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The Growth of Students' Function Limit Concepts Understanding in Solving Controversial Problems Based on Pirie Kieren's Theory

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ABSTRACT

Almost all students understand the limit of a function only up to an intuitive definition and have difficulty understanding the concept of a limit function formally. This study aims to describe the growth of student understanding of functions limit concept in solving controversial problems based on Pirie Kieren's theory. There were twelve Calculus class students in the short semester as participants. The students selected were those who had taken calculus courses. Students are given the task of solving controversial problems to understand the concept of limit functions. There was only one student who showed a growing understanding of the concept of the limit of a function and was interviewed for further exploration. This research is a qualitative descriptive research. Therefore, the researchers analyzed the results of students' work through data reduction, data presentation, and conclusion drawing. The result shows that through controversial problems, students' understanding grows to an inventising level. However, students did 'fold back' at the observing level. At this level, students look at or re-read their notebooks to recall previously owned concepts. For further research it is suggested that researchers can design a learning process that can help grow student understanding through controversial problems.

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A. INTRODUCTION

Mathematical understanding is an important thing that someone must have in learning mathematics and problem-solving (Rahayuningsih et al., 2022; Yang et al., 2021). Mathematical understanding is dynamic, shifting, changing, evolving, sustainable, and growing to build connected understanding (Pirie & Kieren, 1994). Therefore, mathematical understanding can grow over time (Gulkilik et al., 2020). This mathematical growth can be seen through the Pirie-Kirien theory (Pirie & Kieren, 1994; Syafiqoh et al., 2018). Pirie and Kieren's theory of understanding acts as a lens for observing the growth of understanding in a learning process (Gokalp & Bulut, 2018). There are eight layers of understanding in the Pirie-Kieren theory, including primitive knowing, image making, an image having, property noticing, formalizing, structuring, observing, and inventising as shown in Figure 1 (Gulkilik et al., 2020; Pirie & Kieren, 1994; Yao, 2020; Yao & Manouchehri, 2020).

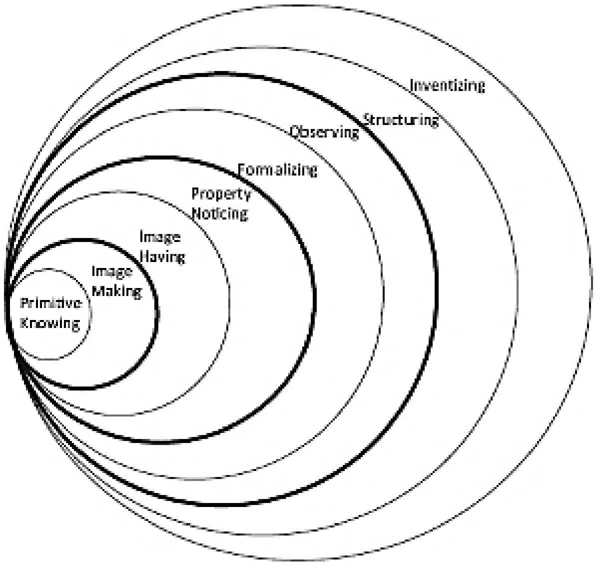


Figure 1. The Pirie-Kieren theory of mathematical understanding (Pirie & Kieren, 1994)

Each layer of Pirie-Kieren's understanding is a continuous and non-linear process (Pirie & Kieren, 1994). Someone whose understanding is already at the formalizing layer may return to the image-having layer. The process of going back in understanding into the previous layer is known as the ‘folding back’ (Patmaniar et al., 2021; Pirie & Kieren, 1994; Susiswo et al., 2019; Yao, 2020; Yao & Manouchehri, 2020, 2022). Someone can fold back to find a solution to the problems they face. By doing folding back, one's understanding of the previous stage becomes "thicker". Therefore, mathematical understanding grows when someone solves a problem Patmaniar et al. (2021) and learn something new (Ahmadpour et al., 2019). The mathematical problems that can help the growth of understanding are in the form of controversial problems. This is because controversial problems cause a clash of thoughts that can encourage someone to examine the problem more deeply (Subanji et al., 2021).

Controversial problems are problems that generate debate because of different points of view (Simic-Muller et al., 2015; Subanji et al., 2021). Controversy can occur when someone faces a problem that is different from normal problem Subanji et al. (2021) and contrary to the existing scheme in mathematics (Walida et al., 2022). This problem arises due to an understanding of a problem that has not been resolved, causing conflict in one's mind (Subanji et al., 2021; Walida et al., 2022). Although this controversial problem can lead to conflict, it can make students' understanding of the problem deeper (Mueller & Yankelewitz, 2014).

In fact, most students are less able to explore the mathematical understanding they have during learning (Gokalp & Bulut, 2018). The problems given are just ordinary problems so that students are less challenged by the problems. This is in accordance with the findings of researchers in the field. Most students are still often given procedural problems. Students' understanding cannot be explored further because students' problem solving only relies on memorization. Apart from that, students cannot explain the completion process well. Therefore, controversial problems can be given to students Walida et al. (2022) to see the development of the students' mathematical understanding.

