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Determining the Inverse of a Matrix over Min-Plus Algebra

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ABSTRACT

Linear algebra over the semiring $\mathbb{R}_\varepsilon$ with $\otimes$ (plus) and $\oplus$ (maximum) operations which is known as max-plus algebra. One of the isomorphic with this algebra is a min-plus algebra. Min-plus algebra that is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$, with $\otimes'$ (plus) and $\oplus'$ (minimum) operations. Given a matrix whose components are elements of $\mathbb{R}_{\varepsilon'}$ is called min-plus algebra matrices. Any matrix can be connected by an inverse. In conventional algebra, a square matrix is said an invertible matrix if the $\det(A) \neq 0$. In contrast to max-plus algebra, a matrix is said to have inverse condition if it meets certain conditions. Some concepts from the max-plus algebra can be transformed to the min-plus, because of their structural similarity. This means that the inverse matrix concept in max-plus can be constructed into a min-plus version. Thus, this study will explain the inverse of a matrix over the min-plus algebra, property of multiplying two invertible matrices, and connection between invertible matrix and linear mapping used the literature study method, with literature sources such as books, journals, articles, and theses. The data analysis technique used in this research is qualitative data analysis technique. Then, this article has a principal result that is matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ has a right inverse if and only if there are permutations of σ and the value of $\lambda_i < \varepsilon'$, $i \in \{1, 2, 3, \dots, n\}$ such that $A = P_\sigma \otimes' D(\lambda_i)$ which is the inverse of matrices. Furthermore, if B is the correct inverse that satisfies $A \otimes' B = E$ then $B \otimes' A = E$ and B is uniquely determined by A.

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A. INTRODUCTION

Max-plus algebra is a linear algebra over the semiring $\mathbb{R}_\varepsilon$, where $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ and $\varepsilon = -\infty$ with operations of a maximum ($\oplus$) and plus ($\otimes$) (Bermanei, 2021). The algebraic structure of max-plus that is semifield and idempotent commutative semiring which is denoted by $\mathbb{R}_{max} = (\mathbb{R}_\varepsilon; \oplus, \otimes)$ (Brunsch et al., 2012; Valverde-Albacete & Peláez-Moreno, 2020). Operations in max-plus can be extended into matrices. Given a matrix of dimensions n row and n column, with its components being the set $\mathbb{R}_\varepsilon$ it can be referred to as a matrix in max-plus, and represented as $\mathbb{R}_\varepsilon^{n\times n}$ (Farlow, 2009).

In conventional algebra, given a matrix $A \in \mathbb{R}^{n\times n}$ whose elements are real numbers. Related to this matrix, Rellon et al. (2023) stated that a matrix A of size $n \times n$ is called an invertible matrix and also called a nonsingular matrix if and only if $\det(A) \neq 0$. If it $\det(A) = 0$ then the matrix has no inverse and is called a singular matrix. Therefore, not all matrices in conventional algebra have an inverse. In contrast to the inverse matrix of max-plus algebra studied by (Farlow, 2009). It doesn't necessarily have an inverse to the $\otimes$ operation, so matrix

A is an invertible matrix if it meets certain conditions. One of the isomorphic tropical semirings with max-plus is a min-plus algebra (Jamshidvand et al., 2019).

Min-plus algebra is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$, where $\mathbb{R}$ be a set of real numbers and $\varepsilon' = +\infty$ with the minimum ($\oplus'$) and plus ($\otimes'$) operations (Nowak, 2014; Siswanto, Nurhayati, et al., 2021). If $a, b \in \mathbb{R}_{\varepsilon'}$, $a \oplus' b = \min(a, b)$, $a \otimes' b = a + b$ (Suroto, 2022). Furthermore, min-plus algebra has an idempotent commutative semiring structure and represented as $\mathbb{R}_{min} = (\mathbb{R}_{\varepsilon'}; \oplus', \otimes')$ (Akian et al., 2007; Rosyada et al., 2021). Min-plus algebra can also be applied in everyday life, this has been discussed by several researchers including the fastest route modeling (Suprayitno, 2017; Susilowati & Fitriani, 2019), fiber networks (Li & Zhao, 2012), and petri nets (Farhi et al., 2009). Let a matrix of size $n \times n$, whose elements are from $\mathbb{R}_{\varepsilon'}$ commonly known as matrices in min-plus algebra or $\mathbb{R}_{\varepsilon'}^{n \times n}$.

