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Characteristic Polynomial and Eigenproblem of Triangular Matrix over Interval Min-Plus Algebra

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ABSTRACT

A min-plus algebra is a linear algebra over the semiring $\mathbb{R}_{\varepsilon'}$, equipped with the operations "$\oplus$'=min" and "$\otimes= +$". In min-plus algebra, there is the concept of characteristic polynomial obtained from permanent of matrix. Min-plus algebra can be extended to an interval min-plus algebra, which is a set $I(\mathbb{R})_{\varepsilon'}$ equipped with the operations $\overline{\oplus}$' and $\overline{\otimes}$. Matrix over interval min-plus algebra has some special forms, one of which is a triangular matrix. The concept of characteristic polynomial and eigenproblem can be applied to a triangular matrix. There is a more concise formula for solving the eigenproblem of triangular matrix because this matrix is a special form of matrix. This research will discuss the characteristic polynomial and solving eigenproblem of triangular matrix over interval min-plus algebra using its characteristic polynomials. The research method used is a literature study. From the research results, the permanent formula and characteristic polynomial formula of the triangular matrix are obtained. It is also obtained that the smallest corner of the characteristic polynomial is the principal eigenvalue and the vector eigen corresponding to the principal eigenvalue can be obtained through the matrix A_λ. For readers who are interested in this topic, can research about characteristic polynomial and eigenproblem of matrices with other special forms over min-plus interval algebra.

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A. INTRODUCTION

In mathematics, the topic of max-plus algebra is often studied. Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ with $\mathbb{R}$ being the set of all real numbers and $\varepsilon = -\infty$ equipped with the operations $\oplus$ and $\otimes$ (Gyamerah et al., 2016). Operation $\oplus$ is a maximum operation and operation $\otimes$ is a plus operation. If $p, q \in \mathbb{R}_\varepsilon$, then $p \oplus' q = \max\{p, q\}$ and $p \otimes q = p + q$. Max-plus algebra can be denoted as $(\mathbb{R}_\varepsilon,\oplus,\otimes)$ which is a semiring with neutral element ε and unit element $e = 0$. Can be formed $m \times n$ matrix set whose elements are members of $\mathbb{R}_\varepsilon$ called set of matrices over max-plus algebra. This set of matrices can be denoted as $\mathbb{R}_\varepsilon^{m\times n}$. Max-plus algebra has been applied in several problems (Haneefa & Siswanto, 2021)(Nurwan & F. Payu, 2022)(Prastiwi & Listiana, 2017). For application in time intervals, max-plus algebra can be extended to interval max-plus algebra. Interval max-plus algebra is the set $I(\mathbb{R})_\varepsilon$ equipped with operations $\overline{\oplus}$ and $\overline{\otimes}$ (Siswanto *et al.*, 2019). A matrix over interval max-plus algebra is a matrix whose entries are members of $I(\mathbb{R})_\varepsilon$. The set of these matrices is denoted by $I(\mathbb{R})_\varepsilon^{m\times n}$.

Besides max-plus algebra, there is another semiring that is also discussed in mathematics, namely min-plus algebra. A min-plus algebra is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$ with $\mathbb{R}$ being the set of all real numbers and $\varepsilon' = +\infty$ equipped with the operations $\oplus'$ and $\otimes$ (Nowak, 2014; Watanabe & Watanabe, 2014). If $p, q \in \mathbb{R}_{\varepsilon'}$ then $p \oplus' q = \min\{p, q\}$ and $p \otimes q = p + q$. The set of matrices over min-plus algebra is the set of matrices whose entries are members of $\mathbb{R}_{\varepsilon'}$ and is denoted by $\mathbb{R}_{\varepsilon'}^{m\times n}$. The application of min-plus algebra in several problems has been studied (Susilowati & Fitriani, 2019)(Farhi, 2023). Min-plus algebra can also be extended to interval min-plus algebra. An interval min-plus algebra is a set $I(\mathbb{R})_{\varepsilon'}$ equipped with operations $\overline{\oplus'}$ and $\overline{\otimes}$ (Awallia et al., 2020). Matrices whose entries are members of $I(\mathbb{R})_{\varepsilon'}$ are called matrices over interval min-plus algebra. The set of these matrices is denoted by $I(\mathbb{R})_{\varepsilon'}^{m\times n}$.

