

# Numerical Solution of the Advection-Diffusion Equation Using the Radial Basis Function

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## ABSTRACT

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The advection-diffusion equation is a form of partial differential equation. This equation is also known as the transport equation. The purpose of this research is to approximate the solution of advection-diffusion equation by numerical approach using radial basis functions network. The approximation is performed by using the multiquadrics basis function. The simulation of the numerical solution is run with the help of the Matlab program. The one-dimensional advection-diffusion equation used is  $\frac{\partial u}{\partial t} + C \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}$  with given initial conditions, boundary conditions, and exact solution  $u(x, t)$ . The numerical solution approximation using the radial basis function network with  $dt = 0.004$  and  $dx = 0.02$  produces the value at each discretization point is close to the exact solution. In this study, the smallest error between numerical solution and the exact solution is obtained  $2.18339 \times 10^{-10}$ .



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## A. INTRODUCTION

The equation of diffusion-advection is composed of two equations, namely the diffusion equation and the advection equation. The advection equation is a linear wave equation of first order and belongs to the category of hyperbolic differential equations that describe the pattern of the spread of a gas or liquid substance. The diffusion equation is a differential equation that describes the movement of a substance with high concentration to an area with low concentration, and this differential equation belongs to the category of partial differential equations (Leveque, 2022). The application of the diffusion-advection equation is mostly used in fluid dynamics (Sulpiani & Widowati, 2013). The techniques used for simulating this equation are beneficial for many different fields, including electromagnetism, finance, food processing, soil adsorption of pollutants, the transfer of air and river water, etc (Ara et al., 2021). The role of the diffusion-advection equation is crucial in the industry, particularly in predicting the concentration of pollutants (Syafi'i, 2013). The mathematical form of the one-dimensional diffusion-advection equation is as follows:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2} \quad (1)$$

with initial condition

$$c(x, 0) = f(x) \quad 0 \leq x \leq L$$

and boundary conditions

$$c(0, t) = g(t) \quad 0 \leq t \leq T$$

$$c(L, t) = h(t) \quad 0 \leq t \leq T$$

where  $f(x)$ ,  $g(t)$  and  $h(t)$  are known functions, while  $c$  and  $D$  respectively represent the advection velocity and diffusion process (Mojtabi & Deville, 2015).

Several previous studies have solved equation (1) using finite difference or finite element methods, as seen in articles such as an apparently simple but precise finite difference method for the progressive diffusion equation Sanjaya & Mungkasi (2017), for the numerical solution of the advection-diffusion problem, use the Galerkin-finite element approach. Sharma et al. (2011), using a new cubic trigonometric B-spline method, advection-diffusion problems are numerically solved Nazir et al. (2016), a Semi-Lagrangian approach is used to calculate a numerical solution to the advection-diffusion equation Bahar et al. (2018), comparative analysis of the finite difference, fourth order finite difference, finite volume, and differential quadrature methods in the explicit condition for the advection-diffusion equation (Gharehbaghi et al., 2017). The difference between this research and previous research is the approach to solving the exact equation of diffusion of advection using radial basis function. Previous research related to numerical solutions with the radial basis function is as follows: gaussian radial basis functions method for linear and nonlinear convection-diffusion models in physical phenomena Wang et al. (2021) Zakharov et al. (2014), numerical solution of differential equations by neural networks with radial basis functions Li Jianyu et al. (2012), functional integral equations can be approximated using radial basis function Firouzdor et al. (2016), differential quadrature based on radial basis functions for the one-dimensional heat equation (Aliy et al., 2021). Radial basis functions (RBFs) method is commonly used to represent topographical surfaces and other intricate 3D shapes (Chenoweth, 2009).