Research conducted by Gyamerah et al. (2016); Komenda et al. (2018); Schutter et al. (2020) discusses max-plus algebra. This means that many have developed related to the concept of max-plus algebra. Then $\mathbb{R}_{max}$ was expanded into min-plus algebra by Rahayu et al. (2021); Suroto (2022); Watanabe et al. (2018) who have explained the definition and properties of min-plus algebra up to its matrix. Rellon et al Rellon et al. (2023) has discussed the term invertible matrices in conventional algebra and Farlow (2009) has discussed in the max-plus algebra, Farlow has shown that the concepts of max-plus algebra are not strictly analogous to conventional algebra. Therefore, the structures in min-plus algebra and max-plus algebra are similar to conventional algebra. So, some concepts and properties in conventional algebra have versions in min-plus and max-plus algebra, but the inverse matrices in the min-plus version have not been discussed by previous researchers.

Based on the description above, this research aims to develop what has been discussed in Farlow's and Rollen's research into min-plus algebra and how it is similar and or different from the results of the max-plus algebra. Therefore, this research to determine the right inverse and left inverse matrix theorems in min-plus algebra after defining invertible matrix, permutation matrix, and diagonal matrix. This research also discusses the properties of multiplying two invertible matrix and the connection between invertible matrix and linear mapping.

B. METHODS

This research is a literature review using previous references from other studies, such as books, journals, articles, and theses. The data analysis technique used in this research is qualitative data analysis technique, qualitative research seeks to answer 'how' rather than 'what' (Mattimoe et al., 2021). The steps taken in this research can be shown in Figure 1.

