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Domination Numbers in Graphs Resulting from Shackle Operations with Linkage of any Graph

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ABSTRACT

The domination number is the number of dominating nodes in a graph that can dominate the surrounding connected nodes with a minimum number of dominating nodes. This domini number is denoted by $\gamma(G)$. In this research, we will examine the domination number of the distance between two graphs resulting from the shackle operation with any graph as linkage. This differs from previous research, namely the domination of numbers at one and two distances. This study emphasizes how the results of operations on the shackle are connected to the shackle graph as any graph connects the copy. Any graph here means all graphs are connected and generally accepted. The method used in this research is pattern recognition and axiomatic deductive methods. The pattern detection method examines patterns where a graph's number of dominating points can dominate the connected points around it with a minimum number of dominating nodes. Meanwhile, axiomatic deductive is a research method that uses the principles of deductive proof that apply to mathematical logic by using existing axioms or theorems to solve a problem. The Result of graph $\boldsymbol{S_n}$ with $\boldsymbol{t}$ copies and $\boldsymbol{S_m}$ as linkage, then the two-distance domination number in the graph resulting from the shackle operation is $\boldsymbol{\gamma_2(Shack(S_n, S_m, t)) = t-1}$; graph $\boldsymbol{S_n}$ with $\boldsymbol{t}$ copies and $\boldsymbol{C_m}$ as linkage, then the two-distance domination number in the graph resulting from the shackle operation is $\boldsymbol{\gamma_2(Shack(S_n, C_m, t)) = \begin{cases} t, for\ 3 \le m \le 6 \\ \left\lceil \frac{n}{5} \right\rceil (t-1), for\ m \ge 7 \end{cases}}$; graph $\boldsymbol{C_n}$ with $\boldsymbol{t}$ copies and $\boldsymbol{S_m}$ as linkage, then the two-distance domination number in the graph resulting from the shackle operation is

$$\gamma_2(Shack(C_n, S_m, t)) = \begin{cases} t-1, for\ n = 3 \\ t, for\ 4 \le n \le 5 \\ \left\lceil \frac{n}{5} \right\rceil t, for\ n \ge 6 \end{cases}$$

This research provides benefits and adds to research results in the field of graph theory specialization of two-distance domination numbers in the result graph of shackle operation with linkage any graph.

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A. INTRODUCTION

Graph theory is a branch of mathematics that can be applied in various fields. Fields that can be applied to graph theory include the computer field, industry, government, security, and others (Joedo et al., 2022), (Verma et al., 2022), and (Ali et al., 2022). In general, graphs can be interpreted as non-empty sets called nodes, and sets that can be empty are called edges (Ponraj et al., 2022). One of the exciting topics in graph theory is domination numbers. Domination numbers have been around for a long time, namely since 1850. Domination numbers first appeared among chess fans in Europe (Mahapatra & Pal, 2022), (Vecchio et al., 2017), (C. Zhang et al., 2023)and (Khan et al., 2023). Strengthened by the problem that arises and must be faced by determining the number of queens that must be placed on the 8 x 8 chess board so that the queen can control all squares on the chess. In such cases, the number of queens on the chessboard must be minimal (X. Zhang et al., 2022), (Milano et al., 2022), (Xue & Li, 2023) .

Mathematically, the domination number can be the number of dominating nodes in a graph that can dominate the surrounding connected nodes with a minimum number of dominating nodes (de Berg & Kisfaludi-Bak, 2020), (Movahedi et al., 2021). This domini number is denoted by $\gamma(G)$. Domination numbers have been widely used in life, for example, placing police monitoring posts on certain roads, placing electric cars on plantation land, and placing CCTV at certain angles so that they can reach the surrounding area at a certain distance (Ashraful Alam et al., 2022), (Kang & Shan, 2020). By implementing a dominating set, the placement of police posts, electric cars, and CCTV will be more efficient, and the number can be minimized. Domination numbers have sub-subs, such as one distance domination number, two distance domination number, three distance domination number, and n^{th} distance domination number (Queiroz et al., 2023), (Zhuang, 2023). It was also researched regarding the domination number of the results of operations on graphs (amalgamation operations, shackle operations, joint operations, etc.). Previous research has been researched with the titles Two Distance Domination Numbers in Graphs Resulting from Shackle Operations (Umilasari, 2017), Two Distance Domination Numbers in Jahangir $J_{m,n}$ Graphs (Wahyuni & Utoyo, 2017), Two Distance Domination Numbers in Graph of Amalgamation Operation Results (Saifudin, 2017). The following is some other research related to dominance numbers with the title On Dominating Graph of Graphs, Median Graphs, Partial Cubes and Complement of Minimal Dominating Sets (Mofidi, 2023), The Number of 2-dominating Sets, and 2-domination Polynomial of a Graph (Movahedi et al., 2021), Signed and Minus Dominating Functions in Graphs (Kang & Shan, 2020), Lower Bounds for Dominating Set in Ball Graphs and for Weighted Dominating Set in Unit-Ball Graphs (de Berg & Kisfaludi-Bak, 2020), On a Countable Family of Boundary Graph Classes for the Dominating Set Problem (Dakhno & Malyshev, 2023), On the Number of Minimum Total Dominating Sets in Trees (Taletskii, 2023), The oddball domination number is connected to the sunlet graph and the bishop graph (Poniman & Fran, 2020), The domination number of the king's graph (Arshad et al., 2023), Hamilton Paths in Dominating Graphs of Trees and Cycles (Adaricheva et al., 2022) and On the 2-Domination Number of Complete Grid Graphs (Shaheen et al., 2017).