Research on controversial problems in mathematics is still focused on reasoning Simic-Muller et al. (2015); Subanji et al. (2021), higher-order thinking Rosyadi et al. (2022); Suryawan et al. (2023), metacognitive Walida et al. (2022), and creative thinking (Subanji et al., 2023). The results of Subanji et al. (2021) research are the characteristics of controversial mathematical reasoning which consists of three levels, namely initial, exploration, and clarification. Rosyadi et al. (2022) researched students' higher-order thinking abilities through presenting controversial problems. The results of their research showed that students reached the analysis and evaluation stage, but there was one student who was able to reach the creating stage. Walida et al. (2022) researched students' metacognitive strategies when solving controversial mathematics problems. Meanwhile, Subanji et al. (2023) research produced five levels of creative models by providing controversial questions, including: pre-imitation, imitation, modification, combination, and construction.

From the results of previous research, controversial mathematics problems have not been studied in terms of the growth of student understanding. Meanwhile, controversial problems can be used to dig deeper into students' understanding (Mueller & Yankelewitz, 2014). Controversial problems have not been studied further related to students growing understanding. It is important to examine the growth of students' understanding through the provision of controversial problems. Therefore, the aim of this research is to describe the growth of students' function limit concepts understanding in solving controversial problems based on Pirie Kieren's Theory.

B. METHODS

This research is a qualitative descriptive research that describes the growth of students' understanding of function limit concepts in solving controversial problems based on Pirie Kieren's theory (Cohen et al., 2018). There were twelve students of the Calculus course in the short semester, at one of the public universities in Malang, who were retaking the Calculus course. The characteristics of the twelve students are represented in Table 1.

Table 1. The Characteristics of The Twelve Students

Name	Gender	Growth in Limit Understanding
SM1	Male	Inventising
SM2, SM3	Male	Observing
SM4	Female	Observing
SM5	Female	Formalizing
SM6	Female	Property noticing
SM7, SM8, SM9, SM10	Male	No understanding grows
SM11, SM12	Female	No understanding grows

In this research, eight worksheets were provided based on layers of understanding of Pirie Kieren's theory. An interview guideline was also provided. The worksheets contained controversial problems, while the interview guideline contained questions according to the layers of understanding of Pirie and Kieren's theory, namely primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, to inventising (Pirie & Kieren, 1994). The procedure of this study is represented in Figure 2.

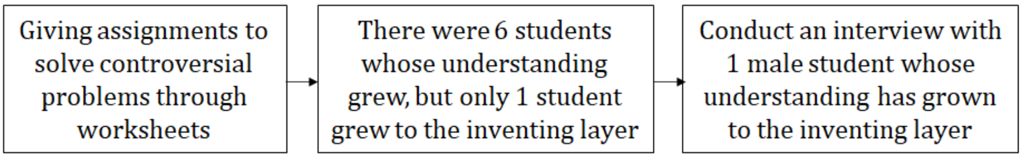
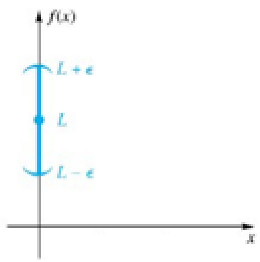
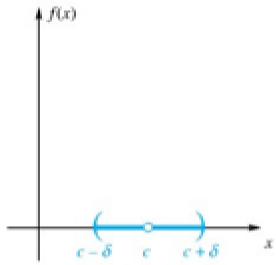
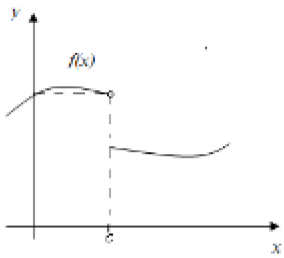


Figure 2. The Research Procedure

The students were tasked to solve the following controversial problems of Limits of Functions in the worksheet, as shown in Table 2.

Table 2. Design of Controversial problem-Solving Tasks based on Pirie Kieren's Theory

Task no	Layers of Understanding on Pirie Kieren's Theory	Controversial Problems
I	Primitive knowing	1. Consider the following statement. "$\lim_{x\to c} f(x) = L$ *if and only if* $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$" Do you agree that this statement is the definition of a limit? Explain! 2. A student stated that "If it is known that any $f(x)$, for x approaches a different c, then the value $\lim_{x\to c_1} f(x) = \lim_{x\to c_2} f(x)$" Do you agree with this student? Why is that?
II	Image making	1. Pay attention to a student solution regarding the limit as follows: $\lim_{x\to 3}\frac{x^2-x-6}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2) = 5.$ At the solution, the student explained that $\lim_{x\to 3}\frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2)$ because $\frac{x-3}{x-3} = 1$. Do you agree with the student's explanation? Explain in detail! 2. One student explained that: "The value of $\lim_{x\to 1}\frac{x^2-1}{x-1}$ does not exist, because the value $f(x)$ is undefined for $x = 1$" Do you agree with the student's explanation? Give reasons by using table visualizations and graphical images of the function!
III	Image having	1. A student draws a solution to the inequality $0 < \lvert x-2\rvert < 1$ as follows.  Is the student's solution true? Explain in detail! 2. Jiso and Jeni are discussing the absolute value inequalities depicted in Figures A and B.