In conventional algebra, for an $n \times n$ square matrix, its determinant can be found. However, in max-plus algebra the matrix determinant is replaced by the matrix permanent since there is no inverse to the operation $\oplus$. The permanent of matrix over max plus algebra $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ is $perm(A) = \bigoplus_{\sigma\in P_n} a_{1\sigma(1)} \otimes \ldots \otimes a_{n\sigma(n)}$, with P_n being the permutation group on $\{1, 2, \ldots, n\}$ (Rosenmann *et al.*, 2019). The concept of permanent has also been discussed in other semirings, including interval max-plus algebra by Siswanto et al. (2021) and min-plus algebra (Siswanto et al., 2021). From the permanent of matrix, the characteristic polynomial of a matrix can be obtained.

The characteristic polynomial and eigenproblem of a matrix are interconnected concepts. In max-plus algebra, the characteristic polynomial of matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ is $\chi_A(x) = perm(A \oplus x \otimes I)$ (Hook, 2015; Nishida et al., 2020). The concept of this characteristic polynomial in other semirings, that is, in interval max-plus algebra has been studied by Siswanto et al. (2021) and in min-plus algebra has been studied (Maghribi *et al.*, 2023). A characteristic polynomial is one of the methods to solve the eigenproblem of a matrix. The eigen problem aims to find the eigenvalue and eigenvector corresponding to the eigenvalue (Jiang, 2022). In max-plus algebra, eigenvalues can be obtained from the greatest corner of the characteristic polynomial. This eigenvalue is called the principal eigenvalue. The eigenproblem can be written by the equation $A \otimes x = \lambda \otimes x$. Moreover, λ is called the eigenvalue of matrix A and x is called the eigenvector corresponding to the eigenvalue λ (Siswanto, 2023). The application of the eigenproblem in max-plus algebra has been studied in (Al Bermanei et al., 2023; De Schutter et al., 2020; Maharani & Suparwanto, 2022; Permana et al., 2020; Subiono et al., 2018).

The concepts of characteristic polynomials and eigenproblems can be applied to special matrices, one of which is a triangular matrix. The characteristic polynomial of triangular matrix over interval max-plus algebra has been studied (Wulandari & Siswanto, 2019). The characteristic polynomial and eigenproblem of triangular matrix over min-plus algebra have been studied (Maghribi, 2023). In this article, we will discuss the characteristic polynomial of triangular matrix over interval min-plus algebra and solving eigenproblem of triangular matrix over interval min-plus algebra using its characteristic polynomials.

B. METHODS

The research method used is a literature study. This research begins by explaining the definition of triangular matrix over interval min-plus algebra. Next, determine the formulas for the permanent and characteristic polynomials of a triangular matrix. After obtaining the characteristic polynomial formula, the eigenvalue can be determined, which in turn can be obtained the eigenvector corresponding to the eigenvalue. First, some definitions and theorems used for the discussion in this article are given. The following is a definition of a triangular matrix over min-plus algebra and theorem about the permanent of triangular matrix over min-plus algebra.

Definition 1. (Maghribi, 2023) A matrix is called a triangular matrix over min-plus algebra if all entries of above or below the diagonal of the matrix are equal to ε'.

Theorem 1. (Maghribi, 2023) If $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$, then the permanent of triangular matrix A is

$$perm(A) = a_{11} \otimes a_{22} \otimes \dots \otimes a_{nn}. \tag{1}$$

Next, definitions and theorem regarding eigenproblems over min-plus algebra are given.

Definition 2. (Rahayu *et al.*, 2021) Given $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. If $\lambda(A) \in \mathbb{R}_{\varepsilon'}$ and $v \in \mathbb{R}_{\varepsilon'}$ such that v has at least one finite entry and

$$A \otimes v = \lambda \otimes v \tag{2}$$

then λ is an eigenvalue of A and v is the corresponding eigenvector.

Definition 3. (Rahayu *et al.*, 2021) Given $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and λ defined as in Definition 2, the matrix A_λ is defined as

$$A_\lambda = a_{ij} - \lambda, \tag{3}$$

matrix A_λ^+ is defined as

$$A_\lambda^+ = A_\lambda \oplus' A_\lambda^{\otimes 2} \oplus' \dots \oplus' A_\lambda^{\otimes n}, \tag{4}$$

A_λ^* is defined as

$$A_\lambda^* = I \oplus' A_\lambda^+. \tag{5}$$

Theorem 2. (Maghribi *et al.*, 2023) If $(A_\lambda^+)_{vv} = 0$ then the v-th columns in matrix A_λ^+ are eigenvectors corresponding to $\lambda(A)$.