This research will study the domination number of the distance between two graphs resulting from the shackle operation with any graph's linkage. This differs from previous research: distance one and two domination numbers. This research emphasizes how the results of operations on the shackle are connected to the shackle graph as any graph connects its copy. Any graph here means that all graphs are connected and apply in general. Thus, the resulting formula generally applies to lower and upper limits. It can be drawn from the explanation above that the title of this research is "Domination Numbers in Graphs Resulting from Shackle Operations with Linkage of Any Graph."

A shackle graph is denoted by $\mathrm{Shack}(G_1, G_2, \cdots, G_k)$, a shackle graph formed from k copies of graph G is denoted by $\mathrm{Shack}(G, k)$ with $k \geq 2$ and k is an integer (Maryati et al., 2010), (Huntala et al., 2023), (Prayitno, 2015). The shackle operation in this study consists of point shackle and side shackle (Kristiana et al., 2022). The node shackle operation is denoted by $\mathrm{Shack}(G, v, t)$ meaning that the graph is constructed from any graph G with t copies and v as the node linkage, the resulting graph of the node shackle operation can be seen in Figure 1. Meanwhile, the side shackle operation is denoted by $\mathrm{Shack}(G, e, t)$, which means that the graph is constructed from t copies of any graph G, and e is edge linkage (Maghfiro et al., 2023). The graph resulting from the edge shackle operation can be seen in Figure 2.

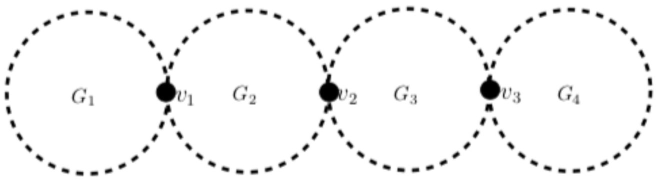

Figure 1. Graph of Shackle Operation Results Point $\mathrm{Shack}(G, v, 4)$

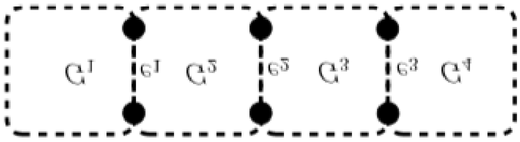

Figure 2. Graph of Shackle Operation Results Side $\mathrm{Shack}(G, e, 4)$

B. METHODS

The methods used in this research are pattern detection and axiomatic deductive methods. The pattern detection method examines patterns where a graph's number of dominating points can dominate the connected points around them with a minimum number of dominating nodes. Meanwhile, axiomatic deductive is a research method that uses the principles of deductive proof that apply to mathematical logic by using existing axioms or theorems to solve a problem. Then, the method will determine the domination number with the minimum domination point. The research focuses on determining the domination number of the distance between two graphs resulting from shackle operations with any linkage of shack graphs $\mathrm{Shack}(G, H, k)$. From Figure 3, the design of this research will be explained as follows.

1. Study of Literature

At this stage, the researcher read references from previous research regarding graph theory books, articles related to graph theory, articles related to domination numbers, any graphs that previous researchers had not researched.

2. Determine the Graph to be used

The graphs that will be used include: From a scientific perspective, we will look for graphs resulting from shackle operations with any linkage of Shack graphs (G, H, k).