		 Gambar A  Gambar B Jiso states that Figure A represents the absolute value inequality $\|f(x) - L\| > \varepsilon$ and Figure B represents the absolute value inequality $\|x - c\| > \delta$. However, Jeni denies that it should be $\|f(x) - L\| < \varepsilon$ dan $\|x - c\| > \delta$. Who do you agree more with?
IV	Property Noticing	1. Ria, a university student, is looking at the graphic image of $f(x)$ below.  She states that "for every $\varepsilon > 0$ however small, we can find $\delta > 0$ such that $0 < \|x - c\| < \delta \Rightarrow \|f(x) - L\| < \varepsilon$".
V	Formalizing	1. Tia explains the formal definition of Limit of a Function as follows. *"A function f that is defined in the domain $x_1 < x = c < x_2$ is said to approach L if $x = c$, and written $\lim_{x \to c} f(x) = L$, if given any positive number ε, then there is a positive integer δ so that $\|f(x) - L\| < \varepsilon$ provided $0 < \|x - c\| < \delta$."* Is Tia's explanation true? Give good reasons!
VI	Observing	1. Salma, Nabila, and Novia are having a discussion regarding the formal definition of Limit of a Function. Salma says that $\lim_{x \to 2}(-x) = -2$, while Nabila says that $\lim_{x \to 2}(-x) = 2$, and Novia says $\lim_{x \to 2}(-x)$ does not exist. Who do you think is right? Prove it by using the formal definition of Limit of Function! 2. Roni is presenting and explaining the proof $\lim_{x \to 1}(2x - 1) = 1$ "We can choose $\delta = \frac{\varepsilon}{2}$ or even greater, then for every $\varepsilon > 0$ there exist $\delta = \frac{\varepsilon}{2}$ so that $0 < \|x - 1\| < \delta \Rightarrow \|(2x - 1) - 1\| < \varepsilon$" Paul disputes Roni's statement, then states that $\delta > \frac{\varepsilon}{2}$ is impossible. Do you, as their classmate, agree more with Roni's opinion or Paul's? Explain using the definition of Limit of a Function!
VII	Structuring	1. Anggi is proving a Limit of a Function theorem as follows.

		Suppose k is a constant, and f is a function so that it has a limit at point c. So $$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$$ *Proof:* *The theorem is proven as follows. If* $k = 0$ *is valid, so it is proved for* $k \neq 0$. *Because f has a limit at point c, then for any* $\varepsilon > 0$, *there exists* $\delta > 0$, *so that* $$0 < \lvert x - c \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \frac{\varepsilon}{\lvert k \rvert}.$$ *We use* $\delta = \frac{\varepsilon}{k}$ for any $\varepsilon > 0$. Take note: $$\lvert kf(x) - kL \rvert = k\lvert f(x) - L \rvert < k\frac{\varepsilon}{k} = \varepsilon.$$ Is the proof of the theorem that Anggi made true? Explain!
VIII	Inventising	1. Alya and her group mates are discussing the Limit theorem as follows. f and g are continuous functions that have a limit at c then holds $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$ *Alya posed a question to her group mates whether the theorem applies to all* $f(x)$ *and* $g(x)$. *Then one of her group mates, Reza, said that this applies to all* $f(x)$ *and* $g(x)$. Is Reza's statement true? Explain in detail! 2. Pay attention to the theorem below! $$\lim_{x \to c} \sqrt[n]{f(x)},$$ for n natural numbers. Is the limit theorem above true? Write questions related to the theorem above! Then discuss with your group members to get the answer!

The data is analyzed by ’making sense’ out of image and text, representing the data, and making an interpretation of the larger meaning of the data (Creswell & Creswell, 2017). Making a sense out of images and text is done by identifying worksheet answers where one student, namely SM1, is found to be suitable for interview to find out the growth of his understanding. Then the ’making sense’ is also done by reducing the results of the interview. ’Representing the data’ is done by presenting the results of the growth of the subject's understanding in a narrative and visual manner concerning the theoretical indicators of Pirie and Kieren (Pirie & Kieren, 1994), as shown in Table 3.

