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Construction of Ordinal Numbers and Arithmetic of Ordinal Numbers

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ABSTRACT

The purpose of this paper is to introduce the idea of how to construct transfinite numbers and study transfinite arithmetic. The research method used is a literature review, which involves collecting various sources such as scientific papers and books related to Cantorian set theory, infinity, ordinal or transfinite arithmetic, as well as the connection between infinity and theology. The study also involves constructing the objects of study, namely ordinal numbers such as finite ordinals and transfinite ordinals, and examining their arithmetic properties. The results of this research include the methods of constructing both finite and transfinite ordinal numbers using two generation principles. Both finite and transfinite ordinal numbers are defined as well-orderings that are also transitive sets. Arithmetic of finite ordinal numbers is well-known, but the arithmetic of transfinite ordinal numbers will be introduced in this paper, including addition, multiplication, and exponentiation.

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A. INTRODUCTION

Cantor's mathematical work was not focused on developing set theory but on developing the notion of infinity (Sambin, 2019). The topic of infinity, before Cantor elaborated it in detail in his mathematical model, was considered controversial by philosophers, theologians, and mathematicians. According to Dauben (1979), as a philosopher, Aristotle rejected the idea of actual infinity because it could lead to contradictions. Aristotle illustrated the understanding of the nature of positive numbers, namely, if two positive numbers are given, say a and b, then the property of $a < a + b$ and $b < a + b$. Aristotle rejected infinity because it could lead to a contradiction; if suppose b is an infinite number denoted by $b = \infty$, then the property $b = \infty = a + \infty = a + b$ which contradicts the nature of $b < a + b$. Some Christian theologians also reject true infinity because the idea of true infinity challenges the Godhead and the absolute infinity of God. According to Thomas Aquinas, the only true infinity is God (Roszak & Huzarek, 2019), so there is no true infinity other than God. Mathematicians followed the philosophers, and one of the mathematicians who rejected the idea of true infinity was Cantor's teacher Leopold Kronecker.

In developing the concept of true infinity, Cantor classified three types of collections: finite sets, transfinite infinity, and absolute infinity (Zermelo, 2012). Finite sets are understood as sets that have a finite cardinality or a finite number of elements (Engler, 2023; Read & Greiffenhagen, 2023). Transfinite infinity, on the other hand, is a type of set that is also a true infinity and its cardinality can be expressed as Aleph (the first letter of the Hebrew alphabet denoted by $\aleph$) (Jourdain, 1910). For example, the set of natural numbers in Cantorian set theory tradition is an example of transfinite infinity and is denoted by $\omega = \{0,1,2, \dots\}$, with a cardinality of $|\omega| = \aleph_0$ (read as Aleph null). Absolute infinity, on the other hand, is understood as a collection that has an infinite number of elements, but its cardinality cannot be expressed by any Aleph. An example of absolute infinity is the proper class of all sets, denoted by S. Here, the notion of a proper class in the context of set theory refers to a collection that has never been an element of any other class. Additionally, a non-proper class or set is a collection that becomes an element of a certain class.

The notion of true infinity extends beyond mathematical studies and enters the realm of metaphysics and theology. St. Augustine stated that finite numbers exist in the mind of God Augustine (2015), and naturally, the collection of all finite numbers, such as the set of natural numbers, also exists in the mind of God (Der Veen & Horsten, 2013). The metaphysical notion of infinity is critically examined in Drozdek (1995). Cantor also expanded upon St. Augustine's metaphysical notions by asserting that transfinite infinity and absolute infinity also exist in the mind of God. Mathematically, since the cardinality of the proper class S cannot be expressed by any Aleph, it means that the cardinality of the proper class S represents an absolute infinity that is inherently indeterminable mathematically, as Cantor expressed Dauben (1979), as follows: *"The absolute can only be acknowledged, but never known, not even approximately known".*

Transfinite infinity also exists in the mind of God, as affirmed by Cantor in a letter he wrote to Jeiler in 1895 (Thomas-Bolduc, 2016), stated as follows: "*... there are transfinite cardinal numbers and transfinite ordinal numbers, which possess a mathematical regularity as definite and as humanly research-able as the finite numbers and forms. All these particular modes of the transfinite exist from eternity as ideas in the divine intellect*" (Thomas-Bolduc, 2016).