3. Determine the set of Points on Each Graph

The names of the points are first determined to make it easier to determine the domination number of two $\gamma_2(\mathrm{Shack}(G, H, k))$, the names of the points are first determined. Each naming of these points can be a basis for other points so that the minimum dominating point of each graph is formed, which is the dominating number.

4. Determine the Domination Number $\gamma_2(\mathrm{Shack}(G, H, k))$

After determining the naming of the points of the graph, the time comes when the domination number can be said to be the number of dominating points in a graph that can dominate the connected points around them and with a minimum number of dominating points. The two-distance domination number is denoted by $\gamma_2(\mathrm{Shack}(G, H, k))$.

5. Deconstruction of Graphs

After determining the minimum point of the graph as the dominator, construction will be carried out to determine the general formula for the graph. The minimum cardinality between the domination sets in graph G is called the dominating number of graph G denoted by $\gamma_2(\mathrm{Shack}(G, H, k))$.

6. Conduct Score Analysis $\gamma_2(\mathrm{Shack}(G, H, k))$.

At this stage, an analysis of the domination numbers that have been proven will be carried out, whether there is a connection with other graphs so that the general formula for the graph is found.

7. Drawing Conclusions

Last, after all the circuits have been passed, a conclusion will be drawn as a dominating number for the distance between two graphs resulting from Shackle Operation with Linkage of Any Graph. The research stages to obtain the domination number on the graph are shown in Figure 3.