Next, definitions and theorems regarding characteristic polynomial and eigenproblem of triangular matrix over min-plus algebra are given.

Theorem 3. (Maghribi, 2023) The min-plus characteristic polynomial of a triangular matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is

$$\chi_A(x) = (x \oplus' a_{11}) \otimes (x \oplus' a_{22}) \otimes \ldots \otimes (x \oplus' a_{nn}). \tag{6}$$

Theorem 4. (Maghribi, 2023) The smallest corner of the characteristic polynomial of the triangular matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ which is also the principal eigenvalue is

$$\lambda(A) = \bigoplus'_{i\in\mathbb{N}} \; a_{ii}. \tag{7}$$

Theorem 5. (Maghribi, 2023) Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ be a triangular matrix that satisfies the following conditions

1. There is more than one diagonal element that is greater than or equal to λ(A) and
2. All entries above or below the diagonal are greater than or equal to λ(A).

The columns of matrix A_λ containing diagonal element 0 are the eigenvectors of A corresponding to λ(A).

Next, definitions and theorems regarding interval min-plus algebra and eigenproblem over interval min-plus algebra are given.

Definition 5. (Awallia *et al.*, 2020) Interval min-plus algebra is the set of $I(\mathbb{R})_{\varepsilon'}$ defined as

$$I(\mathbb{R})_{\varepsilon'} = \left\{x = [\underline{x}, \overline{x}] \middle| \underline{x}, \overline{x} \in \mathbb{R}_{\varepsilon'}, \underline{x} \leq \overline{x} < \varepsilon'\right\} \cup \{[\varepsilon', \varepsilon']\}, \varepsilon' = \infty. \tag{8}$$

The interval min-plus algebra is equipped with the operations $\oplus'$ and $\otimes$ that for all $x, y \in I(\mathbb{R})_{\varepsilon'}$ holds

1. $x \overline{\oplus'} y = \left[\underline{x} \oplus' \underline{y}, \overline{x} \oplus' \overline{y}\right]$ and
2. $x \overline{\otimes} y = \left[\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}\right]$.

Definition 6. (Awallia *et al.*, 2020) Matrices over interval min-plus algebra is defined as the set of matrices whose entries are members of $I(\mathbb{R})_{\varepsilon'}$ denoted by $I(\mathbb{R})_{\varepsilon'}^{m\times n}$ that is

$$I(\mathbb{R})_{\varepsilon'}^{m\times n} = \left\{A = [A_{ij}] \middle| A_{ij} \in I(\mathbb{R})_{\varepsilon'}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\right\}. \tag{9}$$

Definition 7. (Awallia *et al.*, 2020) Matrices over interval min-plus algebra have binary operations $\oplus'$ and $\otimes$ defined that for all $A, B \in I(\mathbb{R})_{\varepsilon'}^{m\times n}$ and $k \in I(\mathbb{R})_{\varepsilon'}$ holds

$$\left[A \overline{\oplus'} B\right]_{ij} = A_{ij} \overline{\oplus'} B_{ij} \text{ dan } \left[k \overline{\otimes} A\right]_{ij} = k \overline{\otimes} A_{ij} \tag{10}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Definition 8. (Awallia *et al.*, 2020) For every $A \in I(\mathbb{R})_{\varepsilon'}^{m\times p}$ and $B \in I(\mathbb{R})_{\varepsilon'}^{p\times n}$ is defined

$$\left[A \overline{\otimes} B\right]_{ij} = \overline{\oplus'}_{k=1}^{p} A_{ik} \otimes B_{kj} \tag{11}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Definition 9. (Awallia *et al.*, 2020) Given an interval matrix $A \in I(\mathbb{R})_{\varepsilon'}^{m\times n}$ with $\underline{A}$ and $\overline{A}$ being its lower bound matrix and upper bound matrix, respectively. The matrix interval of A denoted by $\left[\underline{A}, \overline{A}\right]$ is defined as

$$\left[\underline{A}, \overline{A}\right] = \left\{A \in I(\mathbb{R})_{\varepsilon'}^{m\times n} \,\middle|\, \underline{A} \leq A \leq \overline{A}\right\} \tag{12}$$

and the set of matrix intervals of A denoted by $I\left(\mathbb{R}_{\varepsilon'}^{m\times n}\right)_b$ is defined as

$$I\left(\mathbb{R}_{\varepsilon'}^{m\times n}\right)_b = \left\{A = \left[\underline{A}, \overline{A}\right] \middle| A \in I(\mathbb{R})_{\varepsilon'}^{m\times n}\right\}. \tag{13}$$