Table 3. Descriptors and Indicators at the Mathematical Understanding Layer based on Pirie Kieren's Theory

Layers of Understanding on Pirie Kieren's Theory	Descriptor	Indicator
Primitive knowing	Stating the definitions of the terms found in the problem.	Explaining the intuitive definition of the limit of a function through solving controversial problems.
Image making	Getting an idea or picture that will be used in solving the problem.	Using the intuitive definition of limit on examples of limit of a function through solving controversial problems.
Image having	Using ideas or images of problem-solving without using examples.	Using the concept of absolute value to identify the epsilon and delta values at the limit of a function by solving a controversial problem.
Property noticing	Verifying the relationship between the definitions	Identifying the existence of function limit values in the image by involving the mathematical symbols of epsilon and delta through solving controversial problems
Formalizing	Finding their own concept and using the concept found to solve the given problem.	Explaining the formal definition of the limit of a function through solving controversial problems.
Observing	Making a formal statement from the patterns found to solve the given problem.	Demonstrating the existence of δ and proving $\|f(x) - L\| < \varepsilon$ using a formal definition of limits through resolving controversial issues.
Structuring	Associating the relationship between one formula and another and being able to prove it based on logical arguments.	Proving the limit theorem by solving a controversial problem.
Inventising	Obtaining a complete structured understanding and creating new questions that can grow into a new concept.	Asking a question about a limit function theorem if the premises of the theorem are reduced.

Based on the achievements of the student's mathematical understanding layer referring to Table 3, the researcher makes 'an interpretation of the larger meaning of the data' by making detailed descriptions supported by findings on problem-solving data and interviews.

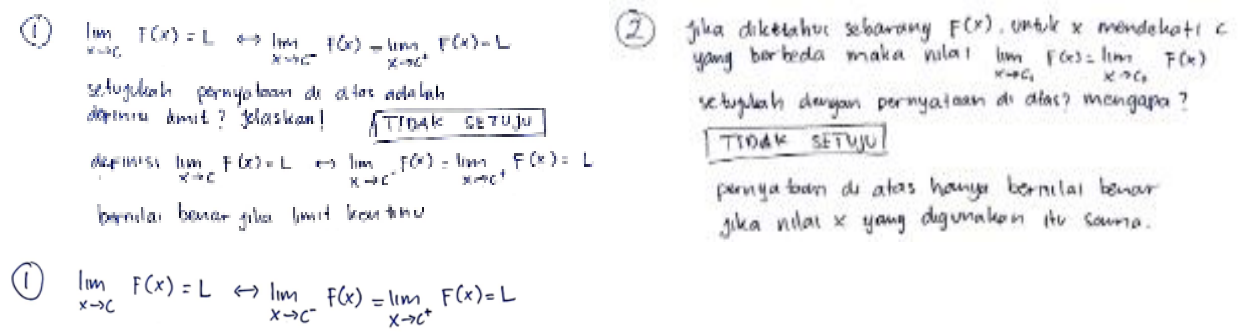
C. RESULT AND DISCUSSION

The research subjects were 12 students who were studying calculus material in the short semester. The growth of students' mathematical understanding of limit material is very diverse. However, of the 12 students, only 1 male student whose mathematical understanding reached inventising layer. The following is a portrait of SM1's answers to controversial questions. One male student, namely SM1, shows a growing understanding of the concept of limits. SM1 shows the growth of understanding of the limit concept from primitive knowing to inventising layer.

However, at the observing level, SM1 experienced a folding back in stating the formal definition of limits.

1. SM1 Explains the Intuitive Definition of Limit of a Function through Solving Controversial problems

The deepest layer of understanding in Pirie Kieren's theory is primitive knowing. This level is the initial process of understanding a new concept, combining previous knowledge with the knowledge being studied. SM1 is considered to have an appropriate understanding of primitive knowing if SM1 is able to explain the intuitive definition of limit of a function by solving controversial problems. SM1's resolution of 2 controversial problems is shown in Figure 3.



Do you agree that this statement is the definition of a limit? Explain!

No, I disagree

The definition of

The value is true if the limit continues

If it is known that any $f(x)$, for x approaches a different c, then the value $\lim_{x \to c_1} f(x) = \lim_{x \to c_2} f(x)$

Do you agree with this student? Why is that?

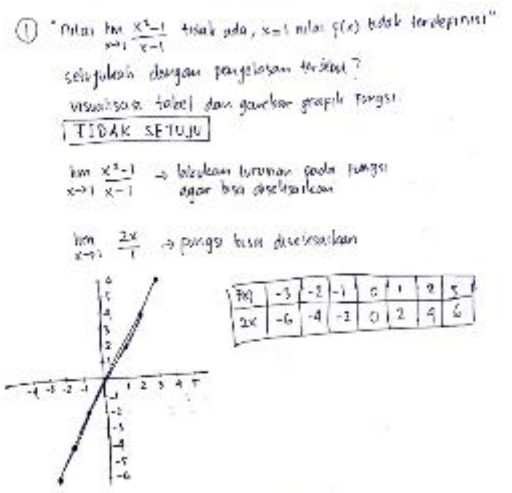
No, I disagree. The statement is true if the values of x are equal

