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Characteristic Min-Polynomial and Eigen Problem of a Matrix over Min-Plus Algebra

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ABSTRACT

Let $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, with $\mathbb{R}$ being a set of all real numbers. The algebraic structure $(\mathbb{R}_\varepsilon,\oplus,\otimes)$ is called max-plus algebra. The task of finding the eigenvalue and eigenvector is called the eigenproblem. There are several methods developed to solve the eigenproblem of $A \in \mathbb{R}_\varepsilon^{n\times n}$, one of them is by using the characteristic max-polynomial. There is another algebraic structure that is isomorphic with max-plus algebra, namely min-plus algebra. Min-plus algebra is a set of $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{+\infty\}$ that uses minimum $(\oplus')$ and addition $(\otimes)$ operations. The eigenproblem in min-plus algebra is to determine $\lambda \in \mathbb{R}_{\varepsilon'}$ and $v \in \mathbb{R}_{\varepsilon'}^n$ such that $A \otimes v = \lambda \otimes v$. In this paper, we provide a method for determining the characteristic min-polynomial and solving the eigenproblem by using the characteristic min-polynomial. We show that the characteristic min-polynomial of $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is the permanent of $I \otimes x \oplus' A$, the smallest corner of $\chi_A(x)$ is the principal eigenvalue $(\lambda(A))$, and the columns of A_λ^+ with zero diagonal elements are eigenvectors corresponding to the principal eigenvalue.

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A. INTRODUCTION

Max-plus algebra is defined as the set of $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, with $\mathbb{R}$ being a set of all real numbers and $\varepsilon = -\infty$ under the maximum $(\oplus)$ and addition $(\otimes)$ binary operations (Subiono, 2015). The algebraic structure of triple $(\mathbb{R}_\varepsilon,\oplus,\otimes)$ is an idempotent commutative semiring with the unit element $e = 0$ (Gyamerah et al., 2015). Max-plus algebra can be used to examine a manufacturing system as well as to schedule and maintain the stability of rail network systems (Subiono, 2015). Similarly, consider the set of matrices sized $m \times n$, the entries of which are the elements of $\mathbb{R}_\varepsilon$. This set is known as the set of matrices over max-plus algebra, denoted $\mathbb{R}_\varepsilon^{m\times n}$ (Jones, 2021). In max-plus algebra, a form of polynomial is also known, which is called max-polynomial (Butcovic, 2010; Wang et al., 2021).

In linear algebra, any square matrix can be related to the determinant. In particular, the determinant of a matrix has been studied by Liesen & Mehrmann (2015), Sobamowo (2016), and Kim & Pipattanajinda (2017). However, due to the absence of additive inverse, the idea of determinant in max-plus algebra has no direct analog in conventional algebra. Therefore, the determinant of matrices over max-plus algebra can be represented by permanent. The max-plus permanent of $A \in \mathbb{R}_\varepsilon^{n\times n}$ is $perm(A) = \oplus_{\sigma\in P_n} a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \ldots \otimes a_{n\sigma(n)}$, where P_n is the set of all permutations of the set $N = \{1,2,\ldots,n\}$ (Yonggu & Hee, 2018). Permanent is used for various problems, one of which is a way to solve the eigenproblem of matrices over max-plus algebra.

Eigenproblem is one of the most studied topics in linear algebra, especially max-plus algebra. For each $A \in \mathbb{R}_{\varepsilon}^{n\times n}$, the eigenproblem is the task of finding the eigenvalue and eigenvector in such a way that $A \otimes v = \lambda \otimes v$, where $v \in \mathbb{R}_{\varepsilon}^{n}$ is the eigenvector and $\lambda \in \mathbb{R}_{\varepsilon}$ is the eigenvalue (Mufid & Subiono, 2014). The principal eigenvalue of A is written $\lambda(A)$. Various methods have been elaborated to solve the eigenproblem of $A \in \mathbb{R}_{\varepsilon}^{n\times n}$. It can be solved using graph theory (Tunisa et al., 2017; Umer et al., 2019). Besides that, a method for obtaining the eigenvalues of a matrix over max-plus algebra using the characteristic max-polynomial has also been devised. The characteristic max-polynomial is the permanent of matrix $A \oplus x \otimes I$, with I being the identity matrix over max-plus algebra. Then from these eigenvalues can be obtained the eigenvectors corresponding to the eigenvalues (Rosenmann et al., 2019; Nishida et al., 2020).