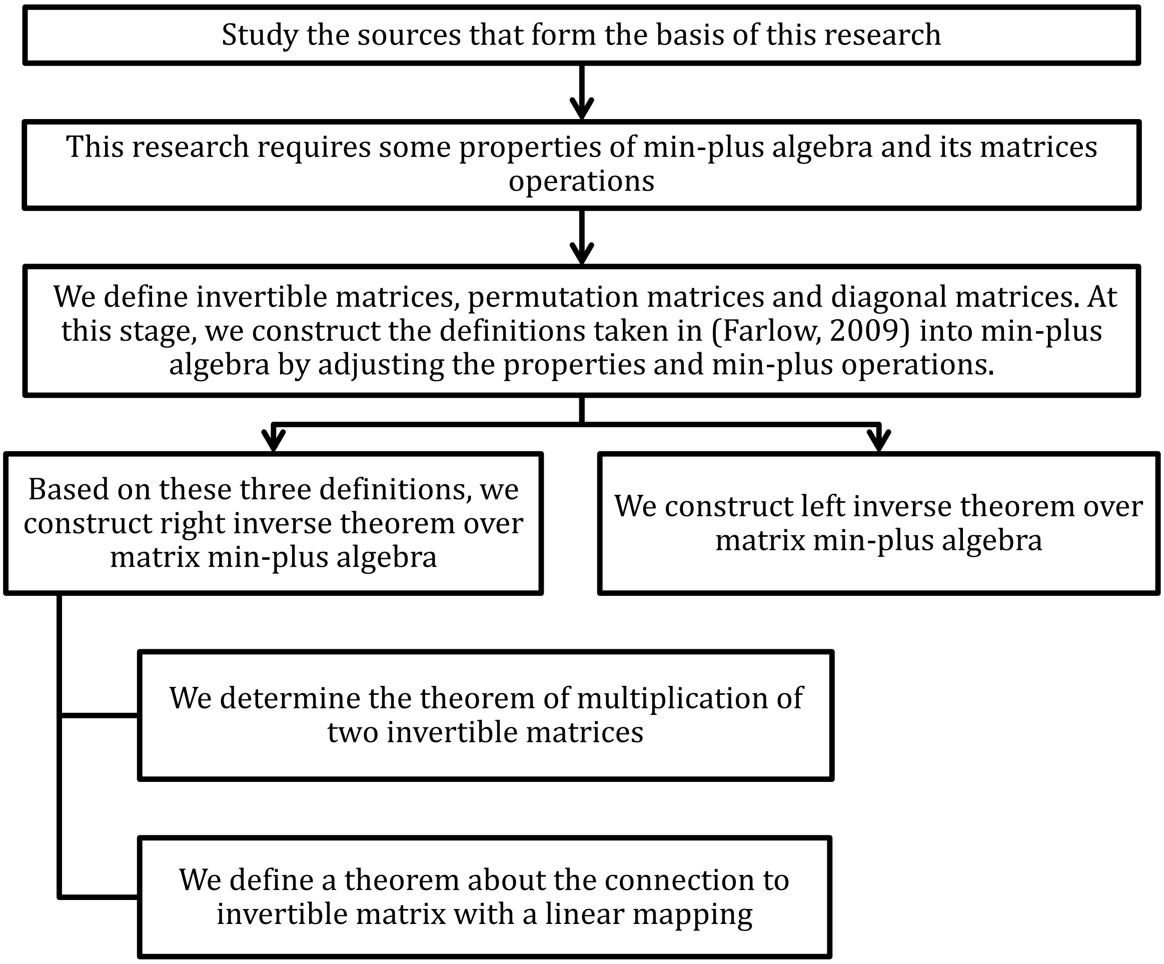

Figure 1. Research Flowchart

1. Min-Plus Algebra

The min-plus algebra is also known as idempotent semifield and idempotent commutative semiring with the notation by $\mathbb{R}_{min}$. Element identity (neutral element) for minimum ($\oplus'$) operations is $\varepsilon' = +\infty$ and element identity for plus ($\otimes'$) operations is $e = 0$ (Tam, 2010). According to (Awallia et al., 2020) if $a, b, c, x, y \in \mathbb{R}_{\varepsilon'}$ then min-plus algebraic properties apply.

a. Both operations $\oplus'$ and $\otimes'$ are associative

$$(a \oplus' b) \oplus' c = a \oplus' (b \oplus' c)$$
$$(a \otimes' b) \otimes' c = a \otimes' (b \otimes' c)$$

b. We have commutative which is defined as

$$a \oplus' b = b \oplus' a \text{ and } a \otimes' b = b \otimes' a.$$

c. Has an identity element (neutral element) $\varepsilon' = +\infty$ with respect to $\oplus'$ is an

$$b \oplus' \varepsilon' = \varepsilon' \oplus' b = b.$$

While the identity $e = 0$ with respect to $\otimes'$, ie

$$b \otimes' e = e \otimes' b = b + 0 = b.$$

d. There exists the unique inverse $y = -x$ of x with respect to $\otimes'$ if $x \neq \varepsilon'$, we have

$$x \otimes' y = x \otimes' (-x) = 0 = e.$$

e. The identity $\varepsilon' = +\infty$ with respect to $\oplus'$ operation absorbs $\otimes'$ operation, we have

$$x \otimes' \varepsilon' = \varepsilon' \otimes' x = \varepsilon'.$$

f. Has the distributive property of $\otimes'$ operation to $\oplus'$ operation,

$$a \otimes' (b \oplus' c) = (a \otimes' b) \oplus' (a \otimes' c).$$

g. Idempotent to the operation $\oplus'$ such that it occurs

$$x \oplus' x = x.$$

Definition 1 (Watanabe et al., 2018)**.** The k power of x is determined by

$$x^{\otimes' k} = \underbrace{x \otimes' x \otimes' \ldots \otimes' x}_{k\ times} \qquad x \in \mathbb{R}_{\varepsilon'} \text{ and } k \in \mathbb{N}.$$

The k power of x in min-plus, reduced to the multiplication $x^{\otimes' k} = kx.$

2. Matrix over Min-Plus Algebra

The set of matrices of dimensions m row and n column wich have entries in $\mathbb{R}_{\varepsilon'}$, is known as min-plus algebra matrices and is denoted by $\mathbb{R}_{\varepsilon'}^{m\times n}$ (Siswanto, Gusmizain, et al., 2021). The various operations are defined in $\mathbb{R}_{\varepsilon'}^{m\times n}$ (Watanabe & Watanabe, 2014).