High-order accuracy, geometrical flexibility, computational efficiency, and ease of implementation are all desirable qualities in a numerical approach for PDE problems. The approaches that are often employed typically meet one or two of the requirements, but not all of them. Finite difference techniques can be made high-order accurate, but they need a structured grid. Radial basis functions (RBFs) are one of a method for solving PDEs. A function  $\phi: R^n \rightarrow R^1$  is called radial if there exists one variable functions  $\varphi: \{0, \infty\} \rightarrow R$  such that  $\phi(x) = \varphi(\|x\|)$  with  $\| \cdot \|$  representing the Euclidean norm (Bhatia & Arora, 2016). An RBFs depends on the distance to a center point  $c_j$  and is of the form  $\varphi(\|x - c_j\|)$  (Larsson & Fornberg, 2003). In this study, the author employed a radial basis function network to approximate the analytical solution of the diffusion-advection equation. The radial basis function network is a simple artificial neural network that consists of three layers, namely the input layer, hidden layer, and output layer. Each hidden unit is a specific activation function called a basis function. These basis functions are used to activate the radial basis function network (Mai-Duy & Tran-Cong, 2003). The basis function utilized in this research is the one-dimensional Multiquadric function (1D).

$$\phi(x, c) = \sqrt{(x - c)^2 + a^2} \quad (2)$$

with  $c$  being the center point at  $x$  and  $a$  being the variance of the center (Sarboland & Aminataei, 2015). The interpolant, which is a linear combination of translations of a basis function that only depends on the Euclidean distance from its center, is the main characteristic of the Multiquadric method. As a result, this basis function is symmetric about its center (Golbabai et al., 2016).

A radial basis function network is a function  $f: R^n \rightarrow R^1$  composed of a set of weights  $\{w_j\}_{j=1}^m$  and a set of radial basis functions

$$\varphi_j(x) = \varphi(\|x - c_j\|) \tag{3}$$

with  $\|\cdot\|$  representing the normal vector. The representation of the radial basis function network approximation for a single variable function  $u(x)$  is as follows (Luga et al., 2019):

$$u(x) \approx \hat{u}(x) = \sum_{j=1}^N w_j \phi(x, c_j). \tag{4}$$

Based on the explanation above, the purpose of this research is to solve the one-dimensional diffusion-advection equation with a numerical approach using Radial basis functions (RBFs).

**B. METHODS**

This research constitutes a literature study. A literature study is a research-based on information and data derived from written works such as books, scientific articles, and other sources (Sugiyono, 2015). The use of a literature review as a research method is more important than ever. A more or less systematic method of compiling and summarizing prior research can be broadly characterized as a literature review (Snyder, 2019). Based on the literature study that has been reviewed, the steps in determining the approximation of the advection-diffusion equation solution are as follows:

1. Determining the basis function  $\phi(x, c)$ .
2. Calculating the weight values  $w_j$ .
3. A radial basis function network can be formed based on the set of weight values  $w_j$  and the set of basis functions  $\phi$ .
4. Assuming that the set of basis functions forms a basis vector denoted as  $A$ . The set of weight values ( $w_j$ ) is denoted by  $W$ , and the function  $f(x)$  for  $x = \{x_1, x_2, \dots, x_m\}$  is denoted by  $B$ . Thus, we obtain

$$\begin{aligned} AW &= B \\ (A^{-1}A)W &= A^{-1}B \\ W &= A^{-1}B \end{aligned} \tag{5}$$

where

$$A = \begin{bmatrix} \phi_1(x_1, c_1) & \phi_2(x_1, c_2) & \dots & \phi_N(x_1, c_N) \\ \phi_1(x_2, c_1) & \phi_2(x_2, c_2) & \dots & \phi_N(x_2, c_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_m, c_1) & \phi_2(x_m, c_2) & \dots & \phi_N(x_m, c_N) \end{bmatrix}$$

$$W = [w_1, w_2, \dots, w_N]^T$$

$$B = [y_1, y_2, \dots, y_m]^T$$

5. The analysis of error

Error represents how close the approximate solution is to the true solution (exact solution). In this study, the error calculation used is by calculating the square difference (sum square error) between the numerical result obtained and the exact function solution.

$$|e| = |f(x) - \hat{f}(x)|$$

for a number of  $m$  points

$$SSE = \frac{\sum_{j=1}^m (f(x) - \hat{f}(x))^2}{m}$$

(Mai-Duy & Tran-Cong, 2003).