Siswanto *et al.* (2016), Myšková (2016), and Siswanto *et al.* (2021) have expanded max-plus algebra into interval max-plus algebra for the application of time in intervals. Moreover, there is another algebraic structure that is isomorphic with max-plus algebra, namely min-plus algebra. Let $\mathbb{R}$ be a set of all real numbers, and $\varepsilon' = +\infty$. Min-plus algebra is defined in $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{+\infty\}$ under the minimum ($\oplus'$) and addition ($\otimes$) binary operations (Nowak, 2014). The algebraic structure of triple $(\mathbb{R}_{\varepsilon'},\oplus',\otimes)$ is an idempotent commutative semiring with the unit element $e = 0$ (Nowak, 2014). Similarly, consider the set of matrices sized $m \times n$, the entries of which are the elements of $\mathbb{R}_{\varepsilon'}$. This set is known as the set of matrices over min-plus algebra, denoted $\mathbb{R}_{\varepsilon'}^{m\times n}$ (Watanabe & Watanabe, 2014). It is also known as tropical semiring (Maclagan & Sturmfels, 2015).

Eigenproblems are also found in min-plus algebra. It is done by determining the eigenvalue and eigenvector in such a way that $A \otimes v = \lambda \otimes v$, where $v \in \mathbb{R}_{\varepsilon'}^{n}$ is the eigenvector and $\lambda \in \mathbb{R}_{\varepsilon'}$ is the eigenvalue (Nowak, 2014). Solving eigenproblems in min-plus algebra is useful for determining the shortest path of a travel route (Suprayitno, 2017). As research develops, the eigenproblem of matrices over min-plus algebra can be solved by graph theory. A graph G is a finite non-empty set V of objects called vertices together with a possibly empty set E of two-element subsets of V called edges (Chartrand et al., 2015). A graph in which each edge is assigned a weight is called a weighted graph (Chartrand et al., 2015). A graph that contains directed edges (arcs) is called a digraph (Bang-Jensen & Gutin, 2018).

Every matrix over min-plus algebra may be represented by a graph, and vice versa (Nowak, 2014; Watanabe & Watanabe, 2014). Therefore, according to Watanabe *et al.* (2017), characteristic polynomials are related to graphs with min-plus matrices made up of vertices and directed edges with weights. Considering the previous description, we will devise a method for solving the eigenproblem in min-plus algebra by using the characteristic min-polynomial. The way to determine the characteristic min-polynomial will be covered in this research, especially for irreducible matrices. Furthermore, the characteristic min-polynomial of a matrix can be utilized to find its eigenvalue. From this eigenvalue, the eigenvectors corresponding to the eigenvalue can be obtained.

B. METHODS

This research method is a literature study. This study aims to expand the research on min-plus algebra based on research in linear algebra and max-plus algebra, and also determine the connections or direct analogues between them. The research begins by defining a polynomial form in min-plus algebra, then determining the characteristic min-polynomial referring to the definition of permanent according to Siswanto, Gusmizain, *et al.* (Siswanto, Gusmizain, et al., 2021). The next step will be determining the eigenvalue of matrices over min-plus algebra through its characteristic min-polynomial. The last step is finding the eigenvectors corresponding to the eigenvalue. We want to recall some valuable definitions and theorems for the following discussion.