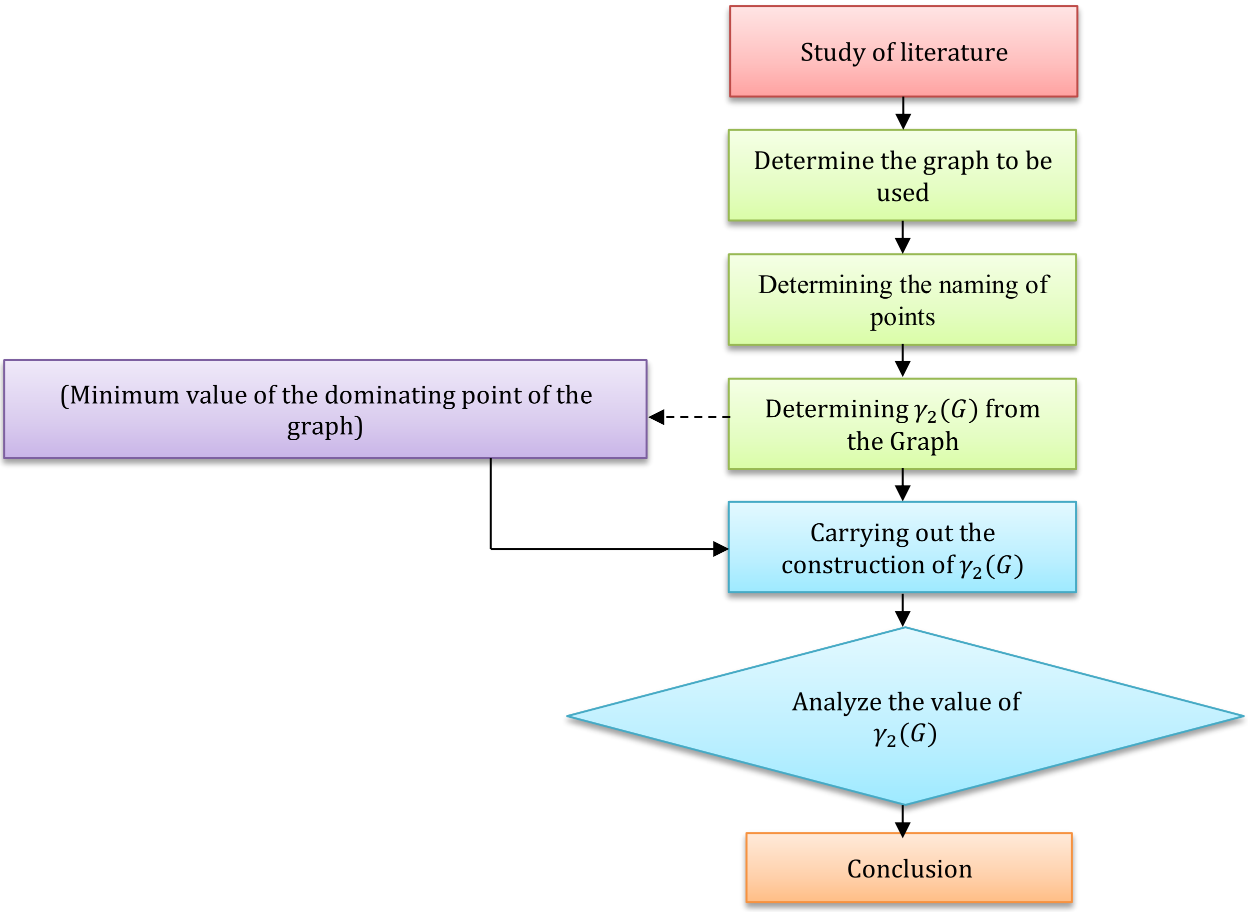

Figure 3. Research Flow

C. RESULT AND DISCUSSION

The following are the results and discussion related to this research, starting with the operational definition along with the resulting theorems.

1. Operational Definition of Shackle Operation Results Graph

a. $Shack(S_n, S_m, t)$ is the graph resulting from the shackle operation on graph S_n with t copies and S_m is the graph between S_i and S_{i+1}. If $|V(S_n)| = n + 1$ and $|V(S_m)| = m + 1$, then $|V(Shack(S_n, S_m, t))| = nt + mt - 2t - m + 3$.

b. $Shack(S_n, C_m, t)$ is the graph resulting from the shackle operation on graph S_n with t copies and C_m is the graph between S_i dan S_{i+1}. If $|V(S_n)| = n + 1$ and $|V(C_m)| = m$, then $|V(Shack(S_n, C_m, t))| = (n-1)t + m(t-1) + 2 = nt - t + mt - m + 2$

c. $Shack(C_n, S_m, t)$ is the graph resulting from the shackle operation on graph C_n with t copies and S_m is the graph between C_i dan C_{i+1}. If $|V(C_n)| = n$ and $|V(S_m)| = m + 1$, then $|V(Shack(C_n, S_m, t))| = nt + (m-1)(t-1) = nt + mt - m - t + 1$

2. Theorems resulting from the Shackle Operation Result Graph

Theorem 1. Given a graph S_n is t copies and S_m is linkage, then the two distance domination number in the graph resulting from the shackle operation is

$$\gamma_2(Shack(S_n, S_m, t)) = t - 1 \tag{1}$$

Proof. $Shack(S_n, S_m, t)$ is the graph resulting from the shackle operation from graph S_n with t copies and S_m as a linkage. The $Shack(S_n, S_m, t)$ graph has $nt + mt - 2t - m + 3$ nodes and a dominating node can dominate a maximum of $1 + m + 2n$ nodes. Next, it will be proven that $\gamma_2(Shack(S_n, S_m, t)) = t - 1$ is the minimum number of dominators which will be proven through two cases, namely $v_{m+1} \in D_2$ and v_{m+1} are not elements of D_2 where v_{m+1} is the central node in the sub graph S_m and D_2 is the set of dominating nodes.

Case 1. $v_{m+1} \in D_2$

In the case of $v_{m+1} \in D_2$ for the number of 2^{nd} copies up to the t^{th} copy, the number of domination numbers in the distance of two in $Shack(S_n, S_m, t)$ will form an arithmetic sequence as follows.

The 1st term for $t = 2 \rightarrow 1$
The 2nd term for $t = 3 \rightarrow 2$
The 3rd term for $t = 4 \rightarrow 3$
$\vdots$
The kth $term\ for\ t = t \rightarrow t - 1$

From this sequence we will get the domination number of the two distances in the graph $Shack(S_n, S_m, t)$ which is $t - 1$. Next, it will be proven using mathematical induction as follows.

$$\gamma_2(Shack(S_n, S_m, t)\,) = t - 1 \tag{2}$$

It will be proven that the 1st term is correct

$$\begin{aligned} \gamma_2(Shack(S_n, S_m, 2)\,) &= 2 - 1 \\ \Leftrightarrow \qquad 1 &= 1 \end{aligned} \tag{3}$$

Assume that the k^th term is correct

$$\gamma_2(Shack(S_n, S_m, k)\,) = k - 1 \tag{4}$$

It will be proven that for the $k + 1$ th term it is also correct.

$$\begin{gathered} \gamma_2(Shack(S_n, S_m, k + 1)\,) = \gamma_2(Shack(S_n, S_m, k)\,) + difference \\ \Leftrightarrow k + 1 - 1 = k - 1 + 1 \\ \Leftrightarrow k = k \end{gathered} \tag{5}$$

Thus, in the case $v_{m+1} \in D_2$, the graph $Shack(S_n, S_m, t)$ has a domination number of $t - 1$.

