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Game Chromatic Number of Tadpole Graph, Broom Graph, and Tribune Graph

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ABSTRACT

Graph coloring game is one of application in graph theory. The goal in this article is determine game chromatic number of tadpole graph, broom graph, and tribune graph. The graphs are simple, connected, and undirected and thus eligible for playing graph coloring game. Given two players with the first player is called A and second player is called B coloring vertex of graph G with a set of colors $C = \{c_1, c_2, c_3, ..., c_k\}$. A must to make sure that all vertex of G has colored and B must try to prevent A coloring of all vertex. The first step was taken by A as first player, two players take turns coloring vertex of graph G, with rule that every vertex have different color from the neighbourhood. If all vertex of graph G have been colored, then A win or B win if some vertex hasn't colored. The smallest number of k if A has a strategy to win at graph G with k color, then k called game chromatic number which is denoted by $\chi_g(G)$. The strategy to win this game is coloring the biggest degree of vertex in graph first. The result obtained from this paper is A win the game with the strategy of first coloring the largest degrees of vertex. So, exact of game chromatic number of tadpole graph is 3, broom graph is 2 or 3 with several conditions, and tribune graph is 3 or 4 with several conditions.

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A. INTRODUCTION

Graph theory is a branch of discipline mathematics which study about vertices and edges. Graph theory was discovered by Swiss mathematician, Leonhard Euler at 1736 when he solved Konigsberg bridge problem which popular in Europe (Abdurahman et al., 2024). Generally, the simple graph denoted by G with $V(G)$ is a set of vertex, $E(G)$ is a set of edge of G, and $(V(G), E(G))$ is a finite set (Maghfiro et al., 2023). Application of graph theory has implemented on daily life and many research on graph is often developed until now. Generally, graph serve for modelling of problem to make it easier. Example, graph can be implemented for fuzzy chemical molecular graphs (Lu et al., 2023), efficiency analysis planetary gear transmission (Xue & Li, 2023), and approximation graph coloring with hybrid quantum-classical algorithms (Bravyi et al., 2022). One of the unique topic in graph theory is game chromatic number. This topic has been invented by Hans L. Bodlaender on 1989. Game chromatic number is the result after playing graph coloring game. Given two players with first player has named A and B as second player. They will coloring vertex of graph G with a set of colors $\{1, 2, 3, \dots, k\}$. A must make sure that all vertex of G are colored, B must prevent it. The rule is they must obey to coloring vertex and with a different color from neighbortborhood vertex. If all vertex of G has been colored, then A win and otherwise B win. The smallest k such that A win the game vertex coloring of G is called game chromatic number, denoted by $\chi_g(G)$.

This paper discuss the vertices coloring of a graph. The rule is simple that every vertices must be colored by different color with neighbourhood vertices in graph. The number of smallest k such that all vertices of G has been colored with k color, then the chromatic number of G is k denoted by $\chi(G) = k$. The problem of this article is how to find the minimum value of game chromatic number and win the game. So, the strategy must be use to make it win easier. The strategy is coloring the vertex with the biggest degree on graph. When this vertex has been coloring, then opposite player can't coloring this vertex and choose another vertex and player has a chance to make all vertices colored by some color. Therefore, all vertices has been colored and player winning the game with minimum game chromatic number. Some graph in articles are determined of game chromatic number with graph coloring game like game chromatic number of jellyfish graph, snail graph, and octopus graph (Abdurahman et al., 2024), game chromatic number of splitting graph of path and cycle (Akhtar et al., 2019), and the game chromatic numbers of corona product graphs (Yusuf et al., 2023). These articles discuss about how to determine game chromatic number of graph with find the minimum colors of graph coloring game.