Definition 1. (Nowak, 2014) The representation from graph to matrix over min-plus algebra is to represent the directed graph $G = (V, E)$ by a matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$, with the entry a_{ij} denoting the weight of the arc from vertex j to vertex i. If two vertices are not connected directly by an arc, the corresponding matrix entry is ε'.

Definition 2. (Nowak, 2014) Matrix A over min-plus algebra is irreducible if its corresponding graph $G(A)$ is strongly connected.

Theorem 1. (Chaturvedi & McConnell, 2017; Nowak, 2014) Let $C(A)$ represent the set of all circuits in a graph $G(A)$. Then the minimum average circuit weight which is also the principal eigenvalue is

$$\lambda(A) = \min_{p\in C(A)} \frac{\|p\|_w}{\|p\|_l} \tag{1}$$

where p is the closed path (cicruit), $\|p\|_w$ is the circuit weight, and $\|p\|_l$ is the circuit length.

Theorem 2. (Nowak, 2014; Watanabe et al., 2017) Any irreducible matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ possesses unique and finite eigenvalue.

Definition 3. (Subiono, 2015) The minimum average circuit weight is called the critical circuit of $G(A)$.

Definition 4. (Nowak, 2014; Watanabe & Watanabe, 2014) Critical graph of $G(A)$, denoted $G^c(A) = \left(V^c(\,G(A)), E^c(\,G(A))\right)$ composes of all vertices and arcs which belong to critical circuits in $G(A)$.

Lemma 1. (Nowak, 2014) Let $G(A)$ be a graph containing at least one circuit. As a result, any circuit in $G^c(A)$ is critical.

Definition 5. (Nowak, 2014) We define the several operations in $\mathbb{R}_{\varepsilon'}^{n\times n}$.

1. If $A, B \in \mathbb{R}_{\varepsilon'}^{n\times n}$ then $(A \oplus' B)_{ij} = min\{a_{ij}, b_{ij}\}$.
2. If $A, B \in \mathbb{R}_{\varepsilon'}^{n\times n}$ then $(A \otimes B)_{ij} = \oplus'^{\,n}_{k=1}(a_{ik} \otimes b_{kj})$.
3. If $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $\alpha \in \mathbb{R}_{\varepsilon'}$ then we have $(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij}$.
4. We have an identity matrix over min-plus algebra with respect to $\otimes$, where

$$I = \begin{pmatrix} 0 & \cdots & \varepsilon' \\ \vdots & \ddots & \vdots \\ \varepsilon' & \cdots & 0 \end{pmatrix},$$

such that $A \otimes I = A = I \otimes A$.

5. Given $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $k \in \mathbb{N}$, the k-power of A is defined by

$$\underbrace{A^{\otimes k} = A \otimes \ldots \otimes A}_{k \text{ times}}.$$

Definition 6. (Siswanto, Gusmizain, et al., 2021) For a matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ the permanent of A is defined as

$$perm(A) = \bigoplus_{\sigma \in P_n}{}' \bigotimes_{i=1}^{n} (a_{i\sigma(i)})$$

with σ and P_n is a set of all permutations from $\{1,2, \ldots, n\}$.

Theorem 3. (Nowak, 2014) Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. If the average weight of any circuit in $G(A)$ is greater than or equal to zero,

$$A^{+} = \bigoplus_{k=1}^{\varepsilon'}{}' A^{\otimes k}. \tag{2}$$

Definition 7. (Rahayu et al., 2021) Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $\lambda(A)$ be defined as it is in Theorem 1. A matrix A_λ can be defined as

$$A_\lambda = a_{ij} - \lambda(A), \tag{3}$$

The matrix A_λ^{+} is further defined as

$$A_\lambda^{+} = A_\lambda \oplus' A_\lambda^{\otimes 2} \oplus' \ldots \oplus' A_\lambda^{\otimes n}, \tag{4}$$

Then, define the matrix A_λ^{*} as follows

$$A_\lambda^{*} = I \oplus' A_\lambda^{+}. \tag{5}$$

C. RESULT AND DISCUSSION

1. Characteristic Min-Polynomial

The characteristic polynomial over min-plus algebra will be discussed in this section. Before discussing further about this topic, we would like to form a polynomial in min-plus algebra and its smallest corner. The following will be presented the definition of this polynomial, called min-polynomial, and the theorem regarding the smallest corner of min-polynomial.

