(\Gamma_{H,D_{2n}}) \setminus \{e\}} d(e,y) + \sum_{x,y \in \{a,a^2,\ldots,a^{n-1}\}} d(x,y) + \\
&\quad \textstyle\sum_{x,y \in \{b,ab,\ldots,a^{n-1}b\}} d(x,y) + \sum_{x \in \{a,a^2,\ldots,a^{n-1}\}} d(x, a^i b) + \\
&\quad \textstyle\sum_{\substack{x \in \{a,a^2,\ldots,a^{n-1}\} \\ y \in \{b,ab,\ldots,a^{n-1}b\} \setminus \{a^i b\}}} d(x,y) \\
&= (2n-1)(1) + \binom{n-1}{2}(2) + \binom{n}{2}(2) + (n-1)(1) + (n-1)(n-1)(2) \\
&= 2n - 1 + (n^2 - 2n + 2) + (n^2 - n) + (n-1) + 2(n^2 - 2n + 1) \\
&= 4n^2 - 4n + 2 \\
&= 4p^{2k} - 4p^k + 2 \blacksquare
\end{aligned}$$

Theorem 3.12. Let D_{2n} be a dihedral group with $n = 2^k$, where $k \geq 2$ and $k \in N$ and let H be a subgroup. Then, the Wiener index is

$$W(\Gamma_{H,D_{2n}}) = 2^{2k+2} - 2^{k+1} + 1$$

Proof. Based on Theorem 3.1 then obtained

$$\begin{aligned}\textstyle\sum_{x,y\in V(\Gamma_{H,D_{2n}})} d(x,y) &= \textstyle\sum_{y\in V(\Gamma_{H,D_{2n}})\backslash\{e\}} d(e,y) + \sum_{x,y\in V(\Gamma_{H,D_{2n}})\backslash\{e\}} d(x,y)\\ &= (2n-1)(1) + \binom{2n-1}{2}(2)\\ &= 2n-1+(4n^2-6n+2)\\ &= 4n^2-4n+1\\ &= 4(2^k)^2-4(2^k)+1\\ &= 2^{2k+2}-2^{k+1}+1 \blacksquare\end{aligned}$$

d. The Harary Index

The Harary index is a graph invariant that is defined as the sum of the reciprocals of the distances between all pairs of vertices in a given graph. The Harary index is named after Frank Harary, who introduced it in 1976. It is a measure of the connectivity or accessibility of a graph, Formally, it is defined as follows.

Definition 3.8. (Zahidah et al., 2021)
Let a connected graph G, the Harary index of G is defined as:

$$H(G) = \sum_{x,y\in V(G)} \frac{1}{d(x,y)}$$

Where the sum is taken over all pairs of vertices x, y in G. Here are some theorems on the Harary index based on each case.

Teorema 3.14. Let D_{2n} is the dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and H is a subgroup where $H = \langle a\rangle$, then the Harary index $\Gamma_{H,D_{2n}}$ is

$$H(\Gamma_{H,D_{2n}}) = \frac{3}{2}p^{2k} - \frac{1}{2}$$

Proof. Based on Theorem 3.1 then obtained

$$\begin{aligned}\textstyle\sum_{x,y\in V(\Gamma_{H,D_{2n}})} \frac{1}{d(x,y)} &= \textstyle\sum_{y\in V(\Gamma_{H,D_{2n}})\backslash\{e\}} \frac{1}{d(e,y)} + \sum_{x,y\in\{a,a^2,\dots,a^{n-1}\}} \frac{1}{d(x,y)} +\\ &\quad \textstyle\sum_{x,y\in\{b,ab,\dots,a^{n-1}b\}} \frac{1}{d(x,y)} + \sum_{\substack{x\in\{a,a^2,\dots,a^{n-1}\}\\ y\in\{b,ab,\dots,a^{n-1}b\}}} \frac{1}{d(x,y)}\\ &= (2n-1)(1) + \frac{1}{2}\binom{n-1}{2} + \frac{1}{2}\binom{n}{2} + n(n-1)(1)\\ &= n^2+n-1+\frac{1}{4}(n^2-3n+2)+\frac{1}{4}(n^2-n)\\ &= \frac{3}{2}n^2-\frac{1}{2}\\ &= \frac{3}{2}(p^k)^2-\frac{1}{2}\\ &= \frac{3}{2}p^{2k}-\frac{1}{2} \blacksquare\end{aligned}$$