So, it is clear that Cantor has two notions of true infinity, namely transfinite infinity and absolute infinity, which ontologically exist in the mind of God. Both notions of true infinity also have ontological differences, as stated in Drozdek (1995), where transfinite infinity is found in the real world and the human mind, while absolute infinity is only expressed in God. Considering that transfinite infinity can be found in the real world and in the human mind, it confirms that transfinite infinity not only exists in the mind of God but also exists outside the mind of God, namely in the real world and in the human mind. When transfinite infinity exists in the human mind, it is treated like natural numbers, inheriting arithmetic properties. According to Oppy (2014), Cantor developed the arithmetic of transfinite infinity, also known as transfinite arithmetic, with theological motivations. Unfortunately, the theological motivations behind transfinite arithmetic are not well-known among mathematicians, so mathematicians have mostly focused on the arithmetic aspects of transfinite infinity. As an introduction to further research on the metaphysical and theological concepts of transfinite infinity, the purpose of this paper is to examine how to construct transfinite numbers and study transfinite arithmetic. Some studies related to the metaphysical and theological concepts of transfinite infinity used to understand the infinity of God can be found in (Russell, 2011), and further research on transfinite arithmetic used to understand the infinity of God can be found in (Grim, 1991). The ideas regarding the theological implications of transfinite arithmetic are an ongoing area of research, as explained in Oppy (2014).

B. METHODS

The method employed in this research is a literature review. The steps involved in this method are as follows: First, gathering various sources such as scholarly articles and relevant books on Cantor's transfinite set theory, infinity, ordinal or transfinite arithmetic, and their relationship with theology. Second, conducting the construction of the objects under investigation, namely ordinal numbers such as finite ordinals and transfinite ordinals, while examining their arithmetic properties. Subsequently, reviewing and analyzing theories related to the construction of ordinal numbers, constructing a conceptual framework for such construction, and applying the construction method to build ordinal numbers. These steps encompass understanding Cantor's theory, transfinite sets, and the concept of infinity in the construction of ordinal numbers, as well as the application of arithmetic rules to ordinal numbers. Finally, summarizing the results of the construction of ordinal numbers and their arithmetic analysis, evaluating the success of the construction method employed, and presenting the main findings of this research. These steps, which can also be visualized in Figure 1 (adapted from Williams, J. K. (2018)), will provide a deeper understanding of the construction of ordinal numbers and their arithmetic, ultimately making a significant contribution to this field, as shown in Figure 1.

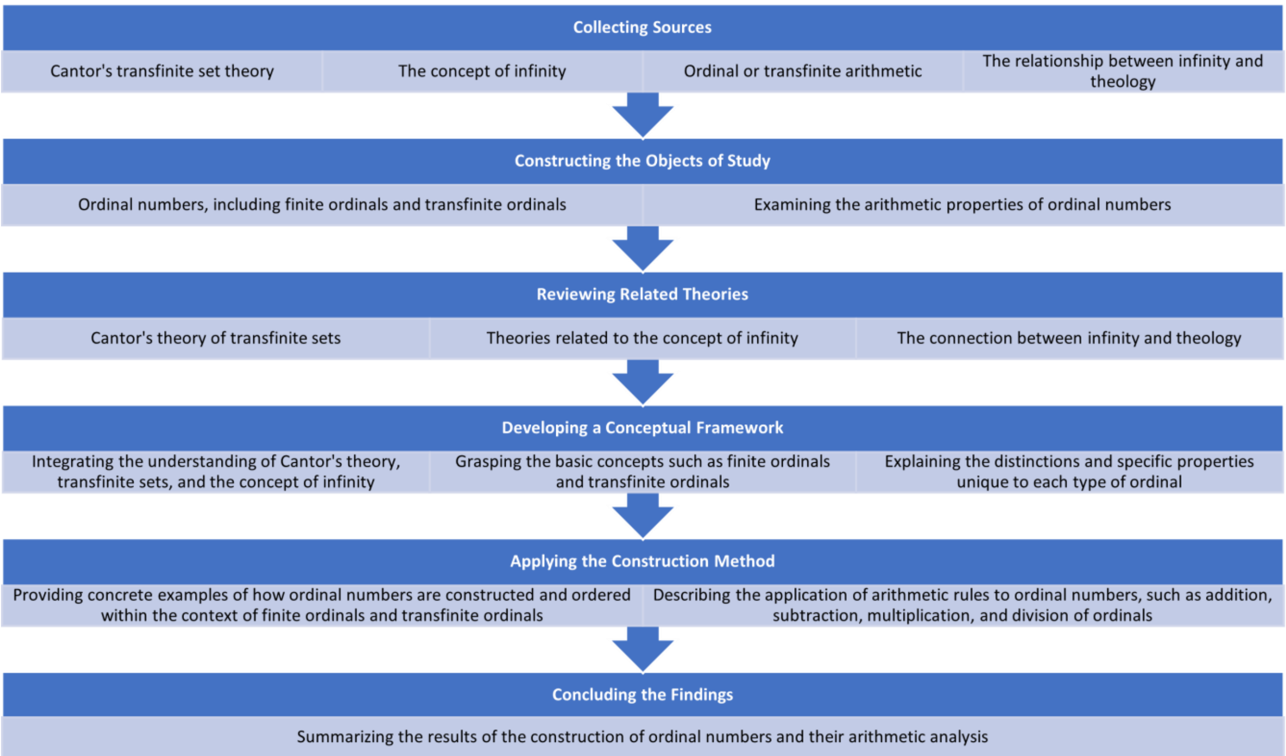

Figure 1. Steps of Literature Review Research

C. RESULT AND DISCUSSION

This section will discuss how to construct a number that extends the natural numbers, let's call it an ordinal number (Goldrei, 1996). The natural numbers themselves are also ordinal numbers, and let's refer to them as finite ordinal numbers or finite ordinals. The extension of finite ordinal numbers into infinite ordinal numbers will be referred to as infinite ordinal numbers or transfinit ordinals (Dasgupta, 2014). The construction of these ordinal numbers, both finite and transfinit, is based on a principle introduced by Cantor called the principle of generation (Sheppard, 2014).