Definition 8. The min-polynomial is expressed by the form

$$p(z) = \bigoplus_{r=1}^{s}{}' c_r \otimes z^{j_r}, \qquad c_r, j_r \in \mathbb{R}$$

where the number j_s is called the degree of $p(z)$, $s + 1$ is called its length, and z^{j_r} denotes the j_r power of z concerning the addition ($\otimes$) operation.

Theorem 4. The smallest corner of $p(z)$ is

$$\bigoplus_{r=0}^{s-1}{}' \frac{c_r - c_s}{j_s - j_r}.$$

Proof. A corner exists since $p > 0$. Let γ be the smallest corner of $p(z)$. Then

$$c_s \otimes z^{j_s} \leq c_r \otimes z^{j_r}$$

for all $r = 0,1, \dots, s$ and for all $z < \gamma$. At the same time there is an $r < s$ such that

$$c_s \otimes z^{j_s} > c_r \otimes z^{j_r}$$

for all $z \geq \gamma$. Hence $\gamma = \bigoplus'_{r=0,1,\dots,s-1} \gamma_r$, where γ_r is the intersection point of $c_s \otimes z^{j_s}$ and $c_r \otimes z^{j_r}$, that is,

$$\gamma_r = \frac{c_r - c_s}{j_s - j_r}.$$

■

The min-polynomial and its smallest corner are important for the following discussion. After getting the concept of min-polynomial, we will discuss the characteristic min-polynomial. The characteristic min-polynomial is a form of polynomial that is invariant under square matrix congruence over min-plus algebra. The following definition explains how to determine the characteristic min-polynomial.

Definition 9. Let $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$. Then the characteristic min-polynomial of A is defined to be

$$\chi_A(x) = perm(I \otimes x \oplus' A) = perm\begin{pmatrix} x \oplus' a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & x \oplus' a_{nn} \end{pmatrix}.$$

According to the concept of permanent, it immediately follows from this definition that $\chi_A(x)$ is of the form

$$x^n \oplus' \delta_1 \otimes x^{n-1} \oplus' \dots \oplus' \delta_{n-1} \otimes x \oplus' \delta_n. \tag{6}$$

Based on the theorem regarding the smallest corner of a min-polynomial, we may now extend the term smallest corner to the characteristic min-polynomial. By Theorem 4, we know that the smallest corner of a min-polynomial $p(z)$ is

$$\bigoplus_{r=0}^{s-1}{}' \frac{c_r - c_s}{j_s - j_r}.$$

If $p(z) = \chi_A(x)$ where $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ then $p = n$, $j_r = r$, and $c_r = \delta_{n-r}$ for $r = 0,1,\dots,n$ with $c_0 = \delta_0 = 0$. Hence the smallest corner of $\chi_A(x)$ is

$$\bigoplus_{r=0}^{n-1}{}' \frac{\delta_{n-r}}{n-r}$$

or, equivalently

$$\bigoplus_{k=1}^{n}{}' \frac{\delta_k}{k}. \tag{7}$$

2. Solve the Eigenproblem

a. Eigenvalue

Theorem 5. If $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ then the smallest corner of $\chi_A(x)$ is $\lambda(A)$.