Definition 10. (Awallia, 2020) Given $\in I(\mathbb{R})_{\varepsilon'}^{n\times n}$. An interval scalar $\lambda(A) \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is called interval min-plus eigenvalue of matrix A if there exists an interval vector $v \in I(\mathbb{R})_{\varepsilon'}^{n}$ with $v \neq \overline{\varepsilon'}_{n\times 1}$ such that $A \overline{\otimes} v = \lambda \overline{\otimes} v$. The vector v is called the interval min-plus eigenvector of matrix A corresponding to λ.

Theorem 6. (Awallia, 2020) Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$, $A \approx \left[\underline{A}, \overline{A}\right] \in I\left(\mathbb{R}_{\varepsilon'}^{n\times n}\right)_b$ and $\lambda(A)$ be an eigenvalue of matrix A. If $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$ with $\underline{\lambda}(\underline{A}) < \varepsilon'$ and $\overline{\lambda}(\overline{A})) < \varepsilon'$ then the columns of A_λ^+ with the lower bound of its diagonal elements as the eigenvectors of matrix A and the upper bound of its diagonal elements 0, are the eigenvectors of matrix A corresponding to the eigenvalues of $\lambda(A)$.

C. RESULT AND DISCUSSION

1. Characteristic polynomial of triangular matrix

This section discusses the characteristic polynomial of triangular matrix over interval min-plus algebra. First, define the triangular matrix over interval min-plus algebra, then determine the permanent formula of the triangular matrix, which is then used to determine the characteristic polynomial formula of the triangular matrix. The following is given the definition of a triangular matrix over interval min-plus algebra, the theorem about the permanent formula of the triangular matrix, and the theorem about the characteristic polynomial formula of the triangular matrix.

Definition 11. A matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is called a triangular matrix if all entries of above or below the diagonal are $[\varepsilon', \varepsilon']$.

Theorem 7. The permanent of a triangular matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is

$$perm(A) = [\underline{a}_{11}, \overline{a}_{11}] \otimes [\underline{a}_{22}, \overline{a}_{22}] \otimes \ldots \otimes [\underline{a}_{nn}, \overline{a}_{nn}]. \quad (14)$$

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be an upper triangular matrix whose diagonal elements are $[\underline{a}_{11}, \overline{a}_{11}], [\underline{a}_{22}, \overline{a}_{22}], \ldots, [\underline{a}_{nn}, \overline{a}_{nn}]$. Matrix A can be written as

$$A = \begin{bmatrix} [\underline{a}_{11}, \overline{a}_{11}] & [\underline{a}_{12}, \overline{a}_{12}] & \cdots & [\underline{a}_{1n}, \overline{a}_{1n}] \\ [\varepsilon', \varepsilon'] & [\underline{a}_{22}, \overline{a}_{22}] & \cdots & [\underline{a}_{2n}, \overline{a}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\varepsilon', \varepsilon'] & [\varepsilon', \varepsilon'] & \cdots & [\underline{a}_{nn}, \overline{a}_{nn}] \end{bmatrix} \quad (15)$$

so that the lower bound matrix of matrix A is obtained

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \varepsilon' & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \underline{a}_{nn} \end{bmatrix} \quad (16)$$

and the upper bound matrix of matrix A is obtained

$$\overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \varepsilon' & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \overline{a}_{nn} \end{bmatrix} \quad (17)$$

such that $A \approx [\underline{A}, \overline{A}] \in I\left(\mathbb{R}_{\varepsilon'}^{n\times n}\right)_b$. Based on Theorem 1, it can be obtained that

$$perm(\underline{A}) = perm \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \varepsilon' & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \underline{a}_{nn} \end{bmatrix} = \underline{a}_{11} \otimes \underline{a}_{22} \otimes \ldots \otimes \underline{a}_{nn} \quad (18)$$