Generation Principle 1. For every ordinal number α there is an ordinal number β whose properties are as follows:

(i) $\alpha < \beta$.
(ii) If given every ordinal number γ whose nature is $\gamma < \beta$ then $\gamma < \alpha$.

Generation Principle 2. For every set X of ordinal numbers, there is an ordinal number α whose properties are as follows:

(i) $\forall x \in X, x < \alpha$.
(ii) There is no ordinal number β such that $\beta < \alpha$ and $\forall x \in X, x < \beta$.

With these two principles of generation, we will proceed to construct ordinal numbers, both finite ordinals and transfinit ordinals. To construct these ordinal numbers, a formal definition of what an ordinal number is will be provided. According to Jech, (2006), ordinal numbers are defined through two concepts: well-ordering sets and transitive sets.

Definition 1.1. The relation $\preccurlyeq$ on set X is said to be ordered if it satisfies the following properties:

(i) $\forall x \in X, x \not\preccurlyeq x$ (anti- reflexive).
(ii) $\forall x, y, z \in X$ if $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$ (transitive).

Then the pair $(X, \preccurlyeq)$ is said to be an ordered set.
From the concept of ordered sets in Definition 1.1, several definitions will be provided regarding the greatest element, smallest element, least upper bound and supremum, as well as the greatest lower bound and infimum (Jech, 2006). Given an ordered set $(X, \preccurlyeq)$, element $a \in X$ is said to be the largest (smallest) element X if $\forall x \in X, x \preccurlyeq a$ ($\forall x \in X, a \preccurlyeq x$), element a is said to be the upper bound (lower bound) of X if $\forall x \in X, x \preccurlyeq a$ ($\forall x \in X, a \preccurlyeq x$), then element a is said to be X supremum (infimum) if a is the smallest upper bound (largest lower bound). Next, the notions of well-ordered set, transitive set, and ordinal number will be defined.

Definition 1.2.

(i) An ordered set $(X, \preccurlyeq)$ it is considered a well-ordered set if every subset of X has the smallest element and $\forall x, y \in X, x \preccurlyeq y$ or $x = y$ or $x \succcurlyeq y$.
(ii) The set Y is said to be a transitive set if every element of Y is a subset of Y.

(iii) An ordinal number is a well-ordered set concerning a certain ordered relation and a transitive set.

To construct ordinal numbers, the first step is to define the concept of a *successor* of a set. A *successor* of a set X is a new set denoted by $X^+ = X \bigcup \{X\}$. Then, based on the concept of *successors*, the smallest finite ordinal number is defined as $0 \stackrel{\text{def}}{=} \emptyset$. Furthermore, ordinal numbers such as $1, 2, 3, \dots, n, \dots$ are defined through the concept of successors as follows:

$$1 = 0^+ = \{\emptyset\} = \{0\}$$
$$2 = 1^+ = \big\{\emptyset, \{\emptyset\}\big\} = \{0,1\}$$
$$3 = 2^+ = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\} = \{0,1,2\}$$
$$\vdots$$
$$n + 1 = n^+ = \{0,1,2, \dots, n\}$$
$$\vdots$$

This can be seen that numbers like $0,1,2,3, \dots, n, n + 1, \dots$ form a set. In ordinal numbers, denote it as n, a well-ordered relation on it is defined as $x < y \stackrel{\text{def}}{=} x \in y$. It is clear that $(n, <)$ forms a *well-ordering* set, and the number n is also a transitive set. As a result, the number n is called an ordinal number, specifically a finite ordinal (because n is a finite set). The process of forming finite ordinal numbers is an implementation of Generation Principle 1 applied to finite ordinal numbers. Then, referring to Generation Principle 2, a new ordinal number is obtained as follows: $\omega = \{0, 1, 2, \dots\}$. Because $\omega = \{0, 1, 2, \dots\}$ is an ordinal number formed through Generation Principle 2 and is also an infinite set, this ordinal number is called a transfinite ordinal number (first). Generation Principle 1 can also be applied to ω, resulting in $\omega + 1 = \omega^+ = \omega \bigcup \{\omega\} = \{0,1,2, \dots, \omega\}$. This can be done until obtaining the following transfinite ordinal numbers:

$$\omega + 1 = \omega^+ = \omega \bigcup \{\omega\} = \{0,1,2, \dots, \omega\}$$
$$\omega + 2 = (\omega + 1)^+ = \{0,1,2, \dots, \omega, \omega + 1\}$$
$$\vdots$$
$$\omega + n = \big(\omega + (n - 1)\big)^+ = \{0,1,2, \dots, \omega, \omega + (n - 1)\}$$
$$\vdots$$