Proof. Let A be a matrix over min-plus algebra and $G(A)$ be a graph representation of A. Given a circuit $C = a_{12}, a_{23}, \dots, a_{r1}$ in graph $G(A)$, let $|C|$ denote the circuit weight, which can be written

$$|C| = a_{12} \otimes a_{23} \otimes \dots \otimes a_{r1},$$

and let $\|C\|$ denote the circuit length, which can be written

$$\|C\| = r.$$

Thus by (1),

$$\lambda(A) = \min_{p \in C(A)} \frac{|C|}{||C||}. \tag{8}$$

Let the eigenvalue of A be obtained from a circuit with length R, accordingly,

$$\lambda(A) = \frac{a_{12} \otimes a_{23} \otimes \ldots \otimes a_{R1}}{R}$$

and

$$R\lambda(A) \leq perm \begin{pmatrix} a_{11} & \cdots & a_{1R} \\ \vdots & \ddots & \vdots \\ a_{R1} & \cdots & a_{RR} \end{pmatrix}. \tag{9}$$

Then from (8), for each circuit in $G(A)$

$$\lambda(A)||C|| \leq |C|. \tag{10}$$

But if P denotes matrix A or any principal submatrices of A, we have, with a self-evident notation,

$$perm(P) = \bigoplus_{p}{}' |C_1| \otimes \ldots \otimes |C_p|, \tag{11}$$

where $|C_1| \otimes \ldots \otimes |C_p|$ is equal to the number of edges of P $\left(order(P)\right)$ for each circuit P in graph $G(A)$. As a result of (10) and (11),

$$\lambda(A)\left(order(P)\right) \leq perm(P), \tag{12}$$

and from (9) and (12), it is satisfying that

$$\lambda(A) = \min_{p \in C(A)} \frac{perm(P)}{order(P)}.$$

Now consider a characteristic min-polynomial of the form (6) and the smallest corner of the characteristic min-polynomial referring to (7). It can be concluded that

$$\delta_k = \bigoplus_{P \in P_k(A)}{}' perm(P)$$
$$\frac{\delta_k}{k} = \min\left(\frac{perm(P)}{order(P)}\right) = \lambda(A).$$

■

b. Eigenvector

Theorem 6. If $(A_\lambda^+)_{vv} = 0$, then the v-th columns of A_λ^+ are eigenvectors corresponding to $\lambda(A)$.

Proof. First, let $\lambda(A)$ be the minimum average circuit weight of $G(A)$. Based on Definition 7, for each circuit of $G(A)$ with average weight γ_r, there exists a comparable circuit in $G(A_\lambda)$ with average weight $\gamma_r - \lambda(A)$ such that the minimum average circuit weight of $G(A_\lambda)$ is 0. In other terms, if the minimum average circuit weight of $G(A)$ is $\lambda(A)$, then the minimum average circuit weight of $G(A_\lambda)$ is 0. Consequently, by Lemma 1, for each $v \in V^c\big(G(A)\big)$,

$$(A_\lambda^+)_{vv} = 0. \tag{13}$$

In other words, it shows that the diagonal elements of matrix A_λ^+ is equal to zero if the circuits v in $G(A)$ is critical. Then on (5), it is known that

$$A_\lambda^* = I \oplus' A_\lambda^+,$$

which shows that

$$(A_\lambda^*)_{iv} = (I \oplus' A_\lambda^+)_{iv} = \begin{cases} \varepsilon' \oplus' (A_\lambda^+)_{iv}, & i \neq v \\ 0 \oplus' (A_\lambda^+)_{iv}, & i = v \end{cases}. \tag{14}$$

According to (13) and (14), it satisfies

$$(A_\lambda^+)_v = (A_\lambda^*)_v \tag{15}$$

Next, consider the following algebraic manipulations of (2), (4), and (5).

$$\begin{aligned} A_\lambda^+ &= A_\lambda \oplus' A_\lambda^{\otimes 2} \oplus' \ldots \oplus' A_\lambda^{\otimes n-1} \oplus' A_\lambda^{\otimes n} \oplus' \ldots \\ &= A_\lambda \otimes \left(I \oplus' A_\lambda \oplus' A_\lambda^{\otimes 2} \oplus' \ldots \oplus' A_\lambda^{\otimes n-1} \oplus' A_\lambda^{\otimes n} \oplus' \ldots\right) \\ &= A_\lambda \otimes (I \oplus' A_\lambda^+) \\ A_\lambda^+ &= A_\lambda \otimes A_\lambda^* \\ A_\lambda^* &= -A_\lambda \otimes A_\lambda^+ \end{aligned} \tag{16}$$