**C. RESULT AND DISCUSSION**

In this research, the equation that will be approximated using the Radial Basis Function is the advection-diffusion equation.

$$\frac{\partial u}{\partial t} + C \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} \tag{6}$$

with initial condition

$$u(x, 0) = f(x) \quad 0 \leq x \leq L$$

and boundary conditions

$$u(0, t) = g(t) \quad 0 \leq t \leq T$$

$$u(L, t) = h(t) \quad 0 \leq t \leq T$$

The numerical solution of equation (6) can be solved using the following steps:

**1. Discretization**

Discretize  $u_t$  with forward difference approximation so that  $u$  becomes a function dependent on variable  $x$

$$u_t + C u_x = D u_{xx}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + C u_x^{n+1} = D u_{xx}^{n+1}$$

Thus, we obtain

$$u_i^{n+1} + C \Delta t u_x^{n+1} - D \Delta t u_{xx}^{n+1} = u_i^n \tag{7}$$

for  $n=1$ , equation (7) can be written as

$$u_i^2 + C \Delta t u_x^2 - D \Delta t u_{xx}^2 = u_i^1$$

with the initial condition substitution, we get

$$u_i^2 + C\Delta t u_x^2 - D\Delta t u_{xx}^2 = f(x) \tag{8}$$

**2. Approximating using Radial Basis Function.**

The next step is to approximate  $u$  using a radial basis function network, following equation (4):

$$u^{n+1} = \sum_{j=1}^N w_j^{n+1} \phi(x, c_j)$$

For  $n = 1$ , we obtain:

$$u^2 = \sum_{j=1}^N w_j^2 \phi(x, c_j)$$

Thus, equation (8) becomes:

$$\sum_{j=1}^N w_j^2 \phi(x, c_j) + C\Delta t \sum_{j=1}^N w_j^2 \phi_x(x, c_j) - D\Delta t \sum_{j=1}^N w_j^2 \phi_{xx}(x, c_j) = f(x) \tag{9}$$

Let

$$\phi(x, c_j) + C\Delta t \phi_x(x, c_j) + D\Delta t \phi_{xx}(x, c_j) = k(x, c_j)$$

Equation (9) can be rewritten as:

$$\sum_{j=1}^N w_j^2 k(x, c_j) = f(x) \quad a \leq x \leq b \tag{10}$$

Based on equation (10), the approximation at boundary conditions is obtained as follows.

$$\sum_{j=1}^N w_j^2 \phi(a, c_j) = g(t) \tag{11}$$

$$\sum_{j=1}^N w_j^2 \phi(b, c_j) = h(t) \tag{12}$$

**3. Calculating the numerical solution  $u^2$**

Based on equations (10), (11), and (12), the linear system of equations can be written in matrix form as follows:

$$\begin{bmatrix} \phi(a, c_1) & \phi(a, c_2) & \dots & \phi(a, c_N) \\ k(x_2, c_1) & k(x_2, c_2) & \dots & k(x_2, c_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(b, c_1) & \phi(b, c_2) & \dots & \phi(b, c_N) \end{bmatrix} \begin{bmatrix} w_1^2 \\ w_2^2 \\ \vdots \\ w_N^2 \end{bmatrix} = \begin{bmatrix} g(a, t^2) \\ f(x_2, t^2) \\ \vdots \\ h(b, t^2) \end{bmatrix}$$

Let

$$A = \begin{bmatrix} \varphi(a, c_1) & \varphi(a, c_2) & \cdots & \varphi(a, c_N) \\ k(x_2, c_1) & k(x_2, c_2) & \cdots & k(x_2, c_N) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(b, c_1) & \varphi(b, c_2) & \cdots & \varphi(b, c_N) \end{bmatrix}$$

$$X = \begin{bmatrix} w_1^2 \\ w_2^2 \\ \vdots \\ w_N^2 \end{bmatrix} \text{ dan } B = \begin{bmatrix} g(a, t^2) \\ f(x_2, t^2) \\ \vdots \\ h(b, t^2) \end{bmatrix}$$

Thus, we can write:

$$AX = B$$

$$X = A^{-1}B$$

The obtained value of  $w_j^2$ , is used to calculate the solution  $u^2$

$$u_1^2 = w_1^2 \phi(a, c_1) + w_2^2 \phi(a, c_2) + \cdots + w_N^2 \phi(a, c_N)$$

$$u_2^2 = w_1^2 \phi(x_2, c_1) + w_2^2 \phi(x_2, c_2) + \cdots + w_N^2 \phi(x_2, c_N)$$

$$u_3^2 = w_1^2 \phi(x_3, c_1) + w_2^2 \phi(x_3, c_2) + \cdots + w_N^2 \phi(x_3, c_N)$$

and so on until  $u_N^2$ .