Many other referrence has supported to develop and make a relevant of this article. There are some referrence in this article use to studying about game chromatic number like (Bharadwaj & Mangam, n.d.), (Furtado et al., 2019), (Samli et al., n.d.), (Fernandez V & Warnke, 2024), (Matsumoto, 2019), (Kierstead, 2005), (Jabbar et al., 2020), (Escoffier et al., n.d.), (Budi et al., 2021), and (Saifudin et al., 2024). In addition, this articles use some graph to find the game chromatic number, the graph are tadpole graph (Manamtam et al., 2022), broom graph (Sriram et al., n.d.), and tribune graph (Anggalia, 2017). Another of referrences is supporting to explore about game chromatic number like (Garrett, 2021), (Inayah et al., 2021), (Saleh et al., 2020), (Bradshaw & Zeng, 2024), (Joedo et al., 2022), (Bradshaw, 2021), (Pavithra & Vijayalakshmi, n.d.), (Ponraj et al., 2022), (Samuel & Vani, 2018), (Liu & Ning, 2023), (Lone & Goswami, 2023), and (Corso et al., n.d.). Based on the research in these referrence, we know that tadpole graph, broom graph, and tribune graph is eligible to use as graph coloring game. The reason is because these graph is eligible graph to playing graph coloring game (simple, connected, and undirected) and determine game chromatic numberTherefore, the purpose of this article is how to find the game chromatic number of tadpole graph, broom graph, and tribune graph.

B. METHODS

This article use a literature study with some theorem from several reference. Basic theorem is using to determine game chromatic number of graph. The method used deductive proof with principal of detemine chromatic number of graph to solve graph coloring game with several theorem. Then, the result can help to determine game chromatic number of graph. The research stage for determine tadpole graph, broom graph, and tribune graph will be explained as follows.

1. The first stage of these method is find the chromatic number of graph. Author knew chromatic number of graph with Welch-Powell algorithm.
2. After find the chromatic number, author must find the biggest degree in graph. The purpose of this stage is helping player to make a strategy during graph coloring game.

3. At this stage, graph coloring game is begin with coloring vertex with the biggest degree. The chromatic number make easier to knowing the minimum of game chromatic number.
4. After this, author use a strategy and playing with opposite to to find game chromatic number of tadpole graph, broom graph, and tribune graph. The game chromatic number is right if total color after graph coloring game is minimum.
5. The final stage is writing the conclusion of game chromatic number of graph. These number can be write after passed all stage.

C. RESULT AND DISCUSSION

Before find the game chromatic number, we must use basic theoritical about graph coloring game. In the articles by T. Bartnicki on 2008 and Abdul Mujib on 2019 (Abdurahman et al., 2024), we know that $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$.

Theorem 1 (Abdurahman et al., 2024) Let G be a graph and $\Delta(G)$ is largest degree of G, then $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$.

Proof. Based on definition of $\chi(G)$, then vertex of graph G can't have been colored by k color if avaliable color less than $\chi(G)$. So, that is impossible when $\chi_g(G)$ less than $\chi(G)$, $\chi(G) \leq \chi_g(G)$. Then, if A as first player could be coloring of all vertices with $v \leq V(G)$, then A always win. The statement is valid if avaliable color is $\Delta(G) + 1$. Therefore $\chi_g(G) \leq \Delta(G) + 1$. Given graph G on Figure 5 with a set of $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$, where element in $E(G)$ is $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_5$, and $e_3 = v_1v_5$.

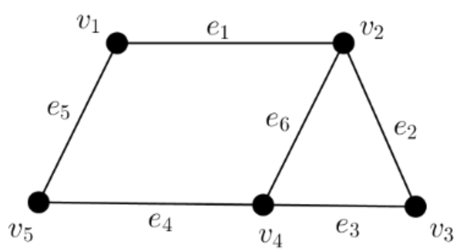

Figure 5. Graph G

If G has been colored by vertex coloring, then $\chi(G) = 3$. In this graph, the game chromatic number of G is $3 \leq \chi_g(G) \leq 4$. Given two players A and B to playing graph coloring game of graph G. Let A coloring v_4 with red color. Then, B coloring v_3 with green color. Next, A coloring v_2 with blue color. After that, B coloring v_1 with green color. Then A coloring v_5 with blue color. The result shown in Figure 6 and $\chi_g(G) = 3$.