Using Generation Principle 2, a new transfinite ordinal number is obtained, namely $. 2 = \{\omega, \omega + 1, \omega + 2, \dots\}$. This process continues indefinitely, resulting in a sequence of ordinal numbers consisting of all finite ordinal numbers and transfinite ordinal numbers, as follows:

$$0,1,2, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega.2, \omega.2 + 1, \omega.2 + 2, \dots, \omega^{\omega}, \dots, \omega^{\omega^{\cdots}}, \dots$$

It is clear that for any given finite ordinal number or transfinite ordinal number, let's say α, the following holds: $\alpha = \{\beta | \beta < \alpha\} = \{\beta | \beta \in \alpha\}$. This naturally forms a well-ordering set under the ordering relation $<$. From this concept, two types of ordinal numbers are obtained, as follows:

(i) Finite ordinal numbers as well as transfinite ordinal numbers, denoted as α, are called *successor* ordinals if there exists an ordinal number β such that $\alpha = \beta^+$
(ii) Finite ordinal numbers as well as transfinite ordinal numbers, denoted as α, are called *limit* ordinals if $\alpha = \sup\{\beta | \beta < \alpha\}$.

Let $\mathcal{O} = \{\text{all ordinal numbers (both finite and transfinite)}\}$. Is $\mathcal{O}$ a set? Before determining whether $\mathcal{O}$ is a set or not, let's first check if $\mathcal{O}$ is transitive. Clearly, $\forall \alpha \in \mathcal{O}$, α is an ordinal number. Since α is an ordinal number (either finite or transfinite), it is evident that all elements of α are ordinal numbers, and therefore $\alpha \subset \mathcal{O}$. Thus, $\mathcal{O}$ is transitive.
Furthermore, it is clear that $\mathcal{O}$ is a *well-ordering* with respect to the relation $\preccurlyeq$ defined on $\mathcal{O}$ as $\alpha \preccurlyeq \beta \overset{\text{def}}{\equiv} \alpha \in \beta$ for every $\alpha, \beta \in \mathcal{O}$. Therefore, it is established that $\mathcal{O}$ is both transitive and a well-ordering.
Now, let's assume that $\mathcal{O}$ is a set. Since $\mathcal{O}$ is transitive and a well-ordering, it follows that $\mathcal{O}$ is an ordinal number. As $\mathcal{O}$ is the collection of all ordinal numbers and $\mathcal{O}$ itself is an ordinal number, it is evident that $\mathcal{O} \in \mathcal{O}$; in other words, $\mathcal{O} \preccurlyeq \mathcal{O}$. This contradicts $\preccurlyeq$ being an anti-reflexive relation. Therefore, $\mathcal{O}$ is not a set; in other words, $\mathcal{O}$= {all ordinal numbers (both finite and transfinite)} is a proper class (a proper class, in the terminology of Cantorian set theory, is usually referred to as an inconsistent multiplicity; the totality of all its elements leads to a contradiction).
By utilizing the fact that $\mathcal{O}$ is a proper class, a condition will be provided for a subclass of $\mathcal{O}$, let's say $\mathcal{C}$, to become $\mathcal{C} = \mathcal{O}$ using a technique known as transfinite induction. The role of transfinite induction will be encountered in proving the properties of transfinite arithmetic.

Transfinite Induction. If $\mathcal{C}$ is a subclass of $\mathcal{O}$ that satisfies the following property:
(i) $0 \in \mathcal{C}$.
(ii) If $\alpha \in \mathcal{C}$ then $\alpha^+ \in \mathcal{C}$.
(iii) If α is an ordinal limit and $\forall \beta < \alpha, \beta \in \mathcal{C}$, then $\alpha \in \mathcal{C}$.
then $\mathcal{C} = \mathcal{O}$.

The study of the fundamentals of arithmetic usually involves the calculation of numbers subjected to various operations such as addition, multiplication, and exponentiation. In this section, we will examine how to perform calculations with ordinal numbers, particularly transfinite ordinals, using operations such as addition, multiplication, and exponentiation, as a starting point for further exploration of ordinal arithmetic (Clark, 2017; Lobo & Cardoso, 2020).

Definition 3.1. Given $\alpha, \beta \in \mathcal{O}$, the ordinal sum $\alpha + \beta$ is defined as follows:
(i) $\alpha + 0 = \alpha$.
(ii) If β is the ordinal successor ($\beta = \gamma^+$ for an ordinal number γ), then $\alpha + \beta = (\alpha + \gamma)^+$.
(iii) If β is the ordinal limit ($\beta = \sup\{\gamma | \gamma \in \beta\}$), then $\alpha + \beta = \sup\{\alpha + \gamma | \gamma \in \beta\}$.