**4. Determine the solutions of  $u^3, u^4, \dots, u^n$**

The steps in calculating  $u^3, u^4, \dots, u^n$  are almost the same as when determining  $u^2$ . The difference is that when calculating  $u^2$  an initial value is required, whereas when determining  $u^3, u^4, \dots, u^n$ , the previous values of  $u$  are used. An example taken in this research is:

$$\frac{\partial u}{\partial t} + 1.0 \frac{\partial u}{\partial x} = 0.01 \frac{\partial^2 u}{\partial x^2} \tag{13}$$

with initial condition

$$u(x, 0) = \exp\left(-\frac{(x+0.5)^2}{0.00125}\right) \quad 0 \leq x \leq 1$$

and boundary conditions

$$u(0, t) = \frac{0.025}{\sqrt{0.000625+0.02t}} \exp\left(-\frac{(0.5-t)^2}{(0.00125+0.04t)}\right)$$

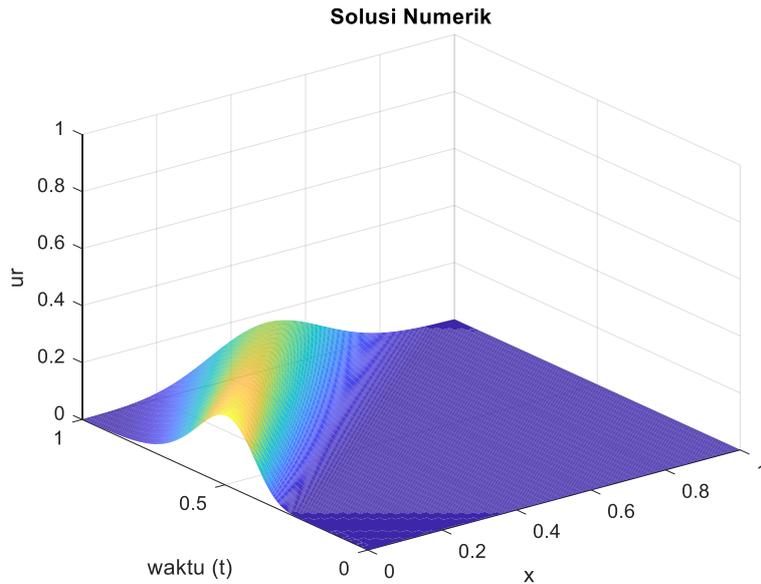
$$u(1, t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(1.5 - t)^2}{(0.00125 + 0.04t)}\right)$$

with  $0 \leq t \leq 1$

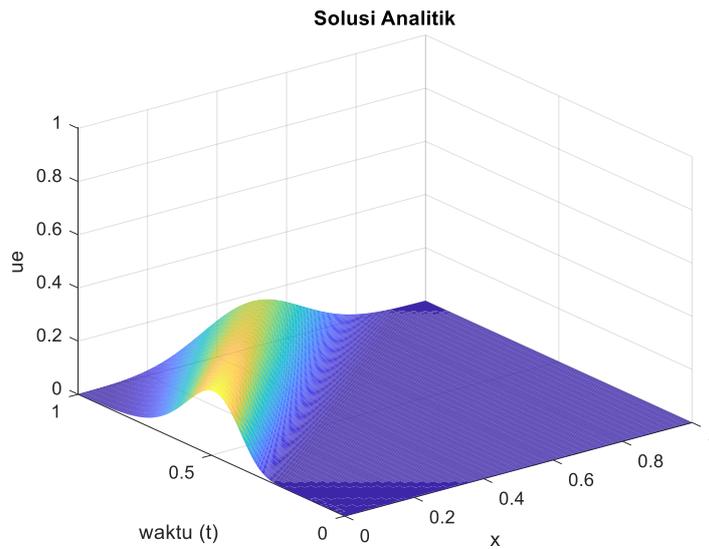
The exact solution of equation (13) is:

$$u(x, t) = \frac{0.025}{\sqrt{0.000625+0.02t}} \exp\left(-\frac{(x+0.5-t)^2}{(0.00125+0.04t)}\right) \text{ (Appadu, 2013).}$$

The author used Matlab program to facilitate the calculation of the equation above and obtain numerical solutions and simulations of equation (13). The numerical solution and analytical solution simulations of equation (13) can be seen in Figure 1 and Figure 2 below.