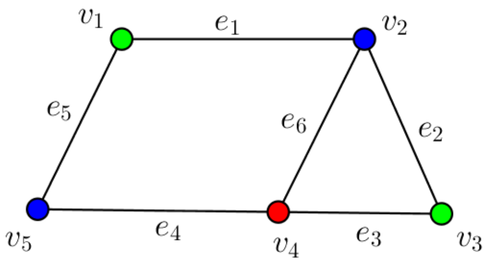

Figure 6. Graph Coloring Game of G

Now we were determining the game chromatic number of tadpole graph, broom graph, and tribune graph with many theorem. Before that, given tadpole graph, broom graph, and tribune graph information. Tadpole graph is a graph obtained by joining a cycle graph C_m to path graph P_n with a bridge. Tadpole graph denoted by $T_{m,n}$ with $m \geq 3$ and $n \geq 2$ (Manamtam et al., 2022). In Figure 1 as an illustration of tadpole graph $T_{m,n}$. The smallest and largest degrees of $T_{m,n}$ is 1 and 3. Broom graph is a specific kind of graph on n vertices, having a path P with d vertices, and pendant vertices with $n-d$, all of these being ajdacent to either the origin u or the terminus v of the path P (Sriram et al., n.d.). Broom graph denoted by $B_{n,d}$ with $d \geq 2$ and $n \geq d$. In Figure 2 as an illustration of broom The smallest and largest degrees of $B_{n,d}$ is 1 and $m+1$. Tribune graph is a graph n times with set of $V(\mathfrak{T}_n)$ and $E(\mathfrak{T}_n)$ (Anggalia, 2017). Given a set of vertices and edges that $V(\mathfrak{T}_n) = \{x_i, z_i, y_j; 1 \leq i \leq 2n; 1 \leq j \leq n+1\}$ and $E(\mathfrak{T}_n) = \{x_i z_i; i\ odd, 1 \leq i \leq 2n\} \cup \{y_j x_{2j-1};\ 1 \leq j \leq n\} \cup \{z_i y_{j+1}; i\ even, 1 \leq i \leq 2n; 1 \leq j \leq n\} \cup \{y_j z_{2j-1};\ 1 \leq j \leq n\} \cup \{z_i z_{i+1}; i\ odd, 1 \leq i \leq 2n\} \cup \{y_j x_{2j}; 1 \leq j \leq n\} \cup \{y_j x_{2j}; 1 \leq j \leq n\} \cup \{x_i z_i; i\ even, 1 \leq i \leq 2n\} \cup \{z_i y_{j+1}; i\ odd, 1 \leq i \leq 2n, i \leq j \leq n\} \cup \{z_i x_{i-1}; i\ even, 1 \leq i \leq 2n\}$. Denoted by $\mathfrak{T}_n$ with $n \geq 1$. The smallest and largest degrees of $\mathfrak{T}_n$ is 2 and 5 for $n \geq 2$ and $\delta(\mathfrak{T}_1) = 2$ and $\Delta(\mathfrak{T}_1) = 4$.