Example 3.2. Suppose two ordinal numbers $\alpha = 2$ and $\beta = \omega + 1$ are given. It is clear that β is the successor ordinal number ($\beta = \omega + 1 = \omega^+$). By Definition 3.1 (ii), it is clear that $2 + (\omega + 1) = \alpha + \beta = (\alpha + \gamma)^+ = (2 + \omega)^+$, then what is $(2 + \omega)^+$?. Since ω is the ordinal limit

$(\omega = \sup\{\gamma|\gamma \in \omega\}$, it is clear that $2 + \omega = \sup\{2 + \gamma|\gamma \in \omega\} = \omega$. Thus $2 + (\omega + 1) = \alpha + \beta = (\alpha + \gamma)^+ = (2 + \omega)^+ = \omega^+ = \omega + 1$.

Example 3.3. Suppose two ordinal numbers $\alpha = \omega$ and $\beta = \omega + 1$ are given. It is clear that β is the successor ordinal number ($\beta = \omega + 1 = \omega^+$). By Definition 3.1 (ii), it is clear that $\omega + (\omega + 1) = \alpha + \beta = (\alpha + \gamma)^+ = (\omega + \omega)^+$, then what is $(\omega + \omega)^+$?. Since ω is the ordinal limit $(\omega = \sup\{\gamma|\gamma \in \omega\}$, it is clear that $\omega + \omega = \sup\{\omega + \gamma|\gamma \in \omega\} = \omega\,.2$. So $\omega + (\omega + 1) = \alpha + \beta = (\alpha + \gamma)^+ = (\omega + \omega)^+ = (\omega.2)^+ = \omega.2 + 1$.

Definition 3.4. Given $\alpha, \beta \in \mathcal{O}$, the ordinal multiplication $\alpha.\beta$ is defined as follows:

(i) $\alpha.0 = 0$.
(ii) If β is the ordinal successor ($\beta = \gamma^+$ for an ordinal number γ), then $\alpha.\beta = (\alpha.\gamma)^+$.
(iii) If β is the ordinal limit ($\beta = \sup\{\gamma|\gamma \in \beta\}$), then $\alpha.\beta = \sup\{\alpha.\gamma|\gamma \in \beta\}$.

Example 3.5. Suppose two ordinal numbers $\alpha = 2$ and $\beta = \omega + 1$. It is clear that β is the successor ordinal number ($\beta = \omega + 1 = \omega^+$). By Definition 3.4 (ii), it is clear that $2.(\omega + 1) = \alpha.\beta = (\alpha.\gamma)^+ = (2.\omega)^+$, then what is $(2.\omega)^+$?. Since ω is the ordinal limit $(\omega = \sup\{\gamma|\gamma \in \omega\}$, it is clear that $2.\omega = \sup\{2.\gamma|\gamma \in \omega\} = \omega$. Thus $2.(\omega + 1) = \alpha.\beta = (\alpha.\gamma)^+ = (2.\omega)^+ = \omega^+ = \omega + 1$.

Example 3.6. Suppose two ordinal numbers $\alpha = \omega$ and $\beta = \omega + 1$ are given. It is clear that β is the successor ordinal number ($\beta = \omega + 1 = \omega^+$). By Definition 3.4 (ii), it is clear that $\omega.(\omega + 1) = \alpha.\beta = (\alpha.\gamma)^+ = (\omega.\omega)^+$, then what is $(\omega.\omega)^+$?. Since ω is an ordinal limit $(\omega = \sup\{\gamma|\gamma \in \omega\}$, it is clear that $\omega.\omega = \sup\{\omega.\gamma|\gamma \in \omega\}$. So $\omega.(\omega + 1) = \alpha.\beta = (\alpha.\gamma)^+ = (\omega.\omega)^+ = \omega.\omega + 1$.

Definition 3.7. Given $\alpha, \beta \in \mathcal{O}$, the ordinal division α^β It is defined as follows:

(i) $\alpha^0 = 1$.
(ii) If β is the ordinal successor ($\beta = \gamma^+$ for an ordinal number γ), then $\alpha^\beta = \alpha^\gamma.\alpha$.
(iii) If β is the ordinal limit ($\beta = \sup\{\gamma|\gamma \in \beta\}$), then $\alpha^\beta = \sup\{\alpha^\gamma|\gamma \in \beta\}$.

Example 3.8. Suppose two ordinal numbers $\alpha = 2$ dan $\beta = \omega + 1$ are given. It is clear that β is the successor ordinal number ($\beta = \omega + 1 = \omega^+$). By Definition 3.7 (ii), it is clear that $2^{\omega+1} = \alpha^\beta = \alpha^\gamma.\alpha = 2^\omega.2$, then what is 2^ω?. Since ω is the ordinal limit $(\omega = \sup\{\gamma|\gamma \in \omega\}$, it is clear that $2^\omega = \sup\{2^\gamma|\gamma \in \omega\} = \omega$. Thus $2^{\omega+1} = \alpha^\beta = \alpha^\gamma.\alpha = 2^\omega.2 = \omega.2$.