Case 2. v_{m+1} is not an element of D_2

In this case, for the number of 2nd copies up to the t^th copy, the number of domination numbers in the distance of two in $Shack(S_n, S_m, t)$ will form an arithmetic sequence as follows:

The 1st term for $t = 2 \rightarrow 2$
The 2nd term for $t = 3 \rightarrow 3$
The 3rd term for $t = 4 \rightarrow 4$
⋮
The k^th $term\ for\ t = t \rightarrow t$

From this sequence we will get the domination number for the two distances in the graph $Shack(S_n, S_m, t)$ which is t. Next, it will be proven using mathematical induction as follows.

$$\gamma_2(Shack(S_n, S_m, t)\,) = t \tag{6}$$

It will be proven that the 1st term is correct

$$\begin{aligned} \gamma_2(Shack(S_n, S_m, 2)\,) &= t \\ \Leftrightarrow \qquad 2 &= 2 \end{aligned} \tag{7}$$

Assume that the k^th term is correct

$$\gamma_2(Shack(S_n, S_m, k)\,) = k \tag{8}$$

It will be proven that for the $k+1$ term it is also correct.

$$\gamma_2(Shack(S_n, S_m, k+1)) = \gamma_2(Shack(S_n, S_m, k)) + beda \\ \Leftrightarrow k+1 = k+1 \tag{9}$$

Thus, in the case that v_{m+1} bis not an element of D_2, the graph $Shack(S_n, S_m, t)$ has a domination number of t. From these two cases it is known that $t-1 \leq t$ then $\gamma_2(Shack(S_n, S_m, t)) = t-1$. Next, it will be proven that t-1 is the minimum number of dominating nodes in the graph $Shack(S_n, S_m, t)$. If $\gamma_2(Shack(S_n, S_m, t)) = t-1-1 = t-2$ then the maximum node that can be dominated based on Lemma is $1+m+2n\left(\left\lceil\frac{nt+mt-2t-m+3}{1+m+2n}\right\rceil - 1\right) \leq 1+m+2n\left(\frac{nt+mt-2t-m+2+1+m+2n}{1+m+2n} - 1\right) = nt+mt-2t-m+2$. This means that the number of nodes that can be dominated is $nt+mt-2t-m+2$, so there is at least one node that is not dominated. Thus $\gamma_2(Shack(S_n, S_m, t)) \neq t-2$, because $t-1$ is the minimum number of dominating nodes then $\gamma_2(Shack(S_n, S_m, t)) = t-1$. To strengthen the evidence, an example Figure 4 is provided as follows. Where $(S_8, S_6, 3)$ is divided into Graph S_n which is a star graph S_8 with a connecting star graph S_6 and $t = 3$ copies.

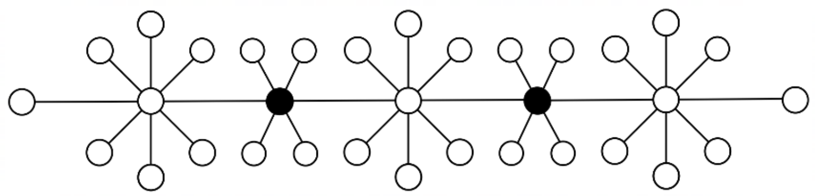

Figure 4. Shack Graph $(S_8, S_6, 3)$ with Black Nodes are Dominant Nodes

Theorem 2. Given a graph S_n has t copies and C_m as linkage, then the two-distance domination number in the graph resulting from the shackle operation is:

$$\gamma_2(Shack(S_n, C_m, t)) = \begin{cases} t, for\ 3 \leq m \leq 6 \\ \left\lceil\frac{n}{5}\right\rceil (t-1), for\ m \geq 7 \end{cases} \tag{10}$$

Proof. $Shack(S_n, C_m, t)$ is the graph resulting from the shackle operation from graph S_n with t copies and C_m as linkage. The $Shack(S_n, C_m, t)$ graph has $(n-1)t + m(t-1) + 2 = nt - t + mt - m + 2$ nodes and a dominating node can dominate a maximum of $n+5$ nodes. Next it will be proven that $\gamma_2(Shack(S_n, C_m, t)) = t$ for $3 \leq m \leq 6$. In this case, the number of 1st copies to the tth copies, the number of domination numbers in the distance of two in $Shack(S_n, C_m, t)$ will form an arithmetic sequence as follows.