and

$$perm(\overline{A}) = perm \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \varepsilon' & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \cdots & \overline{a}_{nn} \end{bmatrix} = \overline{a}_{11} \otimes \overline{a}_{22} \otimes \ldots \otimes \overline{a}_{nn} \quad (19)$$

so that

$$\begin{aligned} perm(A) &= [perm(\underline{A}), perm(\overline{A})] \\ &= [\underline{a}_{11} \otimes \underline{a}_{22} \otimes \dots \otimes \underline{a}_{nn}, \overline{a}_{11} \otimes \overline{a}_{22} \otimes \dots \otimes \overline{a}_{nn}] \\ &= [\underline{a}_{11}, \overline{a}_{11}] \overline{\otimes} [\underline{a}_{22}, \overline{a}_{22}] \overline{\otimes} \dots \overline{\otimes} [\underline{a}_{nn}, \overline{a}_{nn}]. \end{aligned} \tag{20}$$

It can be proved for the lower triangular matrix in the same way.

■

Theorem 8. The interval min-plus characteristic polynomial of a triangular matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is

$$\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})] \tag{21}$$

with

$$\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn}) \tag{22}$$

and

$$\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn}). \tag{23}$$

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be a triangular matrix with $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$. Based on Theorem 3, it is obtained that the characteristic polynomial of $\underline{A}$ is

$$\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn}) \tag{24}$$

and the characteristic polynomial of $\overline{A}$ is

$$\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn}). \tag{25}$$

Since $A \approx [\underline{A}, \overline{A}]$, then can obtained the characteristic polynomial of A is $\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$ with $\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn})$ and $\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn})$.

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2. Eigenproblem of Triangular Matrix

This section discusses the eigenproblem of triangular matrix over interval min plus algebra. The eigen problem aims to find the eigenvalue and eigenvector corresponding to the eigenvalue. The following theorems are given about the smallest corner of the characteristic polynomial of the triangular matrix, the theorem about the eigenvalue of the triangular matrix, and the theorem about the eigenvector corresponding to the eigenvalue.

Theorem 9. The smallest corner of the characteristic polynomial of a triangular matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is

$$\delta_1 = [\underline{\delta}_1, \overline{\delta}_1] \tag{25}$$

with $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ and $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$.

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be a triangular matrix with $A \approx [\underline{A}, \overline{A}]$ where $[\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$. Based on Theorem 8, it is obtained that the characteristic polynomial of A is

$$\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})] \tag{26}$$

with

$$\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \dots \otimes (\underline{x} \oplus' \underline{a}_{nn}) \tag{27}$$

and

$$\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \dots \otimes (\overline{x} \oplus' \overline{a}_{nn}). \tag{28}$$

Furthermore, by Theorem 4, can be obtained that the smallest corner of $\chi_{\underline{A}}(\underline{x})$ is $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ and the smallest corner of $\chi_{\overline{A}}(\overline{x})$ is $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$. Therefore, the smallest corner of the characteristic polynomial $\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$ is $\delta_1 = [\underline{\delta}_1, \overline{\delta}_1]$ with $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ and $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$.

■

Theorem 10. If $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ is a triangular matrix where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$ then the smallest corner of $\chi_A(x)$ is $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$.

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be a triangular matrix with $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$. Based on Theorem 9, the smallest corner characteristic polynomial of A is $\delta_1 = [\underline{\delta}_1, \overline{\delta}_1]$ with $\underline{\delta}_1 = \oplus'^{n}_{i=1} \ \underline{a}_{ii}$ dan $\overline{\delta}_1 = \oplus'^{n}_{i=1} \ \overline{a}_{ii}$. Furthermore, based on Theorem 4, it is obtained that the smallest corner of $\chi_{\underline{A}}(\underline{x})$ is $\underline{\lambda}(\underline{A})$ and the smallest corner of adalah $\overline{\lambda}(\overline{A})$. Since $\chi_A(x) \approx [\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})]$ then the smallest corner of $\chi_A(x)$ is $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$.

■

Theorem 11. If $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ a triangular matrix where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$ and $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$ is the principal eigenvalues of matrix A then the eigenvectors of matrix A corresponding to $\lambda(A)$ are the columns of matrix A_λ with the lower bound of its diagonal elements as the eigenvectors of matrix A and the upper bound of its diagonal elements 0.