a. If $A = (a_{ij}) \in \mathbb{R}_{\varepsilon'}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}_{\varepsilon'}^{m\times n}$ then $(A \oplus' B)_{ij} = \min\{a_{ij}, b_{ij}\}$, for all i, j with the notation $A \oplus' B \in \mathbb{R}_{\varepsilon'}^{m\times n}$.
b. If $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ and $\alpha \in \mathbb{R}_{min}$ then we have scalar multiplication i.e. $(\alpha \otimes' A)_{ij} = \alpha \otimes' a_{ij}$, for all i, j with notation $\alpha \oplus' A \in \mathbb{R}_{\varepsilon'}^{m\times n}$.
c. If $A \in \mathbb{R}_{\varepsilon'}^{m\times k}$ and $B \in \mathbb{R}_{\varepsilon'}^{k\times n}$ then $[A \otimes' B]_{ij} = \oplus'^{k}_{\ell=1}(a_{i\ell} \otimes b_{\ell j})$, for all i, j with the notation $A \otimes' B \in \mathbb{R}_{\varepsilon'}^{m\times n}$.
d. If $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$, we have defined its transpose $A^t \in \mathbb{R}_{\varepsilon'}^{n\times m}$ expressed as $[A^t]_{ij} = a_{ji}$.
e. Denoted $E \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is expressed as

$$[E_n]_{ij} = \begin{cases} e, \text{jika } i = j \\ \varepsilon', \text{jika } i \neq j \end{cases}.$$

Can be see $A \otimes' E_n = E_n = E_n \otimes' A$ for $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$, which means that $E = E_n$ is the identity of the matrix multiplication on $\mathbb{R}_{\varepsilon'}^{n \times n}$.

f. If $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$ and $k \in \mathbb{N}$, the k power of A is determined by

$$A^{\otimes' k} = \underbrace{A \otimes' A \otimes' \dots \otimes' A}_{k \text{ times}}.$$

For $k = 0$ we set $A^{\otimes' 0} = E$.

C. RESULT AND DISCUSSION

This section addresses the inverse matrix in min-plus algebra with investigates whether the concept of inverse matrix can be formulated into min-plus algebra. In research (Farlow, 2009), First define the invertible matrix, the permutation matrix, and the diagonal matrix over max-plus algebra. A similar concept is adapted by changing the coefficients and min-plus operations. In such a way, we can define the invertible matrix, the permutation matrix, and the diagonal matrix over min-plus algebra. Definition 2, Definition 3, and Definition 5 have a similarities with Farlow's definition, the difference is the matrix space.

Definition 2. Matrix $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$ is an invertible if there is a matrice B such that $A \otimes' B = E$, and the inverse of A can be donated $A^{\otimes' -1} = B$.

Definition 3. A matrix of permutation is a matrix in which each i^{th} row and j^{th} column contains exactly one entry which is e and the other entry is ε' . If $\sigma = \{1,2, \dots, n\} \to \{1,2, \dots, n\}$ as a permutation then permutation matrix over min-plus algebra is defined $P_\sigma = [p_{ij}]$ with

$$p_{ij} = \begin{cases} e, & i = \sigma(j) \\ \varepsilon', & i \neq \sigma(j) \end{cases}.$$

So j^{th} column of P_σ has entry e on the $\sigma(j)^{th}$ row. The following is a permutation matrix to explain the same permutation matrix as the identity matrix.

$$P_\sigma = \begin{pmatrix} e & \varepsilon' & \varepsilon' \\ \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \end{pmatrix} \tag{1}$$

Permutation matrix (1) has the same order of entries as the identity matrix. Therefore, the permutation matrix whose entries are diagonal from left to right or right to left is worth e and besides that it has a value of ε' which is also known as the identity matrix (E).

Example 4. Given the matrices A and P_σ size 3×3

$$A = \begin{pmatrix} 2 & -5 & -1 \\ 4 & 7 & 1 \\ -3 & 8 & 6 \end{pmatrix} \text{ dan } P_\sigma = \begin{pmatrix} \varepsilon' & e & \varepsilon' \\ e & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & e \end{pmatrix}.$$

calculate the multiplication between the P_σ and A matrices

$$\begin{aligned} P_\sigma \otimes' A &= \begin{pmatrix} \varepsilon' & e & \varepsilon' \\ e & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & e \end{pmatrix} \otimes' \begin{pmatrix} 2 & -5 & -1 \\ 4 & 7 & 1 \\ -3 & 8 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 7 & 1 \\ 2 & -5 & -1 \\ -3 & 8 & 6 \end{pmatrix}. \end{aligned}$$

Based on the multiplication between P_σ and A, it is shown that a matrix is multiplied from the left by the rows of a matrix P_σ, so that the i^{th} row of matrix A appears as the $\sigma(i)^{th}$ row of the matrix $P_\sigma \otimes' A$. Then, permutation matrix P_σ has an inverse i.e. $P_{\sigma^{-1}}$ with $P_{\sigma^{-1}}$ is the transpose of P_σ that satisfies $P_\sigma \otimes' P_{\sigma\text{^}-1} = E$.