Example 3.9. Suppose two ordinal numbers $\alpha = \omega$ dan $\beta = \omega + 1$ are given. It is clear that β is the successor ordinal number ($\beta = \omega + 1 = \omega^+$). By Definition 3.7 (ii), it is clear that $\omega^{\omega+1} = \alpha^\beta = \alpha^\gamma.\alpha = \omega^\gamma.\omega = \omega^\omega.\omega$.

Some properties that can be explored in ordinal arithmetic with respect to addition, multiplication, and exponentiation are as follows:

Theorem 3.10. If given $\alpha, \beta \in \mathcal{O}$, then $\alpha + \beta = \alpha \cup \{\alpha + \gamma | \gamma < \beta\}$.

Proof:

The proof will use transfinite induction on β.

Write $\mathcal{C} = \{\beta \in \mathcal{O} | \alpha + \beta = \alpha \cup \{\alpha + \gamma | \gamma < \beta\}\}$.

(i) By Definition 3.1 (i), it is clear that $\alpha + 0 = \alpha = \alpha \cup \emptyset$.
So $0 \in \mathcal{C}$.

(ii) Known $\beta \in \mathcal{C}$.
$\alpha + \beta = \alpha \cup \{\alpha + \gamma | \gamma < \beta\}$.
Since $\beta < \beta^+$, it is clear that if $\gamma < \beta$ then $\gamma < \beta^+$.
Consequently
$$\begin{aligned}\alpha \cup \{\alpha + \gamma | \gamma < \beta^+\} &= \alpha \cup (\{\alpha + \gamma | \gamma < \beta\} \cup \{\alpha + \beta\}) \\ &= (\alpha \cup \{\alpha + \gamma | \gamma < \beta\}) \cup \{\alpha + \beta\} \\ &= (\alpha + \beta) \cup \{\alpha + \beta\} \\ &= (\alpha + \beta)^+ \\ &= \alpha + \beta^+ \qquad \text{(Definition 3.1 (ii))}\end{aligned}$$
So $\alpha + \beta^+ = \alpha \cup \{\alpha + \gamma | \gamma < \beta^+\}$.
So $\beta^+ \in \mathcal{C}$.

(iii) Let δ be the ordinal limit and $\forall \beta < \delta, \beta \in \mathcal{C}$.
Since $\beta \in \mathcal{C}$, it is clear that $\alpha + \beta = \alpha \cup \{\alpha + \gamma | \gamma < \beta\}$.
Since δ is the ordinal limit and $\forall \beta < \delta$, it follows that
$$\begin{aligned}\alpha + \delta &= \sup\{\alpha + \beta | \beta < \delta\} \qquad \text{(Definition 3.1 (iii))} \\ &= \sup\{\alpha \cup \{\alpha + \gamma | \gamma < \beta \text{ dan } \beta < \delta\}\} \\ &= \alpha \cup \sup\{\alpha + \gamma | \gamma < \beta \text{ dan } \beta < \delta\} \\ &= \alpha \cup \{\alpha + \gamma | \gamma < \delta\}\end{aligned}$$
So $\alpha + \delta = \alpha \cup \{\alpha + \gamma | \gamma < \delta\}$.
So $\delta \in \mathcal{C}$.

From (i), (ii) and (iii), we get $\alpha + \beta = \alpha \cup \{\alpha + \gamma | \gamma < \beta\}$.

Theorem 3.11. If given $\alpha, \beta \in \mathcal{O}$, then $\alpha.\beta = \{\alpha.\gamma + \delta | \gamma < \beta \text{ and } \delta < \alpha\}$.

Proof:

The proof will use transfinite induction on β.

Write $\mathcal{C} = \left\{\beta \in \mathcal{O} \middle| \alpha.\beta = \{\alpha.\gamma + \delta | \gamma < \beta \text{ and } \delta < \alpha\}\right\}$.

(i) By Definition 3.4 (i), it is clear that $\alpha.0 = 0 = \alpha.0 + 0$.
So $0 \in \mathcal{C}$.

(ii) Known $\beta \in \mathcal{C}$.
Clearly $\alpha.\beta = \{\alpha.\gamma + \delta | \gamma < \beta \text{ dan } \delta < \alpha\}$.
Since $\beta < \beta^+$, it is clear that if $\gamma < \beta$ then $\gamma < \beta^+$.
Consequently $\{\alpha.\gamma + \delta | \gamma < \beta^+ \text{ dan } \delta < \alpha\} = \alpha.\beta^+$.
So $\alpha.\beta^+ = \{\alpha.\gamma + \delta | \gamma < \beta^+ \text{ dan } \delta < \alpha\}$.
So $\beta^+ \in \mathcal{C}$.