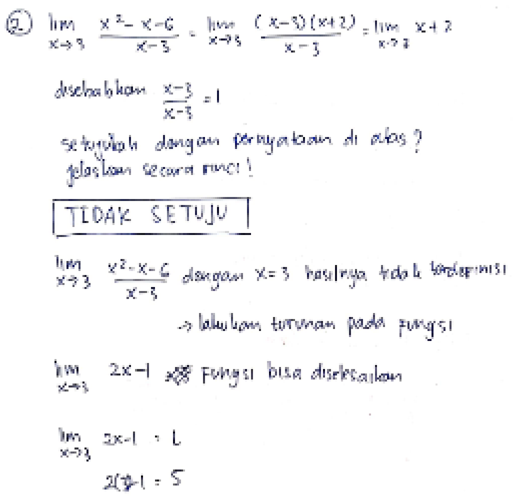
Figure 3. SM1 Problem Solving with Primitive Knowing Achievements

Based on Figure 3, it can be seen that, SM1 was able to solve controversial problems well. Of the 2 controversial problems given, SM1 answered "Disagree", by providing an explanation of the intuitive definition of limit. In the interview process, SM1 stated that "if there are left and right limits, then there are limits" and SM1 stated that the values between the left and right limits may be the same.

2. SM1 Uses the Intuitive Definition of Limits on Examples of limits of functions through Solving Controversial problems

The next layer of understanding in Pirie Kieren's theory is image making. In this level, SM1 was able to cast shadows by performing effective actions. For example, providing alternative solutions through examples. SM1's resolution of 2 controversial problems individually is shown in Figure 4.





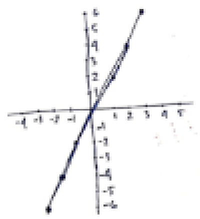
(1)The value of $\mathop{lim}\limits_{x\to 1}\frac{x^2-1}{x-1}$ does not exist, because the value $f(x)$ is undefined for $x = 1$"

Do you agree with the student's explanation? Give reasons by using table visualizations and graphical images of the function!

No, I disagree

$\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ doing the derivative on the function for it can be solved

$\lim\limits_{x\to 1}\frac{2x}{1}$ *function can be done*



(2) $\lim\limits_{x\to 3}\frac{x^2-x-6}{x-3} = \lim\limits_{x\to 3}\frac{(x-3)(x-2)}{x-3} = \lim\limits_{x\to 3} x+2$ because $\frac{x-3}{x-3} = 1$. Do you agree with the statement? Explain in detail!

No, I disagree

$\lim\limits_{x\to 3}\frac{x^2-x-6}{x-3}$ $with\ x = 3\ the\ value\ is\ underfined, doing\ the\ darivative\ on\ the\ function$

$\lim\limits_{x\to 3} 2x-1\ function\ can\ be\ done$

$\lim\limits_{x\to 3} 2x-1 = L$

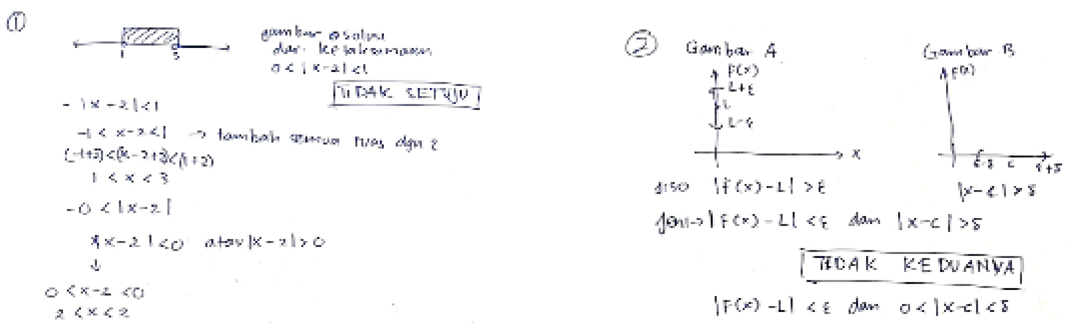
$2(3)-1 = 5$

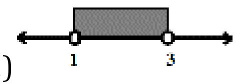
Figure 4. SM1 Problem Solving with Image-Making Achievements

Based on Figure 4, it can be seen that, SM1 was able to solve controversial problems well. Of the 2 controversial problems given, SM1 answered "Disagree". In the interview process, SM1 explained that the visualization of the graphs and tables did not support solving controversial problems. SM1 realized that the solution he made was wrong. SM1 created a graph using the derivative function of the given function. SM 1 argued that the numerator and denominator must be lowered first in order to solve this problem. As for the explanation in the next problem, SM1 stated the same reason, namely the numerator and denominator must be lowered first in order to solve this problem. Apart from that, SM1 also stated that if the limit value does not exist, then the limit function is undefined.

3. SM1 Uses the Concept of Absolute Value to Identify Epsilon and Delta Values at Functional Limits through Solving Controversial Problems

The next layer of understanding in Pirie Kieren's theory is image having. At this level, SM1 was able to customize and manipulate problems without having to solve examples. This level represents the first level of abstraction performed by SM1. SM1's resolution of 2 controversial problems is shown in Figure 5.