Definition 5. If $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}_{\varepsilon'}$, $\lambda_i \neq \varepsilon'$ for each $i = 1, 2, 3, \dots, n$ we defined a diagonal matrix

$$D(\lambda_i) = \begin{pmatrix} \lambda_1 & \varepsilon' & \cdots & \varepsilon' \\ \varepsilon' & \lambda_2 & \cdots & \varepsilon' \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \lambda_n \end{pmatrix}.$$

The diagonal matrix $D(\lambda_i)$ has an inverse, namely $D(-\lambda_i)$ where $-\lambda_i = \lambda_i^{\otimes' -1}$ which satisfies $D(\lambda_i) \otimes' D(-\lambda_i) = E$. Based on the definition that has been given, right inverse of a min-plus algebra matrices is shown in Theorem 6, which is a diagonal matrix that is permuted.

Theorem 6. Matrix $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$ is said to have a right inverse if and only if there are permutations of σ with values of $\lambda_i < \varepsilon'$, $i \in \{1, 2, 3, \dots, n\}$ is such that $A = P_\sigma \otimes' D(\lambda_i)$.

Proof. Given $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$, suppose there is a matrix B that satisfies $A \otimes' B = E$ so that its obtained

1. $min(a_{ik} + b_{ki}) = e = 0$
   for each i there is a k so the $a_{ik} + b_{ki} = e = 0$, we get the function $k = \theta(i)$ with $a_{i\theta(i)}$, $b_{\theta(i)\,i} < \varepsilon'$.
2. $min(a_{ik} + b_{kj}) = \varepsilon' = +\infty$, for each $i \neq j$.
3. Based on (2) obtained $a_{i\theta(j)} = \varepsilon'$.

Because $a_{i\theta(i)} < \varepsilon' = a_{i\theta(j)}$ for $i \neq j$, then θ is an injection and permutation. $a_{i\theta(i)}$ is one entry from $\theta(i)^{th}$column which is not equal to ε'. Suppose that $\hat{A} = P_\theta \otimes' A$ with $\theta(i)^{th}$ row from matrix $\hat{A}$ is the i^{th} row from matrix A, which has entries less than ε' in $\theta(i)^{th}$ column. Thus, all diagonal entries of the matrix A are less than ε'. Therefore

$$P_\theta \otimes' A = \hat{A} = D(\lambda_i) \text{ with } \lambda_i = a_{\theta^{-1}(i)i} < \varepsilon'.$$

Suppose $\sigma = \theta^{-1}$ because $P_\sigma \otimes' P_\theta = P_{\theta^{-1}} \otimes' P_\theta = E$, then $A = P_\sigma \otimes' D(\lambda_i)$.
Otherwise assume $A = P_\sigma \otimes' D(\lambda_i)$ with $\lambda_i \in \mathbb{R}_{\varepsilon'}$ and $\lambda_i < \varepsilon'$. If the assumption is true, then suppose $B = D(-\lambda_i) \otimes' P_{\sigma^{-1}}$ with $-\lambda_i = \lambda_i^{\otimes'-1}$ until obtained

$$A \otimes' B = (P_\sigma \otimes' D(\lambda_i)) \otimes' D(-\lambda_i) \otimes' P_{\sigma^{-1}} = P_\sigma \otimes' E \otimes' P_{\sigma^{-1}} = P_\sigma \otimes' P_{\sigma^{-1}} = E.$$

Because $A \otimes' B = E$, so that B is the right inverse of a matrix A. ■

Example 7. Here is an example of a matrix having a right inverse, given the matrices with size 3×3

$$P_\sigma = \begin{pmatrix} \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \\ e & \varepsilon' & \varepsilon' \end{pmatrix}, D(\lambda_i) = \begin{pmatrix} 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & -3 & \varepsilon' \\ \varepsilon' & \varepsilon' & 5 \end{pmatrix}, \text{and } E = \begin{pmatrix} e & \varepsilon' & \varepsilon' \\ \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \end{pmatrix}.$$

Based on matrix $D(\lambda_i)$, it is known that $\lambda_i < \varepsilon'$, it is determined that matrix A is a permuted diagonal matrix.