The 1st term for $t = 1 \rightarrow 1$
The 2nd term for $t = 2 \rightarrow 2$
The 3rd term for $= 3 \rightarrow 3$
⋮
The kth term for $t = t \rightarrow t$

From this sequence we will get the domination number of the two distances in the graph $Shack(S_n, C_m, t)$ which is t. Next, it will be proven using mathematical induction as follows:

$$\gamma_2(Shack(S_n, C_m, t)) = t \quad (10)$$

It will be proven that the 1st term is correct

$$\gamma_2(Shack(S_n, C_m, 1)) = 1 \quad (11)$$
$$\Leftrightarrow \quad 1 = 1$$

Assume that the k^{th} term is correct

$$\gamma_2(Shack(S_n, C_m, k)) = k \quad (12)$$

It will be proven that for the $k + 1$ th term it is also correct

$$\gamma_2(Shack(S_n, C_m, k + 1)) = \gamma_2(Shack(S_n, C_m, k)) + difference \quad (13)$$
$$\Leftrightarrow k + 1 = k + 1$$

Thus, the graph $Shack(S_n, C_m, t)$ has a domination number of t. Next it will be shown that t is the minimum number of dominating nodes in the graph $Shack(S_n, C_m, t)$. In the graph $Shack(S_n, C_m, t)$ with $3 \leq m \leq 6$, it has a maximum of $(n - 1)t + 6(t - 1) + 2 = nt + 5t - 4$ nodes. Each dominating node can dominate a maximum of $n + 5$ nodes. If $\gamma_2(Shack(S_n, C_m, t)) = t - 1$, then the maximum node that can be dominated based on Lemma is $n + 5 \quad \left(\left\lceil\frac{nt+5t-4}{n+5}\right\rceil - 1\right) \leq n + 5\left(\frac{nt+5t-5+n+5}{n+5} - 1\right) = nt + 5t - 5$. This means that the number of nodes that can be dominated is $nt + 5t - 5$, so there is at least one node that is not dominated. Thus $\gamma_2(Shack(S_n, C_m, t)) \neq t - 1$, because t is the minimum number of dominating nodes then $\gamma_2(Shack(S_n, C_m, t)) = t$.

Next, it will be proven that $\gamma_2(Shack(S_n, C_m, t)) = \left\lceil\frac{n}{5}\right\rceil(t - 1), for\ m \geq 7$ is the minimum number of dominating nodes that can dominate all nodes in the $Shack(S_n, C_m, t)$. If $\gamma_2(Shack(S_n, C_m, t)) = \left\lceil\frac{n}{5}\right\rceil(t - 1) - 1$, then the maximum node that can be dominated based on Lemma is $n + 5 \quad \left(\left\lceil\frac{nt-t+mt-m+2}{n+5}\right\rceil - 1\right) \leq n + 5\left(\frac{nt-t+mt-m+1+n+5}{n+5} - 1\right) = nt - t + mt - m + 1$. This means that the number of nodes that can be dominated is $nt - t + mt - m + 1$, so there is at least one node that is not dominated. Thus $\gamma_2(Shack(S_n, C_m, t)) \neq \left\lceil\frac{n}{5}\right\rceil(t - 1) - 1$, because t is the minimum number of dominating nodes then $\gamma_2(Shack(S_n, C_m, t)) = \left\lceil\frac{n}{5}\right\rceil(t - 1)$.

To strengthen the evidence, an example Figure 5 is provided as follows. Where $(S_6, C_6, 3)$ is divided into Graph S_n which is a star graph S_6 with a connecting Cycle graph C_6 and $t = 3$ copies.

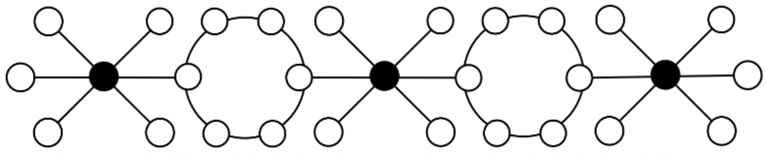

Figure 5. Shack Graph $(S_6, C_6, 3)$ with Black Nodes are Dominant Nodes

Theorem 3. Given a graph C_n with t copies and S_m as linkage, then the two-distance domination number in the graph resulting from the shackle operation is

$$\gamma_2(Shack(C_n, S_m, t)) = \begin{cases} t-1, for\ n = 3 \\ t, for\ 4 \le n \le 5 \\ \left\lceil \frac{n}{5} \right\rceil t, for\ n \ge 6 \end{cases} \tag{14}$$