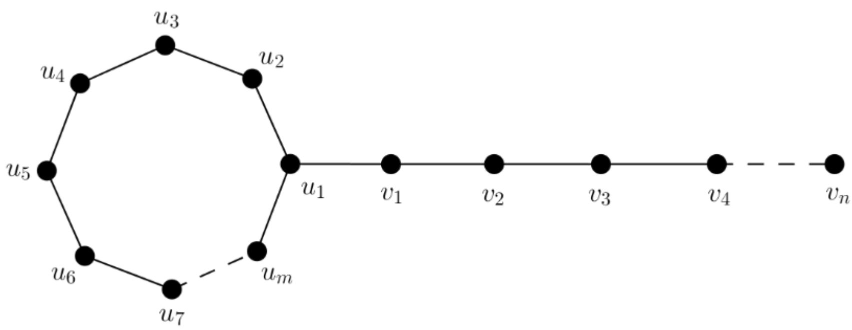

Figure 1. Tadpole Graph $T_{m,n}$

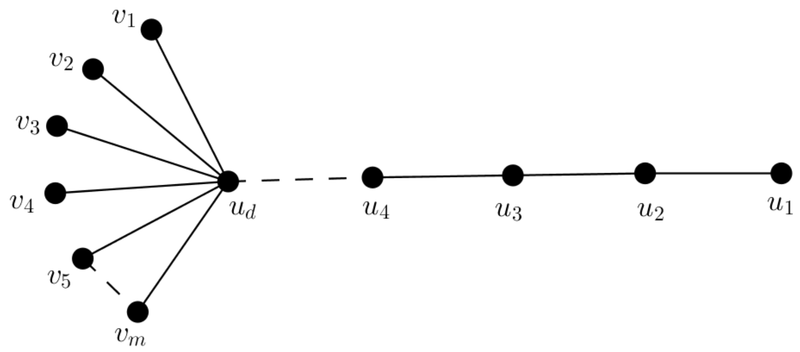

Figure 2. Broom Graph $B_{n,d}$

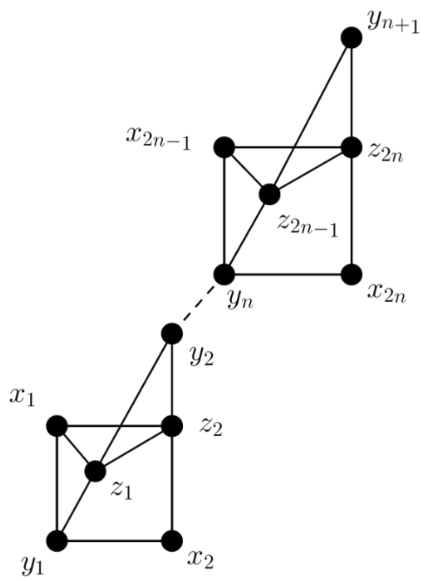

Figure 3. Tribune Graph $\mathfrak{T}_n$

Theorem 2. Let a tadpole graph $T_{m,n}$, then $\chi_g(T_{m,n}) = 3$

Proof. Given a tadpole graph $T_{m,n}$. The chromatic number of tadpole graph using Welch-Powell algorithm is 2. By Theorem 1, $2 \leq \chi_g(T_{m,n}) \leq 3$. Based on strategy, A coloring u_1 with c_1 as first step:

1. If B coloring v_2 with c_2, then A will coloring v_1 with c_3 or other vertex. If A coloring v_1 with c_3, then B will coloring other vertex, so that the remaining vertex can be coloring by three colors.
2. If B coloring other than v_2 without loss of generality example u_3 with c_2, then based on Theorem 2, A as possible will coloring u_2 with c_3 or other vertex. If A coloring u_2 with c_3, then B will coloring other vertex so that remaining vertices can be coloring by three colors.

Based on this case, such that A win and $3 \leq \chi_g(T_{m,n}) \leq 3$. Therefore, $\chi_g(T_{m,n}) = 3$.

Theorem 3. Given a broom graph $B_{n,d}$, then $\chi_g(B_{n,d}) = \begin{cases} 2, for\ d = 1,2 \\ 3, for\ d \geq 3 \end{cases}$

Proof. Given a broom graph $B_{n,d}$. The chromatic number of broom graph using Welch-Powell algorithm is 2. By Theorem 1, $2 \leq \chi_g(B_{n,d}) \leq m + 2$. Based on strategy, A coloring u_d with c_1 as first step:

For $d = 1, 2$, this graph is same of game chromatic number of star graph. Therefore that A win and $2 \leq \chi_g(B_{n,d}) \leq 2$ or $\chi_g(B_{n,1}) = \chi_g(B_{n,2}) = 2$.