$$\begin{aligned} A &= P_\sigma \otimes' D(\lambda_i) \\ &= \begin{pmatrix} \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \\ e & \varepsilon' & \varepsilon' \end{pmatrix} \otimes' \begin{pmatrix} 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & -3 & \varepsilon' \\ \varepsilon' & \varepsilon' & 5 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon' & -3 & \varepsilon' \\ \varepsilon' & \varepsilon' & 5 \\ 1 & \varepsilon' & \varepsilon' \end{pmatrix} \end{aligned} \tag{2}$$

Based on (2), matrix A is a permuted diagonal matrix, so A invertible. Next, we will determine the right inverse of matrix A i.e. matrix B.

$$\begin{aligned} B &= D(-\lambda_i) \otimes' P_{\sigma^{-1}} \\ &= \begin{pmatrix} -1 & \varepsilon' & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' \\ \varepsilon' & \varepsilon' & -5 \end{pmatrix} \otimes' \begin{pmatrix} \varepsilon' & \varepsilon' & e \\ e & \varepsilon' & \varepsilon' \\ \varepsilon' & e & \varepsilon' \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon' & \varepsilon' & -1 \\ 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & -5 & \varepsilon' \end{pmatrix}. \end{aligned}$$

Then check whether $A \otimes' B = E$

$$
\begin{aligned}
A \otimes' B &= \begin{pmatrix} \varepsilon' & -3 & \varepsilon' \\ \varepsilon' & \varepsilon' & 5 \\ 1 & \varepsilon' & \varepsilon' \end{pmatrix} \otimes' \begin{pmatrix} \varepsilon' & \varepsilon' & -1 \\ 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & -5 & \varepsilon' \end{pmatrix} \\
&= \begin{pmatrix} e & \varepsilon' & \varepsilon' \\ \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \end{pmatrix} \\
&= E
\end{aligned}
$$

So, the right inverse of matrix A is obtained, namely

$$B = \begin{pmatrix} \varepsilon' & \varepsilon' & -1 \\ 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & -5 & \varepsilon' \end{pmatrix} \tag{3}$$

We have discussed the theorem related to the right inverse is also the left inverse and the theorem of multiplication of two invertible matrices over max-plus algebra in Farlow's research (Farlow, 2009). The same concept in Farlow's research is constructed into min-plus algebra by adjusting the space and operations on the min-plus algebra so that Theorem 8 and Theorem 10 is obtained.

Theorem 8. Given $A, B \in \mathbb{R}_{\varepsilon'}^{n\times n}$, if $A \otimes' B = E$ then $B \otimes' A = E$ and B is uniquely determined by the matrix A.

Proof. Given matrix $A, B \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Based on Theorem 6, noted that $A = P_\sigma \otimes' D(\lambda_i)$ with $\lambda_i < \varepsilon'$ and permutation σ. Suppose $\hat{B} = D(-\lambda_i) \otimes' P_{\sigma^{-1}}$ which is a left inverse from matrix A. So $\hat{B} \otimes' A = D(-\lambda_i) \otimes' P_{\sigma^{-1}} \otimes' P_\sigma \otimes' D(\lambda_i) = D(-\lambda_i) \otimes' E \otimes' D(\lambda_i) = E$. If $A \otimes' B = E$ then obtained $\hat{B} = \hat{B} \otimes' E = \hat{B} \otimes' A \otimes' B = E \otimes' B = B$, this showing that the matrix B is a uniquely determined and also a left inverse. ■

Example 9. Then the left inverse of the matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ taken from Example 7. If A is invertible then A has a left inverse that is B which satisfies $B \otimes' A = E$. Based on Theorem 8 shows that a right inverse is also a left inverse. Thus that, the left inverse of matrix A is obtained, namely (3). Therefore, check whether $B \otimes' A = E$

$$
\begin{aligned}
B \otimes' A &= \begin{pmatrix} \varepsilon' & \varepsilon' & -1 \\ 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & -5 & \varepsilon' \end{pmatrix} \otimes' \begin{pmatrix} \varepsilon' & -3 & \varepsilon' \\ \varepsilon' & \varepsilon' & 5 \\ 1 & \varepsilon' & \varepsilon' \end{pmatrix} \\
&= \begin{pmatrix} e & \varepsilon' & \varepsilon' \\ \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \end{pmatrix} = E
\end{aligned}
$$

So, its true that matrix B is the right inverse as well as the left inverse of matrix A.

$$B = \begin{pmatrix} \varepsilon' & \varepsilon' & -1 \\ 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & -5 & \varepsilon' \end{pmatrix}.$$

The next section discusses the properties of the product of two invertible matrices and the connection between an ivertible matrix dan a linear mapping.