Proof. $Shack(C_n, S_m, t)$ is the graph resulting from the shackle operation from graph C_n with t copies and S_m as linkage. $Shack(C_n, S_m, t)$ graph has $nt + (m-1)(t-1) = nt + mt - m - t + 1$ nodes and a dominating node can dominate a maximum of $m + 5$ nodes

$$\boldsymbol{for\ n = 3}$$

Next, it will be proven that $\gamma_2(Shack(C_n, S_m, t)) = t - 1$ for $n = 3$. In this case, the number of 1st copies to the tth copies, the number of domination numbers in the distance of two in $Shack(C_n, S_m, t)$ will form an arithmetic sequence as follows.

The 1st term for $t = 2 \rightarrow 1$
The 2nd term for $t = 3 \rightarrow 2$
The 3rd term for $t = 4 \rightarrow 3$
⋮
The kth term for $t = t \rightarrow t - 1$

From this sequence we will get the domination number of the two distances in the graph $Shack(C_n, S_m, t)$ which is $t - 1$. Next, it will be proven using mathematical induction as follows.

$$\gamma_2(Shack(C_n, S_m, t)) = t - 1 \tag{15}$$

It will be proven that the 1st term is correct

$$\gamma_2(Shack(C_n, S_m, t)) = 2 - 1 \tag{16}$$
$$\Leftrightarrow \qquad 1 = 1$$

Assume that the kth term is correct

$$\gamma_2(Shack(C_n, S_m, t)) = k - 1 \tag{17}$$

It will be proven that for the $k+1$ th term it is also correct

$$\gamma_2(Shack(C_n, S_m, k+1)) = \gamma_2(Shack(C_n, S_m, t)) + difference \tag{18}$$
$$\Leftrightarrow k+1-1 = k-1+1$$
$$\Leftrightarrow k = k$$

Thus, the graph $Shack(C_n, S_m, t)$ has a domination number of $t-1$. Next it will be shown that $t-1$ is the minimum number of dominating nodes in the graph $Shack(C_n, S_m, t)$. In the graph $Shack(C_n, S_m, t)$ with $n=3$, it has $nt+(m-1)(t-1) = nt+mt-m-t+1$ nodes and a dominating node can dominate a maximum of $m+5$ nodes. if $\gamma_2(Shack(C_n, S_m, t)) = t-1-1 = t-2$, then the maximum node that can be dominated based on Lemma is $m+5\left(\left\lceil\frac{nt+mt-m-t+1}{m+5}\right\rceil-1\right) \leq m+5\left(\frac{nt+mt-m-t+m+5}{m+5}-1\right) = nt+mt-m-t$. This means that the number of nodes that can be dominated is $nt+mt-m-t$, so there is at least one node that is not dominated. Thus $\gamma_2(Shack(C_n, S_m, t)) \neq t-2$, because $t-1$ is the minimum number of dominating nodes then $\gamma_2(Shack(C_n, S_m, t)) = t-1$. To strengthen the evidence, an example Figure 6 is provided as follows. Where $(C_3, S_6, 3)$ is divided into Graph C_n which is a Cycle graph C_3 with a connecting Star graph S_6 and $t = 3$ copies.

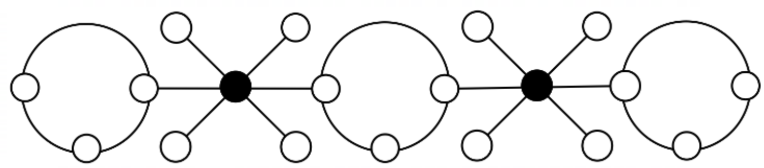

Figure *6*. Graph Shack $(C_3, S_6, 3)$ with the Black Node being the Dominating Node

***for* $4 \leq n \leq 5$**

Next it will be proven that for $\gamma_2(Shack(C_n, S_m, t)) = t$ for $4 \leq n \leq 5$. In this case, the number of 1st copies to the 2nd copy, the number of dominating numbers within two $Shack(C_n, S_m, t)$ will form an arithmetic sequence as follows:

The 1st term for $t = 1 \rightarrow 1$
The 2nd term for $t = 2 \rightarrow 2$
The 3rd term for $t = 3 \rightarrow 3$
$\vdots$
The kth term for $t = t \rightarrow t$