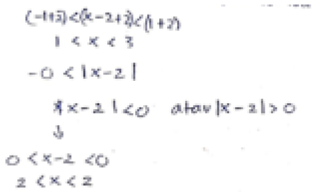


(1)  picture is a solution of inequality $0 < |x - 2| < 1$

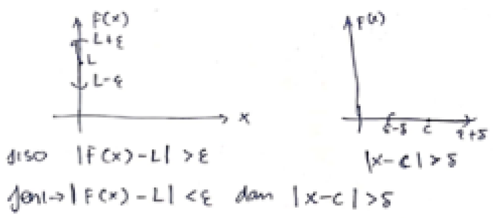
No, I disagree

$-|x - 2| < 1$

$-1 < |x - 2| < 1$ *add up of* 2 *for all statement*



(2) Picture A and Picture B



None of them

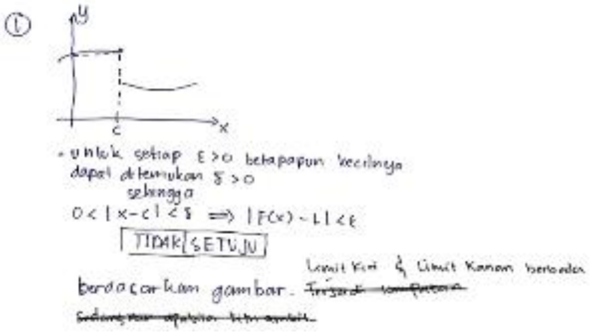
Figure 5. SM1 Problem Solving with Image Having Achievements

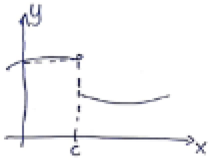
Based on Figure 5, it can be seen that, SM1 was less able to resolve controversial problems properly. Of the 2 controversial problems given, SM1 answered "Disagree" without providing relevant reasons for his answer. In the interview process, SM1 was less able to explain inequalities. SM1 was unable to describe the absolute value inequality in problem number 1.

Meanwhile, in problem number 2, SM1 was able to explain the positions of epsilon and delta. So, it can be concluded that SM1 fulfills the third indicator, it's just that SM1 needed to re-learn about absolute value inequalities.

4. SM1 Identifies the Existence of Limit Values of Functions in the Figures by Involving Epsilon and Delta Mathematical Symbols through Solving Controversial Problems.

Property noticing is the fourth level of understanding in Pirie Kieren's theory. Through teamwork, SM1 was able to study the properties and know the difference between epsilon and delta. SM1's resolution of 1 controversial problem is shown in Figure 6.





For every $\varepsilon > 0$ how small it was, it can be found $\delta > 0$ so,

$0 < |x - c| < \delta \Rightarrow |F(x) - L| < \varepsilon$

No, I disagree

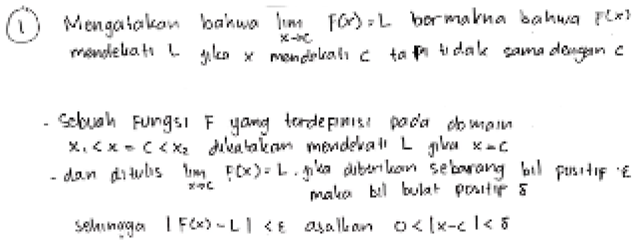
Based on the figure, limit on the left is different from limit on the right

Figure 6. SM1 Problem Solving with Property Noticing Achievements (groups)

Based on Figure 6, it can be seen that, through teamwork, SM1 was able to resolve controversial problems well. Of the controversial problems given, SM1 answered "Disagree". During the interview process, SM1 was able to explain the graphs provided and was able to explain the positions of epsilon and delta.

5. SM1 Explains the Formal Definition of Limit of a Function through Solving Controversial Problems.

Formalizing is the fifth level of understanding in Pirie Kieren's theory. At this level, SM1 was able to understand properties and make a generalization by drawing abstracts about the important features of the limit concept. The concept of limit is understood by the subject as an independent entity. Through teamwork, SM1 was able to solve 1 controversial problem shown in Figure 7.



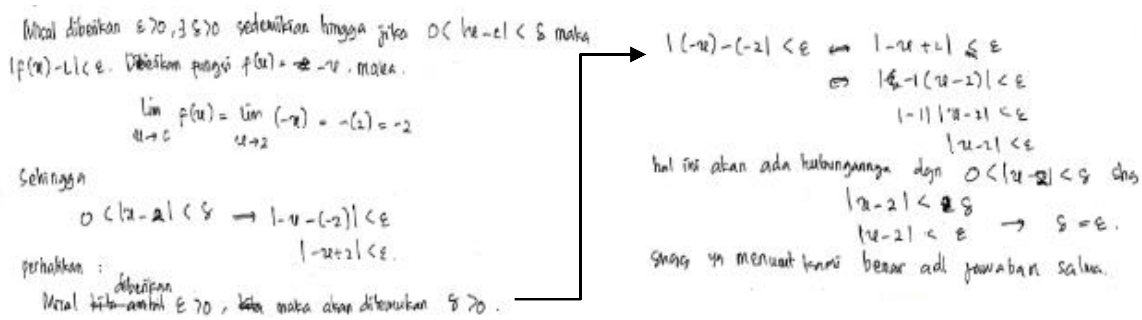
1. Stated that $\lim_{x \to c} f(x) = L$ means F(x) close to L if x x close to c, but it is different from c
   a F function which is defined on the domain $x_1 < x = c < x_2$ said close to L if x = c
   Its written $\lim_{x \to c} f(x) = L$. if it is given ε is positive, so δ is positive
   So, $|F(x) - L| < \varepsilon \; if \; 0 < |x - c| < \delta$