Theorem 10. If matrix $A, B \in \mathbb{R}_{\varepsilon'}^{n\times n}$ invertible then $A \otimes' B$ invertible.

Proof. Given matrix $A, B \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Based on Theorem 6, we can write $A = P_{\sigma^a} \otimes' D({\lambda_i}^a)$ and $B = D\left({\lambda_i}^b\right) \otimes' P_{\sigma^b}$, with ${\lambda_i}^a, {\lambda_i}^b < \varepsilon'$ and permutation σ^a, σ^b. Therefore,

$$\begin{aligned} A \otimes' B &= P_{\sigma^a} \otimes' D({\lambda_i}^a) \otimes' D\left({\lambda_i}^b\right) \otimes' P_{\sigma^b} \\ &= P_{\sigma^a} \otimes' D\left({\lambda_i}^a \otimes' {\lambda_i}^b\right) \otimes' P_{\sigma^b}. \end{aligned}$$

As can be seen that $A \otimes' B$ is a diagonal matrix that has been permuted, such that $A \otimes' B$ invertible. ■

The connection between invertible matrix and linear mapping over max-plus algebra has been explained in Farlow's research. So this connection also applies in min-plus algebra, it is necessary to adjust the matrix space and vector space in $\mathbb{R}_{\varepsilon'}$ and the operations in min-plus algebra, such that we obtain Theorem 11.

Theorem 11. Suppose $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $L_A = \mathbb{R}_{\varepsilon'}^{n} \to \mathbb{R}_{\varepsilon'}^{n}$ with $L_A(x) = A \otimes' x$, then the following statements are equal

1. $A = P_\sigma \otimes' D(\lambda_i)$ for each permutation and $\lambda_i < \varepsilon'$,
2. L_A is surjective,
3. matrix A has a right inverse, $A \otimes' B = E$,
4. matrix A has a left inverse, $B \otimes' A = E$, and
5. L_A is injective.

Proof.

(1 → 2) Let $A = P_\sigma \otimes' D(\lambda_i)$ for each permutation and $\lambda_i < \varepsilon'$, this means that A has a right inverse, namely $\hat{B} = D(-\lambda_i) \otimes' P_{\sigma^{-1}}$ and permutation σ^{-1} which satisfies $A \otimes' \hat{B} = E$. Take any $y \in \mathbb{R}_{\varepsilon'}^{n}$ there exist $x \in \mathbb{R}_{\varepsilon'}^{n}$ with $x = \hat{B} \otimes' y$, so $y = E \otimes' y = \left(A \otimes' \hat{B}\right) \otimes' y = A \otimes' \left(\hat{B} \otimes' y\right) = A \otimes' x = L_A$. Therefore L_A is surjective.

(2 → 3) Know L_A is surjective, for each $y \in \mathbb{R}_{\varepsilon'}^{n}$ there exist $x \in \mathbb{R}_{\varepsilon'}^{n}$ so that $y = L_A = A \otimes' x$. Because L_A is surjective then there is x_i, $i = 1,2, \ldots n$ so that $y_i = A \otimes' x_i$. And formed a matrix $B = x_1, x_2, \ldots, x_n$ then $A \otimes' B = E$. Therefore matrix A has a right inverse.

(3 → 4) It has been proven in Theorem 8.

(4 → 5) Know the matrix A has a left inverse is a $B \otimes' A = E$, will be proven L_A is injective. Suppose $L_A(x_1) \neq L_A(x_2)$, this means

$$A \otimes' x_1 \neq A \otimes' x_2 \Leftrightarrow B \otimes' A \otimes' x_1 \neq B \otimes' A \otimes' x_2 \Leftrightarrow E \otimes' x_1 \neq E \otimes' x_2 \Leftrightarrow x_1 \neq x_2.$$

Therefore L_A is injective.
$(5 \to 1)$ Know L_A is injective, will be proven $A = P_\sigma \otimes' D(\lambda)$ for a permutation σ and $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$, with $\lambda_i < \varepsilon'$. For each i defined,

$$F_i = j \big| a_{ji} < \varepsilon' \text{ and } G_i = j \big| a_{jk} < \varepsilon' \text{ for each } k \neq i.$$