1. If B coloring u_{d-2} with c_2, then A will coloring u_{d-1} with c_3 or other vertex. If A coloring u_{d-1} with c_3, then B will coloring other vertex, so that the remaining vertex can be coloring by three colors.
2. If B coloring other than u_{d-2} without loss of generality example v_1 with c_2, then based on Theorem 2, A as possible will coloring other vertex. If A coloring u_4 with c_1, then B will coloring other vertex or u_2 with c_2. If B coloring u_2 with c_2, then A coloring u_3 with c_3 or other vertex. So that the remaining vertices can be coloring by three colors.

Based on this case, such that A win and $3 \leq \chi_g(B_{n,d}) \leq 3$. Therefore, $\chi_g(B_{n,d}) = 3$.

Theorem 4. Given a tribune graph $\mathfrak{T}_n$, then$\chi_g(\mathfrak{T}_n) = \begin{cases} 3, for\ n = 1 \\ 4, for\ n \geq 2 \end{cases}$

Proof. Given a tribune graph $\mathfrak{T}_\mathrm{n}$, $\mathfrak{T}_1$, and given a set of colors $C = \{c_1, c_2, c_3, c_4\}$. The chromatic number of tribune graph using Welch-Powell algorithm is 3. By Theorem 1, $3 \leq \chi_g(\mathfrak{T}_n) \leq 5$ for $n = 1$ and $3 \leq \chi_g(\mathfrak{T}_n) \leq 6$ for $n \geq 2$

For $n = 1$, based on strategy, A coloring z_1 with c_1 as first step:

1. If B coloring z_2 with c_2, then A will coloring x_1 with c_3 or other vertex. If A coloring x_1 with c_3, then B will coloring x_2 with c_1 or other vertex. If B coloring x_2 with c_1, then the remaining vertex can be coloring by three colors.
2. If B coloring other than z_2 without loss of generality example y_1 with c_2, A will coloring y_2 with c_1 other vertex. If A coloring y_2 with c_1 , then remaining vertex can be coloring by three colors.

Based on this case, such that A win and $3 \leq \chi_g(\mathfrak{T}_n) \leq 3$. Therefore, $\chi_g(\mathfrak{T}_1) = 3$.

For $n \geq 2$, based on strategy, A coloring y_2 with c_1 as first step:

1. If B coloring y_1 with c_1, A as possible will coloring z_1 with c_2 or other vertex. If A coloring z_1 with c_2, then B will coloring z_2 with c_3. If B will coloring z_2 with c_3, then x_1 cannot colored by three colors. So that the remaining vertex can be coloring by four colors.

2. If B coloring other than y_2 without loss of generality example z_1 with c_2, A as possible will coloring y_1 other vertex with c_3. If A coloring y_1 with c_3, then B will coloring alternative vertex example y_3 with c_1. Therefore, the remaining vertex can be coloring by four colors.

Based on this case, such that A win and $4 \leq \chi_g(\mathfrak{T}_n) \leq 4$. Therefore, $\chi_g(\mathfrak{T}_n) = 4$.

D. CONCLUSION AND SUGGESTIONS

Based on the result, A win the graph coloring game because A using the strategy with coloring first of the largest degree. Player A win because when A using the strategy, player B cannot prevent A to make some vertex cannot to be colored. Untill end of the game, all of vertices has been colored, then player A win the game. The game chromatic number of $T_{m,n}$, $B_{n,d}$, and $\mathfrak{T}_n$ $\chi_g(T_{m,n}) = 3$, $\chi_g\left(B_{n,d}\right) = 2$ for $d = 1,2$ and $\chi_g(B_{n,d}) = 3$ for $d \geq 3$, and $\chi_g(\mathfrak{T}_n) = 3$ for $n = 1$ and $\chi_g(\mathfrak{T}_n) = 4$ for $n \geq 2$.

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