Proof. Let $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ be a triangular matrix where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon'}^{n\times n})_b$ and $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$ is the principal eigenvalues of matrix A with $\underline{\lambda}(\underline{A})$ being the eigenvalue of matrix

$\underline{A}$ and $\overline{\lambda}(\overline{A})$ being the eigenvalue of matrix $\overline{A}$. Can be determined the matrix $\underline{A}_{\underline{\lambda}}$ with $\underline{A}_{\underline{\lambda}} = \underline{a}_{ij} - \underline{\lambda}(\underline{A})$ and the matrix $\overline{A}_{\overline{\lambda}}$ with $\overline{A}_{\overline{\lambda}} = \overline{a}_{ij} - \overline{\lambda}(\overline{A})$. Let $\underline{g}_k$ and $\overline{g}_k$, $k = 1, 2, \ldots, n$ be the columns of matrices $\underline{A}_{\underline{\lambda}}$ and $\overline{A}_{\overline{\lambda}}$ respectively. Next, the matrix A_λ is formed, whose columns are determined as follows.

1. If for k in pairs of $\underline{g}_k$ and $\overline{g}_k$ holds $\underline{g}_k \leq \overline{g}_k$, then define the k-column of A_λ as the interval vector $g_k \approx \left[\underline{g}_k, \overline{g}_k\right]$.
2. If for k in pairs of $\underline{g}_k$ and $\overline{g}_k$ holds $\underline{g}_k \nleq \overline{g}_k$, then define $\underline{g}_k^* = -\delta \otimes \underline{g}_k$ with $\delta = maks_i\left(\left(\underline{g}_k\right)_i - \left(\overline{g}_k\right)_i\right)$, $i = 1, 2, .., n$ and the k-column of A_λ as the interval vector $g_k \approx \left[\underline{g}_k^*, \overline{g}_k\right]$.

Based on Theorem 2, it is obtained that the eigenvector of the matrix $\underline{A}$ corresponding to $\underline{\lambda}(\underline{A})$ is any column of the matrix $\underline{A}_{\underline{\lambda}}^+$ whose diagonal element is 0. By Definition 3, can be obtained $\underline{A}_{\underline{\lambda}}^+ = \underline{A}_{\underline{\lambda}} \oplus' \underline{A}_{\underline{\lambda}}^{\otimes 2} \oplus' \ldots \oplus' \underline{A}_{\underline{\lambda}}^{\otimes n}$. Since $\underline{\lambda}(\underline{A}) = \oplus'^{n}_{i=1}\ \underline{a}_{ii}$, then elements of $\underline{a}_{ii} - \underline{\lambda}(\underline{A})$ in the matrix $\underline{A}_{\underline{\lambda}}$ are nonnegative for $\underline{a}_{ii} \neq \lambda(A)$ and 0 for $\underline{a}_{ii} = \lambda(A)$. Furthermore, the elements of $\underline{A}_{\underline{\lambda}}$ that are nonnegative will always be nonnegative for the elements of $\underline{A}_{\underline{\lambda}}$ that are adjacent, and the elements of $\underline{A}_{\underline{\lambda}}$ will be less than or equal to the elements of $\underline{A}_{\underline{\lambda}}^{\otimes n}$. On the other side, the diagonal elements of $\underline{A}_{\underline{\lambda}}$ that equal to 0 are also always equal to 0 for the corresponding diagonal elements of $\underline{A}_{\underline{\lambda}}^{\otimes n}$. Therefore, for a triangular matrix, $\underline{A}_{\underline{\lambda}}^+$ is $\underline{A}_{\underline{\lambda}}$. That is, the eigenvector of the matrix $\underline{A}$ corresponding to $\underline{A}_{\underline{\lambda}}$ is any column of the matrix $\underline{A}_{\underline{\lambda}}$ whose diagonal element is 0. In the same way, it is obtained that $\overline{A}_{\overline{\lambda}}^+$ is $\overline{A}_{\overline{\lambda}}$ and the eigenvector of the matrix $\overline{A}$ corresponding to $\overline{A}_{\overline{\lambda}}$ is any column of the matrix $\overline{A}_{\overline{\lambda}}$ whose diagonal element is 0. Since $\underline{A}_{\underline{\lambda}}^+$ is $\underline{A}_{\underline{\lambda}}$ and $\overline{A}_{\overline{\lambda}}^+$ is $\overline{A}_{\overline{\lambda}}$ then A_λ^+ is A_λ. Furthermore, based on Theorem 6, the eigenvectors of matrix A corresponding to $\lambda(A)$ are the columns of matrix A_λ and the lower bound of its diagonal elements as the eigenvectors of matrix $\underline{A}$ and the upper bound of its diagonal elements 0.