Claim that $F_i \not\subset G_i$. Suppose $F_i \subset G_i$, it will be shown that this contradicts the statement that L_A is injective. Defined $x = [x_k]$ with

$$x_k = \begin{cases} e, k \neq i \\ \varepsilon', k = i \end{cases}.$$

Suppose $b = A \otimes' x = \bigoplus'_{k \neq i} a_{*k}$ where a_{*k} is a k^{th} column from A. Now suppose $j \in F_i$, this means there $k \neq i$ with $a_{jk} < \varepsilon'$. Therefore $b_j \leq a_{jk} < \varepsilon'$. Because $a_{ji} < \varepsilon'$ then $\beta_j < \varepsilon'$ so that $\beta_j \otimes' a_{ji} \geq b_j$. If $j \not\in F_i$ then $a_{ji} = \varepsilon'$. Now suppose $\beta = max_{j \in F_i} \beta_j$, its means $\beta < 0$ and $\beta \otimes' a_{ji} \geq b_j$ for each j . Its said that $A \otimes' [x \oplus' \beta \otimes' e_i] = [A \otimes' x] \oplus' [\beta \otimes' A \otimes' e_i] = b \oplus' \beta \otimes' a_{*j} = b$. Now, $\tilde{x} = x \oplus' \beta \otimes' e_i$, $L_A(\tilde{x}) = L_A(x)$. But $x = \varepsilon' > \tilde{x} = \beta$. Contradiction with injective L_A, its means that the claim is proven to be true.
Because $F_i \not\subset G_i$ means that for each i there is an index $j = \sigma(i)$ with properties $a_{ji} < \varepsilon'$ and $a_{jk} = \varepsilon'$ for each $k \neq i$. In other words, a_{ji} is the only element that is not to ε' in the line $j = \sigma(i)$. But if $j = \sigma(i')$ then $i = i'$, this means that σ is an injective. Consequently, σ is a permutation. Therefore, for each row j there is a unique column with $i(j = \sigma(i))$ so that a_{ji} is a unique element that is not equal to ε'. For any column i and any row k with $k \neq \sigma(i)$, it's known that $k = \sigma(i')$ for each $i' \neq i$. This means that a_{ki} is an element that is not equal to ε' in the k row, so $a_{ji} = \varepsilon'$. Therefore, $a_{\sigma(i)i}$ is an element that is not equal to ε' in column i. Therefore, A is a permuted diagonal matrix, $A = P_\sigma \otimes' D(\lambda)$ where $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_i = a_{\sigma(i)} > \varepsilon'$ for each $i = 1,2, \ldots, n$. ■

Based on the results that have been disusssed, it shows several differences concepts in max-plus and min-plus algebra. To determine the right and left inverse theorems, both use the same steps, namely first defining the invertible matrix, permutation matrix, and diagonal matrix. However, what differentiates the two is the operations and matrix space, where the max-plus algebra matrix space is set $\mathbb{R}_\varepsilon^{n \times n}$ and the min-plus algebra matrix space is set $\mathbb{R}_{\varepsilon'}^{n \times n}$. Furthermore, it is known in Farlow's research which states that a matrix has a right inverse if there are permutation and values of $\lambda_i < \varepsilon'$ so $A = P_\sigma \otimes' D(\lambda_i)$ applies and Farlow also stated that the right inverse is also a left inverse (Farlow, 2009). The two statements in Farlow's research also apply to the min-plus algebra version by adjusting the operations and matrix space. However, what differentiates the inverse concept of max-plus algebra and min-plus algebra is the operation, the matrix space, and the inequality sign on the value of λ_i.

D. CONCLUSION AND SUGGESTIONS

Based on the research, it is shown that the inverse concept in min-plus algebra has similarities to the inverse concept in max-plus, but the difference is the operation and matrix space. The min-plus algebra matrix is said to be an invertible matrix if it meets certain conditions i.e. the matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is said to have a right inverse if and only if there are permutation of σ with the value of $\lambda_i < \varepsilon'$, $i \in \{1, 2, 3, \dots, n\}$ such that $A = P_\sigma \otimes' D(\lambda_i)$. Furthermore, if B is the right inverse that satisfies $A \otimes' B = E$ then $B \otimes' A = E$ with B is uniquely determined by A and a right inverse is also a left inverse. Research on this topic is still widely open for further research by connecting the topic of inverses in min-plus algebra with several types of matrices in min-plus algebra but still paying attention to their properties.

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