**Figure 1.** Numerical Solution Simulation of the Advection Diffusion Equation with  $C = 1, D = 0.01, \Delta t = 0.004$  and  $\Delta x = 0.02$ .



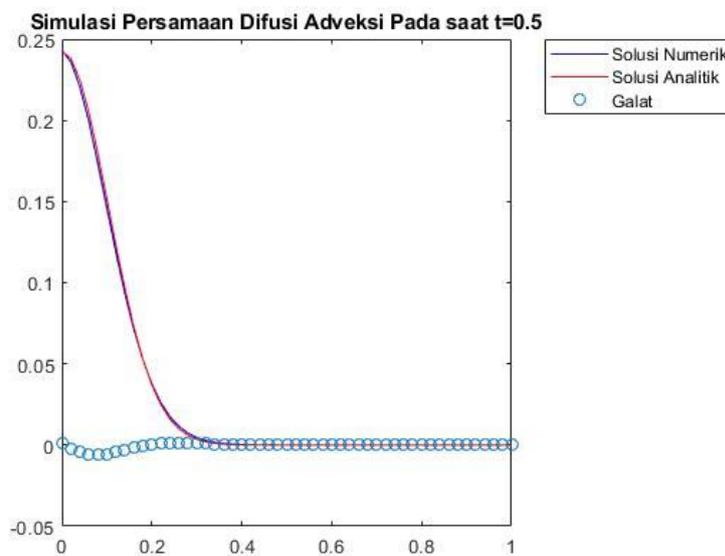
**Figure 2.** Analytical Solution Simulation of the Advection Diffusion Equation with  $C = 1, D = 0.01, \Delta t = 0.004$  and  $\Delta x = 0.02$ .

Based on Figure 1 dan Figure 2, the simulation results of the analytical solution and numerical approach of the one-dimensional diffusion-advection equation with  $C = 1, D = 0.01, dt = 0.004$ , and  $dx = 0.02$ . The table below is comparison of the results of numerical solutions ( $u_{RBF}(x, t)$ ) and analytical solutions ( $u(x, t)$ ) in the first 10 iterations:

**Table 1.** Comparison of Numerical Solutions and Analytical Solutions The One Dimensional Diffusion Advection Equation in the First 10 iterations pada saat  $t = 0.5$

Iterasi	$u_{RBF}(x, t)$	$u(x, t)$
1	0.243269261965732	0.242535625036333
2	0.235825664815324	0.238012948473468
3	0.220635508005349	0.224944676400746
4	0.199078419829179	0.204739173006960
5	0.173406383396275	0.179463552355389
6	0.145913387725380	0.151496174174946
7	0.118703195713279	0.123162121458410
8	0.093427573223786	0.096427908384170
9	0.071196648206432	0.072707367795799
10	0.052567734571106	0.052796385113083

The sum of squared errors (SSE) between the analytical solution and the numerical solution is  $1.1941 \times 10^{-4}$ , This means that the numerical solution with the radial basis function approach is close to the exact solution. Based on the error obtained from the numerical approach solutions and the analytical solutions, the smallest error is  $2.1883 \times 10^{-10}$  and the largest error is  $6.0572 \times 10^{-3}$ . In this study, an approximation of the solution to equation (13) was also simulated at  $t = 0.5$ , as shown in Figure 3 below.



**Figure 3.** Analytical Solution Simulation, Numerical Solution Simulation, and Error of the Advection Diffusion Equation with  $C = 1, D = 0.01, \Delta t = 0.004$  and  $\Delta x = 0.02$  at the time  $t = 0.5$

Figure 3 is the simulation result of the one-dimensional advection diffusion equation with  $C = 1, D = 0.01, \Delta t = 0.004$  and  $\Delta x = 0.02$  at the time  $t = 0.5$ . The figure shows that the numerical solution is close to the exact solution and the error is close to 0.

## D. CONCLUSION AND SUGGESTIONS

One of the methods for computing the numerical solution of the advection diffusion equation is by using a radial basis function network with the multiquadric basis function. In this study, the numerical solution using the radial basis function was able to approximate the analytical solution of the advection diffusion equation with the smallest error is  $2.1883 \times 10^{-10}$  and the largest error is  $6.0572 \times 10^{-3}$ .

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