Thus, we make a substitution from (16) to (15), getting

$$\begin{aligned} (A_\lambda^+)_v &= -A_\lambda \otimes (A_\lambda^+)_v \\ (A_\lambda^+)_v &= -(-\lambda(A) \otimes A) \otimes (A_\lambda^+)_v \\ (A_\lambda^+)_v &= \lambda(A) \otimes (-A) \otimes (A_\lambda^+)_v \\ A \otimes (A_\lambda^+)_v &= \lambda(A) \otimes (A_\lambda^+)_v. \end{aligned}$$

■

D. CONCLUSION AND SUGGESTIONS

The discussions concluded that the characteristic min-polynomial of $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is the permanent of $I \otimes x \oplus' A$, the smallest corner of $\chi_A(x)$ is the principal eigenvalue $\left(\lambda(A)\right)$, and the columns of A_λ^+ with zero diagonal elements are eigenvectors corresponding to $\lambda(A)$.

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REFERENCES

Bang-Jensen, J., & Gutin, G. (2018). *Classes of Directed graphs*. Springer. https://doi.org/10.1007/978-3-319-71840-8

Butcovic, P. (2010). *Max-linear Systems: Theory and Algorithms*. Springer. https://link.springer.com/book/10.1007/978-1-84996-299-5

Chartrand, G., Lesniak, L., & Zhang, P. (2015). *Graphs And Digraphs* (6th ed.). CRC press. https://doi.org/10.1201/b19731

Chaturvedi, M., & McConnell, R. M. (2017). A Note On Finding Minimum Mean Cycle. *Information Processing Letters*, *127*, 21–22. https://doi.org/10.1016/j.ipl.2017.06.007

Gyamerah, S. A., Boateng, P. K., & Harvim, P. (2015). Max-plus Algebra and Application to Matrix Operations. *British Journal of Mathematics & Computer Science*, *12*(3), 1–14. https://doi.org/10.9734/bjmcs/2016/21639

Jones, D. (2021). Matrix roots in the max-plus algebra. *Linear Algebra and Its Applications*, *631*, 10–34. https://doi.org/10.1016/j.laa.2021.08.008

Kim, Y. K., & Pipattanajinda, N. (2017). New method for finding the determinant of a matrix. *Chamchuri Journal of Mathematics*, *9*(2017), 1–12. https://math.sc.chula.ac.th/cjm/content/new-method-finding-determinant-matrix

Liesen, J., & Mehrmann, V. (2015). *Linear Algebra*. Springer. https://doi.org/10.1007/978-3-319-24346-7

Maclagan, D., & Sturmfels, B. (2015). *Introduction to Tropical Geometry*. American Mathematical Society. www.ams.org/bookpages/gsm-161

Mufid, M. S., & Subiono. (2014). Eigenvalues and Eigenvectors of Latin Squares in Max-Plus Algebra. *Journal of the Indonesian Mathematical Society*, *20*(1), 37–45. https://doi.org/10.22342/jims.20.1.178.37-45

Myšková, H. (2016). Interval max-plus matrix equations. *Linear Algebra and Its Applications*, *492*, 111–127. https://doi.org/10.1016/j.laa.2015.10.031

Nishida, Y., Watanabe, S., & Watanabe, Y. (2020). On The Vectors Associated with the Roots of Max-Plus Characteristic Polynomials. *Applications of Mathematics*, *65*(6), 785–805. https://doi.org/10.21136/AM.2020.0374-19