■

D. CONCLUSION AND SUGGESTIONS

From the results and discussion, it is obtained that the permanent of the triangular matrix is the multiplication of its diagonal elements and characteristic polynomial formula of the triangular matrix is $\chi_A(x) \approx \left[\chi_{\underline{A}}(\underline{x}), \chi_{\overline{A}}(\overline{x})\right]$ with $\chi_{\underline{A}}(\underline{x}) = (\underline{x} \oplus' \underline{a}_{11}) \otimes (\underline{x} \oplus' \underline{a}_{22}) \otimes \ldots \otimes (\underline{x} \oplus' \underline{a}_{nn})$ and $\chi_{\overline{A}}(\overline{x}) = (\overline{x} \oplus' \overline{a}_{11}) \otimes (\overline{x} \oplus' \overline{a}_{22}) \otimes \ldots \otimes (\overline{x} \oplus' \overline{a}_{nn})$. Besides that, it is obtained that smallest corner of characteristic polynomial of the triangular matrix is the main eigenvalue and the eigenvector corresponding to the main eigenvalue are the columns of the matrix A_λ with the lower bound of its diagonal elements as the eigenvector of matrix A and the upper bound of its diagonal elements 0. In this research has been discussed on the polynomial characteristics of triangular matrices over interval min-plus algebra. Besides that, it has also discussed the eigenproblem of triangular matrix over interval min-plus algebra. For readers who are interested in this topic, can research about characteristic polynomial and eigenproblem of matrices with other special forms over min-plus interval algebra.

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REFERENCES

Al Bermanei, H., Böling, J. M., & Högnäs, G. (2023). Modeling and scheduling of production systems by using max-plus algebra. *Flexible Services and Manufacturing Journal*, 36, 129-150. https://doi.org/10.1007/s10696-023-09484-z

Awallia, A. R. (2020). *Struktur Aljabar Min-Plus Interval dan Masalah Nilai Eigen Matriks atas Aljabar Min-Plus Interval* [Skripsi]. Universitas Sebelas Maret.

Awallia, A. R., Siswanto, & Kurniawan, V. Y. (2020). Interval min-plus algebraic structure and matrices over interval min-plus algebra. *Journal of Physics: Conference Series*, *1494*(1), 1–7. https://doi.org/10.1088/1742-6596/1494/1/012010

De Schutter, B., van den Boom, T., Xu, J., & Farahani, S. S. (2020). Analysis and control of max-plus linear discrete-event systems: An introduction. *Discrete Event Dynamic Systems: Theory and Applications*, *30*(1), 25–54. https://doi.org/10.1007/s10626-019-00294-w

Farhi, N. (2023). A Min-Plus Algebra System Theory for Traffic Networks. *Mathematics*, *11*(19), 1–23. https://doi.org/10.3390/math11194028

Gyamerah, S., Boateng, P., & Harvim, P. (2016). Max-plus Algebra and Application to Matrix Operations. *British Journal of Mathematics & Computer Science*, *12*(3), 1–14. https://doi.org/10.9734/bjmcs/2016/21639

Haneefa, G. P. T., & Siswanto. (2021). Petri Net Model and Max-Plus Algebra in Outpatient Care at Solo Peduli Clinic, Surakarta. *Journal of Physics: Conference Series*, *1776*(1), 1–7. https://doi.org/10.1088/1742-6596/1776/1/012047

Hook, J. (2015). Max-plus singular values. *Linear Algebra and Its Applications*, *486*, 419–442. https://doi.org/10.1016/j.laa.2015.08.019

Jiang, Y. (2022). Study on eigenvalue and eigenvector introduction. *Journal of Physics: Conference Series*, *2282*(1), 1–8. https://doi.org/10.1088/1742-6596/2282/1/012004

Maghribi, S. M. Al. (2023). *Polinomial Karakteristik dan Masalah Eigen Matriks atas Aljabar Min-Plus* [Skripsi]. Universitas Sebelas Maret.