(iii) Let η be the ordinal limit and $\forall \beta < \eta, \beta \in \mathcal{C}$.
Since $\beta \in \mathcal{C}$, it is clear that $\alpha.\beta = \{\alpha.\gamma + \delta | \gamma < \beta \text{ and } \delta < \alpha\}$.
Since η is an ordinal limit and $\forall \beta < \eta$, it follows that
$$\alpha.\eta = \sup\{\alpha.\beta | \beta < \eta\}$$

$= \sup\{\alpha.\gamma + \delta | \gamma < \beta, \delta < \alpha \text{ dan } \beta < \eta\}$
$= \sup\{\alpha.\gamma + \delta | \gamma < \eta \text{ dan } \delta < \alpha\}$
$= \{\alpha.\beta + \delta | \beta < \eta \text{ dan } \delta < \alpha\}$

Clearly $\alpha.\eta = \{\alpha.\beta + \delta | \beta < \eta \text{ dan } \delta < \alpha\}$.

So $\eta \in \mathcal{C}$.

From (i), (ii), and (iii), we get $\alpha.\beta = \{\alpha.\gamma + \delta | \gamma < \beta \text{ dan } \delta < \alpha\}$.

Theorem 3.12. If given $\alpha, \beta, \gamma \in \mathcal{O}$ then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Proof:

The proof will use transfinite induction on γ.

Write $\mathcal{C} = \{\gamma \in \mathcal{O} | (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)\}$.

(i) Clearly $(\alpha + \beta) + 0 = \alpha + \beta$
$= \alpha + (\beta + 0)$

So $0 \in \mathcal{C}$.

(ii) Known $\gamma \in \mathcal{C}$.

Clearly $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Since $\gamma^+ = \gamma + 1$, it is clear that $(\alpha + \beta) + \gamma + 1 = \big((\alpha + \beta) + \gamma\big) + 1$
$= \big(\alpha + (\beta + \gamma)\big) + 1$
$= \alpha + (\beta + \gamma) + 1$
$= \alpha + (\beta + \gamma + 1)$

Clearly $(\alpha + \beta) + \gamma + 1 = \alpha + (\beta + \gamma + 1)$.

So $\gamma^+ \in \mathcal{C}$

(iii) Let γ be the ordinal limit and $\forall \delta < \gamma, \delta \in \mathcal{C}$.

Since $\delta \in \mathcal{C}$, it is clear that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$.

Clearly $(\alpha + \beta) + \gamma = \sup\{(\alpha + \beta) + \delta | \delta < \gamma\}$
$= \sup\{\alpha + (\beta + \delta) | \delta < \gamma\}$
$= \alpha + (\beta + \gamma)$

Clearly $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

So $\gamma \in \mathcal{C}$.

From (i), (ii), and (iii), we get $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Theorem 3.13. If given $\alpha, \beta, \gamma \in \mathcal{O}$ then $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$.

Proof:

The proof will use transfinite induction on γ.

Write $\mathcal{C} = \{\gamma \in \mathcal{O} | (\alpha.\beta).\gamma = \alpha.(\beta.\gamma)\}$.

(i) Clearly $(\alpha.\beta).0 = 0$
$= \alpha.(\beta.0)$

So $0 \in \mathcal{C}$

(ii) Diketahui $\gamma \in \mathcal{C}$.

Clearly $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$.

Obtained $(\alpha.\beta).\gamma^+ = \big((\alpha.\beta).\gamma\big)^+$ Definition 3.4 (ii)
$= \big(\alpha.(\beta.\gamma)\big)^+$
$= \alpha.(\beta.\gamma)^+$ Definition 3.4 (ii)

$= \alpha.(\beta.\gamma^+)$ Definition 3.4 (ii)

Clearly $(\alpha.\beta).\gamma^+ = \alpha.(\beta.\gamma^+)$.

So $\gamma^+ \in \mathcal{C}$.

(iii) Let γ be an ordinal limit and $\forall\delta < \gamma, \delta \in \mathcal{C}$.

Since $\delta \in \mathcal{C}$, jelas $(\alpha.\beta).\delta = \alpha.(\beta.\delta)$.

Clearly $(\alpha.\beta).\gamma = \sup\{(\alpha.\beta).\delta | \delta < \gamma\}$ Definition 3.4 (iii)

$= \sup\{\alpha.(\beta.\delta) | \delta < \gamma\}$

$= \alpha.(\beta.\gamma)$ Definition 3.4 (iii)

Clearly$(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$.

So i $\gamma \in \mathcal{C}$.

From (i), (ii), and (iii), we get $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$.

Theorem 3.14. If given $\alpha, \beta, \gamma \in \mathcal{O}$ then $(\alpha^\beta)^\gamma = \alpha^{\beta.\gamma}$.

Proof:

The proof will use transfinite induction on γ.

Write $\mathcal{C} = \{\gamma \in \mathcal{O} | (\alpha^\beta)^\gamma = \alpha^{\beta.\gamma}\}$.