Figure 7. SM1 Problem Solving with Formalizing Achievements

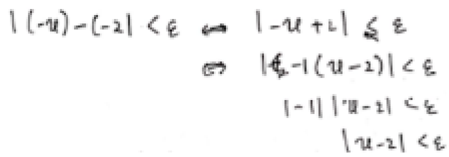
Based on Figure 7, through teamwork, it appears that SM1 was able to resolve controversial problems well. During the interview process, SM1 was able to explain the epsilon and delta functions in defining limits.

6. SM1 Points Out the existence of Delta in His Proving and Uses the Formal Limit Definition through Solving Controversial Problems.

At the Observing level, SM1 must be able to gain consistency in his mind, and accommodate the structure of knowledge to match new knowledge. Through teamwork, SM1 resolves the controversial problems as shown in Figure 8.



Supposing $\varepsilon > 0, \exists\, \delta > 0$ such as if $0 < |x - c| < \delta$ so $|f(x) - L| < \varepsilon$. The given function f(x) = -x
So , $\lim_{x \to c} f(x) = \lim_{x \to 2} (-x) = -(-2) = -2$
So that, $0 < |x - 2| < \delta \to |-x - (-2)| < \varepsilon, |-x + 2| < \varepsilon$
Pay attention to, if $\varepsilon > 0$, it will be found $\delta > 0$



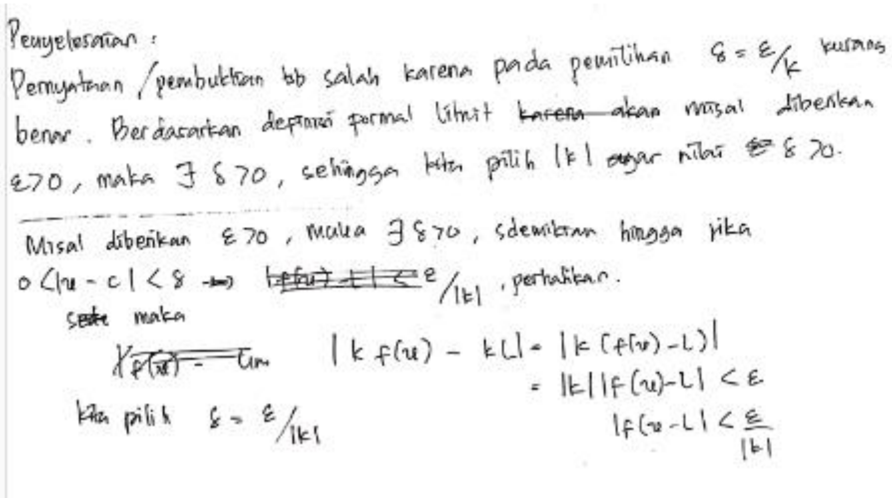
This will have a correlation with $0 < |x - 2| < \delta$ so $|x - 2| < \delta, |x - 2| < \varepsilon \;\to \delta = \varepsilon$
So that, based on our opinion, Salma's answre is right

Figure 8. SM1 Problem Solving with Observing Results

Based on Figure 8, through teamwork, it can be seen that, SM1 was able to solve 1 controversial problem. In the interview process, SM1 was able to explain logically the solution to the controversial problems given. Although, SM1 could not provide an explanation for controversial problem number 2. In the understanding layer of observing, SM1 experienced a folding back. SM1 had difficulty mentioning the formal definition of limit. SM1 needed to recall the information he obtained by looking at the notebook/package book.

7. SM1 Proves the Limit Theorem through Solving Controversial Problems

At the structuring level, SM1 must be able to translate the thinking processes into axiomatic structures. Through the teamwork, SM1 was able to solve 1 controversial problem given shown in Figure 9.

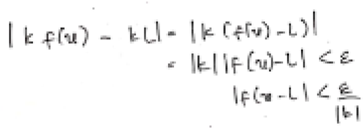


The statement/proof is wrong because of the selection of $\delta = {}^{\varepsilon}/_{k}$ is incorrect.