Theorem 3.15. Let D_{2n} is the dihedral group with $n = p^k$, where p is an odd prime number and $k \in N$, and H is a subgroup where $H = \{e, a^i b\}$ for each $i = 0,1,\ldots,n-1$, then the Harary index $\Gamma_{H,D_{2n}}$ is:

$$H(\Gamma_{H,D_{2n}}) = p^{2k} + p^k - 1$$

Proof. Based on Theorem 3.1 then obtained

$$\begin{aligned}
\sum_{x,y\in V(\Gamma_{H,D_{2n}})} \frac{1}{d(x,y)} &= \sum_{y\in V(\Gamma_{H,D_{2n}})\backslash\{e\}} \frac{1}{d(e,y)} + \sum_{x,y\in\{a,a^2,\ldots,a^{n-1}\}} \frac{1}{d(x,y)} + \\
&\quad \sum_{x,y\in\{b,ab,\ldots,a^{n-1}b\}} \frac{1}{d(x,y)} + \sum_{x\in\{a,a^2,\ldots,a^{n-1}\}} \frac{1}{d(x,a^i b)} + \\
&\quad \sum_{\substack{x\in\{a,a^2,\ldots,a^{n-1}\} \\ y\in\{b,ab,\ldots,a^{n-1}b\}\backslash\{a^i b\}}} \frac{1}{d(x,y)} \\
&= (2n-1)(1) + \binom{n-1}{2}\frac{1}{2} + \binom{n}{2}\frac{1}{2} + (n-1)(1) + (n-1)(n-1)\frac{1}{2} \\
&= \frac{1}{2}n^2 + 2n - \frac{3}{2} + \frac{1}{4}(n^2 - 3n + 2) + \frac{1}{4}(n^2 - n) \\
&= n^2 + n - 1 \\
&= (p^k)^2 + p^k - 1 \\
&= p^{2k} + p^k - 1 \ \blacksquare
\end{aligned}$$

Theorem 3.16. Let D_{2n} is the dihedral group with $n = 2^k$, where $k \geq 2$ and $k \in N$, and H is a subgroup, then the Harary index $\Gamma_{H,D_{2n}}$ is:

$$H(\Gamma_{H,D_{2n}}) = 2^{2k} + 2^{k-1} - \frac{1}{2}$$

Proof. Based on **Theorem 3.1**, then obtained

$$\begin{aligned}
\sum_{x,y\in V(\Gamma_{H,D_{2n}})} \frac{1}{d(x,y)} &= \sum_{y\in V(\Gamma_{H,D_{2n}})\backslash\{e\}} \frac{1}{d(e,y)} + \sum_{x,y\in V(\Gamma_{H,D_{2n}})\backslash\{e\}} \frac{1}{d(x,y)} \\
&= (2n-1)(1) + \binom{2n-1}{2}\frac{1}{2} \\
&= 2n - 1 + \frac{1}{4}(4n^2 - 6n + 2) \\
&= n^2 + \frac{1}{2}n - \frac{1}{2} \\
&= (2^k)^2 + \frac{1}{2}(2^k) - \frac{1}{2} \\
&= 2^{2k} + 2^{k-1} - \frac{1}{2} \ \blacksquare
\end{aligned}$$

D. CONCLUSION AND SUGGESTIONS

The study of topological indices in graphs is an important area of research with wide-ranging applications in various fields. we have investigated and determined several important topological indices of a given graph, including the degree and distance of each vertex, as well as the first and second Zagreb indices, the Wiener index, and the Harary index. These topological indices provide valuable information about the connectivity, complexity, and accessibility of the graph, and they can be used to study a variety of properties and phenomena in different fields.

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