(i) Clearly $(\alpha^\beta)^0 = 1$

$= \alpha^0$

$= \alpha^{\beta.0}$

Clearly $(\alpha^\beta)^0 = \alpha^{\beta.0}$.

So $0 \in \mathcal{C}$.

(ii) Known $\gamma \in \mathcal{C}$.

Clearly $(\alpha^\beta)^\gamma = \alpha^{\beta.\gamma}$.

Since $(\alpha^\beta)^{\gamma^+} = (\alpha^\beta)^\gamma.\alpha$ Definition 3.7 (ii)

$= \alpha^{\beta.\gamma}.\alpha$

$= \alpha^{(\beta.\gamma)^+}$ Definition 3.7 (ii)

$= \alpha^{\beta.\gamma^+}$ Definition 3.4 (ii)

Clearly $(\alpha^\beta)^{\gamma^+} = \alpha^{\beta.\gamma^+}$.

So $\gamma^+ \in \mathcal{C}$.

(iii) Let γ be an ordinal limit and $\forall\delta < \gamma, \delta \in \mathcal{C}$.

Since $\delta \in \mathcal{C}$, it is clear$(\alpha^\beta)^\delta = \alpha^{\beta.\delta}$.

Clearly $(\alpha^\beta)^\gamma = \sup\left\{(\alpha^\beta)^\delta \middle| \delta < \gamma\right\}$ Definition 3.7 (iii)

$= \sup\{\alpha^{\beta.\delta} | \delta < \gamma\}$

$= \alpha^{\beta.\delta}$. Definition 3.7 (iii)

Clearly $(\alpha^\beta)^\gamma = \alpha^{\beta.\delta}$.

So $\gamma \in \mathcal{C}$.

From (i), (ii), and (iii), we get $(\alpha^\beta)^\gamma = \alpha^{\beta.\gamma}$.

Theorem 3.15. If given $\alpha, \beta, \gamma \in \mathcal{O}$ then $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$.

Proof:
The proof will use transfinite induction on γ.
Write $\mathcal{C} = \{\gamma \in \mathcal{O} | \alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma\}$.

(i) Clearly
$$\alpha.(\beta + 0) = \alpha.\beta = \alpha.\beta + 0 = \alpha.\beta + \alpha.0$$
Clearly $\alpha.(\beta + 0) = \alpha.\beta + \alpha.0$.
So $0 \in \mathcal{C}$.

(ii) Known $\gamma \in \mathcal{C}$.
Clearly $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$.
Clearly
$$\begin{aligned} \alpha.(\beta + \gamma^+) &= \alpha.(\beta + \gamma)^+ && \text{Definition 3.1 (ii)} \\ &= \left(\alpha.(\beta + \gamma)\right)^+ && \text{Definition 3.4 (ii)} \\ &= (\alpha.\beta + \alpha.\gamma)^+ \\ &= \alpha.\beta + (\alpha.\gamma)^+ && \text{Definition 3.1 (ii)} \\ &= \alpha.\beta + \alpha.\gamma^+ && \text{Definition 3.4 (ii)} \end{aligned}$$
Clearly $\alpha.(\beta + \gamma^+) = \alpha.\beta + \alpha.\gamma^+$.
So $\gamma^+ \in \mathcal{C}$.

(iii) Let γ be an ordinal limit and $\forall \delta < \gamma, \delta \in \mathcal{C}$.

(iv) Since $\delta \in \mathcal{C}$, it is clear that $\alpha.(\beta + \delta) = \alpha.\beta + \alpha.\delta$.
Clearly
$$\begin{aligned} \alpha.(\beta + \gamma) &= \sup\{\alpha.(\beta + \delta) | \delta < \gamma\} && \text{Definition 3.4 (iii)} \\ &= \sup\{\alpha.\beta + \alpha.\delta | \delta < \gamma\} \\ &= \alpha.\beta + \sup\{\alpha.\delta | \delta < \gamma\} && \text{Definition 3.1 (iii)} \\ &= \alpha.\beta + \alpha.\sup\{\delta | \delta < \gamma\} && \text{Definition 3.4 (iii)} \\ &= \alpha.\beta + \alpha.\gamma \end{aligned}$$
Clearly $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$.
So $\gamma \in \mathcal{C}$.

From (i), (ii), and (iii), we get $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$.

D. CONCLUSION AND SUGGESTION

The conclusion of this paper is the method of constructing finite and transfinite ordinal numbers using two generation principles. Ordinal numbers, both finite and transfinite, are defined as well-orderings that are also transitive sets. Arithmetic with finite ordinal numbers is well-known, but arithmetic with transfinite ordinal numbers, including addition, multiplication, and exponentiation, is introduced in this paper. The suggestion put forward in this paper is that the concept of arithmetic with transfinite ordinal numbers should be further developed in the fields of mathematics, philosophy of religion, and theology.

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