Based on formal limit definition, as if it was give $\varepsilon > 0$ so $\delta > 0$ so that we can choose $|k|$ to make the

value $\delta > 0$

If given $\varepsilon > 0, so\ \exists\delta > 0, so\ that\ if\ 0 < |x - c| < \delta \rightarrow {}^{\varepsilon}/_{k}$. Pay attention



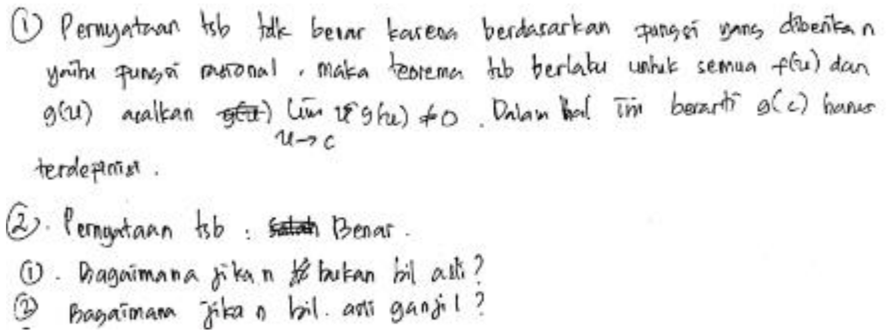
We choose $\delta = {}^{\varepsilon}/_{|k|}$

Figure 9. SM1 Problem Solving with Structuring Achievements

Based on Figure 9, through teamwork, it can be seen that, SM1 was able to resolve controversial problems well. In the interview process, SM1 was able to explain logically the solution to the controversial problems given. SM1 was able to explain limit theory and apply it to obtain a solution to the controversial problem.

8. SM1 Asks a Question about a Function Limit Theorem when the Premises of the Theorem are Reduced

Inventising is the outermost level of the layer of understanding in Pirie Kieren's theory. At this level, SM1 must be able to create new mathematical structures with previous knowledge structures. In this research, SM1 is considered to have the outermost level of understanding if he is able to ask questions about the completeness of the limit theorem. In the interview process, SM1 was able to explain logically about the solution to the controversial problem given in question number 1 as shown in Figure 10.



1. This statement is incorrect because based on the function given, which is rational function, the theorem applied in all f(x) and g(x), as long as $\lim_{x \to c} g(x) \neq 0$ In this case, g(c) must be defined
2. The statement is correct
   a. How if n is not natural number?
   b. How if n is odd natural number?

Figure 10. SM1 Problem Solving with Inventising Achievements

Based on Figure 10, SM1 was able to explain the theory of limits and mentions that the theorem given is incomplete. SM1 is also able to ask questions about the given limit theorem. Student understanding can grow through controversial problem solving activities. Student understanding grows from the primitive knowing to inventising layer, even though it experiences a folding back in the observing layer. The growth of students' understanding of the concept of limits of functions increases but not linearly. This is relevant to previous research which shows that when students' knowledge is insufficient to be used in solving new problems, students broaden their understanding by returning to deeper layers (Martin & Towers, 2016; Patmaniar et al., 2021; Puspitasari & Amir, 2020; Susiswo et al., 2019). Students returning to a deeper layer of understanding does not mean they experience a decrease in understanding, but rather to recall knowledge and apply it in new perspectives (Martin & Towers, 2016; Palha et al., 2013). Likewise, students' mathematical understanding can grow when students learn new things Ahmadpour et al. (2019), including doing problem solving activities (Patmaniar et al., 2021; Putri et al., 2023).

The growth of students' understanding of the concept of limits of functions is shown by the connection of knowledge from initial knowledge, namely intuitive definitions, with new knowledge, namely formal definitions. Meanwhile, mathematical understanding is closely related to knowledge, namely knowledge possessed with new knowledge acquired (Hiebert &

Carpenter, 1992). Primitive knowing is the foundation for the successful growth of mathematical understanding (Putri & Susiswo, 2020).

Controversial problem solving can be used as mediation to foster students' mathematical understanding. Controversial issues often cause conflict with students' understanding structures. However, this triggers students to expand their primitive knowing by exploring controversial problem situations so they can make decisions to choose strategies and determine solutions. Likewise, previous research shows that, when someone does not know directly how to obtain a solution, they need to explore the problem situation to be able to make decisions in determining a solution (Yee & Bostic, 2014). This is supported by other research which states that exploration of problems can support deeper understanding (Mueller & Yankelewitz, 2014). Meanwhile, exploration of this controversial issue can encourage someone to broaden and deepen students' understanding (Subanji et al., 2021).

D. CONCLUSION AND SUGGESTIONS

Students' understanding of function limit concepts can grow when given interventions in solving controversial problems. Students' understanding grows from the primitive knowing layer to inventising layer, even though it may experience a folding back in the observing layer. The folding back experienced by students aims to broaden their understanding by returning to deeper layers and recalling knowledge and applying it to the problems they face. The problems faced by students are controversial problems. This problem can lead students to explore more deeply their understanding of a concept and the contradictions that exist in their thinking schemes. Therefore, controversial problems can be used as a medium for students to strengthen their understanding of a particular concept.

Based on the research results, students still need some intervention from the lecturer when solving controversial problems. In addition, students were seen to be more active in giving their arguments when discussing with their friends. Therefore, for further research it is suggested that researchers can design a learning process that can help grow student understanding through controversial problems.

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