From this sequence we will get the domination number of the two distances in the graph $Shack(C_n, S_m, t)$ which t. Next, it will be proven using mathematical induction as follows:

$$\gamma_2(Shack(C_n, S_m, t)) = t \tag{19}$$

It will be proven that the 1th term is correct

$$\gamma_2(Shack(C_n, S_m, 1)\ ) = 1 \qquad (20)$$
$$\Leftrightarrow \qquad 1 = 1$$

Assume that the k^th term is correct.

$$\gamma_2(Shack(C_n, S_m, k)\ ) = k \qquad (21)$$

It will be proven that $k+1$ ^th is also correct

$$\gamma_2\big(Shack\big((C_n, S_m, k+1)\ \big)\big) = \gamma_2\big(Shack\big((C_n, S_m, k)\ \big)\big) + difference \qquad (22)$$
$$\Leftrightarrow k+1 = k+1$$

Thus, the graph $Shack(C_n, S_m, t)$ has a domination number of t. Next it will be shown that t is the minimum number of dominating nodes in the graph $Shack(C_n, S_m, t)$. In a graph $Shack(C_n, S_m, t)$ with $4 \leq n \leq 5$, has $nt + mt - m - t + 1$ nodes. Each dominating node can dominate a maximum of as many $m+5$ nodes. If $\gamma_2(Shack(C_n, S_m, t)\ ) = t - 1$, then that the maximum node that can be dominated based on Lemma is $m + 5\ \left(\left\lceil\frac{nt+mt-m-t+1}{m+5}\right\rceil - 1\right) \leq m + 5\left(\frac{nt+mt-m-t+m+5}{m+5} - 1\right) = nt + mt - m - t$. This means that the number of nodes that can be dominated is $nt + mt - m - t$, so there is at least one node that is not dominated. Thus $\gamma_2(Shack(C_n, S_m, t)\ ) \neq t - 1$, because is the minimum number of dominating nodes then $\gamma_2(Shack(C_n, S_m, t)\ ) = t$.

$$\boldsymbol{for\ n \geq 6}$$

Next, it will be proven that $\gamma_2(Shack(C_n, S_m, t)\ ) = \left\lceil\frac{n}{5}\right\rceil t, for\ n \geq 6$ there is a minimum number of dominating nodes that can dominate all nodes in the graph $Shack(C_n, S_m, t)$. If $\gamma_2(Shack(C_n, S_m, t)\ ) = \left\lceil\frac{n}{5}\right\rceil t - 1$, then that the maximum node that can be dominated based on Lemma is $m + 5\ \left(\left\lceil\frac{nt+mt-m-t+1}{m+5}\right\rceil - 1\right) \leq m + 5\left(\frac{nt+mt-m-t+m+5}{m+5} - 1\right) = nt + mt - m - t$. This means that the number of nodes that can be dominated is $nt + mt - m - t$, so there is at least one node that Is not dominated. Thus, $\gamma_2(Shack(C_n, S_m, t)\ ) \neq \left\lceil\frac{n}{5}\right\rceil t - 1$, because t is the minimum number of dominating nodes then $\gamma_2(Shack(C_n, S_m, t)\ ) = \left\lceil\frac{n}{5}\right\rceil t$.

D. CONCLUSION AND SUGGESTIONS

The following are the conclusions obtained from the results and previous discussions: graph S_n with t copies and S_m as linkage, then the two-distance domination number in the graph resulting from the shackle operation is

$$\gamma_2(Shack(S_n, S_m, t)\ ) = t - 1$$

graph S_n with t copies and C_m as linkage, then the two-distance domination number in the graph resulting from the shackle operation is

$$\gamma_2(Shack(S_n, C_m, t)\ ) = \begin{cases} t, for\ 3 \leq m \leq 6 \\ \left\lceil \frac{n}{5} \right\rceil (t-1), for\ m \geq 7' \end{cases}$$

And graph C_n with t copies and S_m as linkage, then the two-distance domination number in the graph resulting from the shackle operation is

$$\gamma_2(Shack(C_n, S_m, t)\ ) = \begin{cases} t-1, for\ n = 3 \\ t, for\ 4 \leq n \leq 5 \\ \left\lceil \frac{n}{5} \right\rceil t, for\ n \geq 6 \end{cases}$$

Since research it was one of topic development of Domination Numbers in Graphs others, then we also conclude this research with some open problem below: find the resolving of some family of graphs and find the resolving of some product of graphs. Besides that, use this topic to develop/ combine with some other topic in graph theory. Especially adding to the results of research in the field of specialization in graph theory of two-distance domination of graphs resulting from shackle operations with any graph linkage.

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