Maghribi, S. M. Al, Siswanto, & Sutrima. (2023). *Characteristic Min-Polynomial and Eigen Problem of a Matrix over Min-Plus Algebra*. *7*(4), 1108–1117. https://doi.org/10.31764/jtam.v7i4.16498

Maharani, A. E. S. H., & Suparwanto, A. (2022). Application of System Max-Plus Linear Equations on Serial Manufacturing Machine with Storage Unit. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *16*(2), 525–530. https://doi.org/10.30598/barekengvol16iss2pp525-530

Nishida, Y., Watanabe, S., & Watanabe, Y. (2020). On The Vectors Associated with the Roots of Max-Plus Characteristic Polynomials. *Applications of Mathematics*, *65*(6), 785–805. https://doi.org/10.21136/AM.2020.0374-19

Nowak, A. W. (2014). *The Tropical Eigenvalue-Vector Problem from Algebraic, Graphical, and Computational Perspectives* [Honors Theses, Bates College]. http://scarab.bates.edu/honorstheses/97

Nurwan, N., & F. Payu, M. R. (2022). Max-Plus Algebra Model on Inaportnet System Ships Service Scheme. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *16*(1), 147–156. https://doi.org/10.30598/barekengvol16iss1pp147-156

Permana, A., Siswanto, & Pangadi. (2020). Eigen Problem Over Max-Plus Algebra on Determination of the T3 Brand Shuttlecock Production Schedule. *Numerical: Jurnal Matematika Dan Pendidikan Matematika*, 23–30. https://doi.org/10.25217/numerical.v4i1.702

Prastiwi, L., & Listiana, Y. (2017). The Aplication of Max-Plus Algebra to Determine The Optimal Time of Ikat Kupang Woven Production. *International Journal of Computing Science and Applied Mathematics*, *3*(2), 77–80. http://dx.doi.org/10.12962/j24775401.v3i2.2317

Rahayu, E. W., Siswanto, S., & Wiyono, S. B. (2021). Masalah Eigen dan Eigenmode Matriks atas Aljabar Min-Plus. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *15*(4), 659–666. https://doi.org/10.30598/barekengvol15iss4pp659-666

Rosenmann, A., Lehner, F., & Peperko, A. (2019). Polynomial convolutions in max-plus algebra. *Linear Algebra and Its Applications*, *578*, 370–401. https://doi.org/10.1016/j.laa.2019.05.020

Siswanto. (2023). The Existence of Solution of Generalized Eigenproblem in Interval Max-Plus Algebra. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *17*(3), 1341–1346. https://doi.org/10.30598/barekengvol17iss3pp1341-1346

Siswanto, Gusmizain, A., & Wibowo, S. (2021). Determinant of a matrix over min-plus algebra. *Journal of Discrete Mathematical Sciences and Cryptography*, *24*(6), 1829–1835. https://doi.org/10.1080/09720529.2021.1948663

Siswanto, Kurniawan, V. Y., Pangadi, & Wiyono, S. B. (2021). Characteristic Polynomial of Matrices over Interval Max-plus Algebra. *AIP Conference Proceedings*, *2326*, 1–5. https://doi.org/10.1063/5.0039779

Siswanto, Pangadi, & Wiyono, S. B. (2019). Robust matrices in the interval max-plus algebra. *Journal of Physics: Conference Series*, *1265*(1), 1–7. https://doi.org/10.1088/1742-6596/1265/1/012029

Subiono, Fahim, K., & Adzkiya, D. (2018). Generalized Public Transportation Scheduling Using Max-plus Algebra. *Kybernetika*, *54*(2), 243–267. https://doi.org/10.14736/kyb-2018-2-0243

Susilowati, E., & Fitriani, F. (2019). Determining The Shortest Path Between Terminal and Airport in Yogyakarta Using Trans Jogja with Min Plus Algorithm. *MUST: Journal of Mathematics Education*, *4*(2), 123–134. https://doi.org/10.30651/must.v4i2.2966

Watanabe, S., & Watanabe, Y. (2014). Min-Plus Algebra and Networks. *RIMS Kôkyûroku Bessatsu*, *47*, 41–54. https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/226217/1/B47_004.pdf

Wulandari, A. V., & Siswanto. (2019). Characteristic Polynomial of a Triangular and Diagonal Strictly Double ℝ-astic Matrices over Interval Max-Plus Algebra. *Journal of Physics: Conference Series*, *1306*(1), 1–7. https://doi.org/10.1088/1742-6596/1306/1/012005