Nowak, A. W. (2014). *The Tropical Eigenvalue-Vector Problem from Algebraic, Graphical, and Computational Perspectives* [Honors Theses, Bates College]. http://scarab.bates.edu/honorstheses/97

Rahayu, E. W., Siswanto, S., & Wiyono, S. B. (2021). Masalah Eigen Dan Eigenmode Matriks Atas Aljabar Min-Plus. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *15*(4), 659–666. https://doi.org/10.30598/barekengvol15iss4pp659-666

Rosenmann, A., Lehner, F., & Peperko, A. (2019). Polynomial convolutions in max-plus algebra. *Linear Algebra and Its Applications*, *578*, 370–401. https://doi.org/10.1016/j.laa.2019.05.020

Siswanto, Gusmizain, A., & Wibowo, S. (2021). Determinant of a matrix over min-plus algebra. *Journal of Discrete Mathematical Sciences and Cryptography*, *24*(6), 1829–1835. https://doi.org/10.1080/09720529.2021.1948663

Siswanto, Kurniawan, V. Y., Pangadi, & Wiyono, S. B. (2021). Characteristic polynomial of matrices over interval max-plus algebra. *AIP Conference Proceedings*, *2326*(1). https://doi.org/10.1063/5.0039779

Siswanto, Suparwanto, A., & Rudhito, M. A. (2016). Strongly Regular Matrices and Simple Image Set in Interval Max-Plus Algebra. *JP Journal of Algebra, Number Theory and Applications*, *38*(1), 63–78. https://doi.org/10.17654/NT038010063

Sobamowo, M. G. (2016). On the Extension of Sarrus' Rule to n × n (n > 3) Matrices: Development of New Method for the Computation of the Determinant of 4 × 4 Matrix. *International Journal of Engineering Mathematics*, *2016*, 1–14. https://doi.org/10.1155/2016/9382739

Subiono. (2015). *Aljabar Min-Max Plus dan Terapannya*. Institut Teknologi Sepuluh Nopember.

Suprayitno, H. (2017). Correctness Proof of Min-plus Algebra for Network Shortest-Paths Simultaneous Calculation. In *Journal of Technology and Social Science (JTSS)-61-J. Tech. Soc. Sci* (Vol. 1, Issue 1). http://jtss.e-jikei.org/issue/archives/vol01_no01/9_A010/Camera%20ready%20manuscript_JTSS_A010.pdf

Tunisa, K., Wijayanti, K., & Budhiati Veronica, R. (2017). Nilai Eigen Dan Vektor Eigen Matriks Atas Aljabar Max-Plus. *UNNES Journal of Mathematics*, *6*(2), 189–197. https://journal.unnes.ac.id/sju/index.php/ujm/article/view/20483

Umer, M., Hayat, U., & Abbas, F. (2019). An Efficient Algorithm for Nontrivial Eigenvectors in Max-Plus Algebra. *Symmetry*, *11*(6), 1–9. https://doi.org/10.3390/sym11060738

Wang, C., Xia, Y., & Tao, Y. (2021). Ordered Structures of Polynomials over Max-Plus Algebra. *Symmetry*, *13*(7), 1–16. https://doi.org/10.3390/sym13071137

Watanabe, S., Tozuka, Y., Watanabe, Y., Yasuda, A., & Iwasaki, M. (2017). Two characteristic polynomials corresponding to graphical networks over min-plus algebra. *ArXiv*, 1–16. https://doi.org/10.48550/arXiv.1705.09513

Watanabe, S., & Watanabe, Y. (2014). Min-Plus Algebra and Networks. *Research Institute for Mathematical Sciences Kyoto University*, *47*, 41–54. http://hdl.handle.net/2433/226217

Yonggu, K., & Hee, S. H. (2018). Eigenspaces of Max-Plus Matrices: An Overview. *Journal for History of Mathematics*, *31*(1), 1–17. https://doi.org/10.14477/jhm.2018.31.1.001