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Trace of the Positive Integer Powers $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$

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ABSTRACT

The trace of a matrix is obtained by summing the elements along the main diagonal of a square matrix. The matrix used in this study is a Toeplitz (n-1)-tridiagonal matrix of order n×n. The aim of this research is to determine the general form or formula for the trace of a Toeplitz (n-1)-tridiagonal matrix of order n×n raised to a positive integer power. This research is quantitative, with the research instrument being the collection of data from the multiplication of Toeplitz (n-1)-tridiagonal matrices starting from order 3×3 from powers 2 through 10. This process continues up to order 6×6 from powers 2 through 10, until the pattern becomes apparent. The results of the research are two general forms of the powers of the Toeplitz (n-1)-tridiagonal matrix of order n×n: one for odd positive integer powers and another for even positive integer powers, both of which have been proven using mathematical induction. Furthermore, by using the definition of the trace of a matrix obtained two general forms for the trace of the Toeplitz (n-1)-tridiagonal matrix of order n×n are also derived: one for odd positive integer powers and another for even positive integer powers from the general form of the matrix power. Given the application of these two general forms in example problems with the order 8x8 for powers 12 and 21.

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A. INTRODUCTION

Mathematics is a field of science that studies patterns, structures, spaces, and relationships between concepts using symbols, numbers, and logic. Problem solving with logic, critical and systematic thinking are the goals of mathematics. Mathematical principles are used as the basis of many modern technological innovations, such as cryptography (Kromodimoeljo, 2012) and artificial intelligence (Andriyani, 2024). Mathematics is also a foundation for many other professions and disciplines. Mathematics has many fields of study, including arithmetic, geometry, algebra, trigonometry, calculus, statistics, etc (Andriani, 2012).

Topics discussed in algebra include matrices, systems of linear equations, vector spaces, linear transformations, and more. The most commonly encountered mathematical concept is matrices, because matrices are an important tool for modeling and analyzing linear systems. A matrix is an arrangement of numbers arranged in rows (m) and columns (n) (Anton & Rorres, 2004). Matrices have many types, one of which is the toeplitz matrix. The toeplitz matrix has special properties and structure so that it has its own uniqueness. This toeplitz matrix is $n \times n$ in size whose entries on the main diagonal are of the same value, as well as entries on the subdiagonal of the same value corresponding to the main diagonal (Aryani & Husna, 2019).

The discussion of the toeplitz matrix has been widely discussed and has developed every year until now. As in Putri (2020) in 2020 discussed the symmetrical pentadiagonal toeplitz matrix. As in Hadi (2020) in the same year the toeplitz-Hessenberg matrix was discussed. Then in Aryani et al. (2022) two years later discussed the symmetrical heptadiagonal toeplitz matrix. In Aryani et al. (2022); Hadi (2020) with different types of matrices, the research is equally looking for the general form of the trace in each matrix. Matrix theory is a fundamental theory in algebra, various things can be done from a matrix, such as matrix multiplication, matrix determinant, trace matrix and others.

In the operation of the matrix, the calculation of the trace matrix is relatively easy to do because the trace matrix is obtained by simply summing the main diagonal elements of the square matrix (Aryani et al., 2023). In this definition, determining the trace matrix is not difficult, but if the matrix that has been referred to is of power n, then to determine the trace matrix, first the matrix operation is carried out by multiplying the matrix n times. Therefore, determining the trace of a power matrix becomes quite difficult. It is quite interesting to do research on how to determine the right formula for determining the trace of a raised matrix without having to do matrix multiplication. In (Aryani & Husna, 2019), a trace matrix calculation was carried out on a 3×3 tridiagonal toeplitz matrix with a positive power integer, the following is the original format of the matrix:

$$A_3 = \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix}.$$

The general form of the trace matrix is obtained:

$$tr(A_3)^n = \begin{cases} 3a^n + 4\sum_{r=1}^{\frac{n-1}{2}} \binom{n}{2r} 2^{r-1} a^{n-2r} b^r c^r & \text{, for } n \text{ odd} \\ 3a^n + 4\sum_{r=1}^{\frac{n-1}{2}} \binom{n}{2r} 2^{r-1} a^{n-2r} b^r c^r & \text{, for } n \text{ even} \end{cases}.$$

Then in (Olii et al., 2021), the trace matrix calculation was carried out on the $2-$tridiagonal 3×3 toepitz matrix with positive integer power, using a matrix:

$$\mathrm{A} = \begin{bmatrix} \mathrm{a} & 0 & \mathrm{c} \\ 0 & \mathrm{a} & 0 \\ \mathrm{b} & 0 & \mathrm{a} \end{bmatrix}.$$

The general form of the trace matrix is obtained:

$$tr(A^n) = \begin{cases} a^n + 2\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} a^{n-2i}(bc)^i \ , & n \text{ odd}\ , n > 0,\ \ n \in \mathbb{Z} \\ a^n + 2\sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} a^{n-2i}(bc)^i \ , & n \text{ even}, n \geq 0,\ \ n \in \mathbb{Z} \end{cases},$$

and there are still many studies related to trace matrices with various matrices studied such as in (Rahmawati, Wartono, et al., 2019) discussed the Trace of Integer Power of Real 3×3 Specific Matrices. in (Fitri Aryani et al., 2020) discussed the Trace a 3×3 Special-shaped Matrix of Positive Power Integer. in (Rahmawati et al., 2021) the Trace of Positive Integer Power of Squared Special Matrix was discussed. In (Aryani, Marzuki, et al., 2023) discussed about the Trace of the Adjacency Matrix $n \times n$ of the Cycle Graph To the Power of Six To Ten. It is known that this research is related to the studies presented above, namely determining the general form of the trace of a matrix raised to a positive integer power. However, it differs in the type of matrix used. The study (Olii et al., 2021) is the main reference for this research, but the matrix used in this study is different from the one used in (Olii et al., 2021). The matrix used in this research is a $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$ with the following matrix form:

$$A_n = \begin{bmatrix} a & 0 & 0 & \cdots & 0 & 0 & c \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & a \end{bmatrix}, \quad n \geq 3 \tag{1}$$

B. METHODS

This research uses the literature study method. Here are some stages in this research:

1. The matrix used is an $(n-1)$ −tridiagonal toeplitz matrix of order $n \times n$ with $n \geq 3$ in Equation (1).
2. Find the matrix expansions ${A_n}^2$ to ${A_n}^m$ with n, m is the limit when the pattern is visible.
3. Conjecture the general form of matrix expansion ${A_n}^m$ for m positive integers.
4. Prove the general form of matrix division ${A_n}^m$ for m positive integers by mathematical induction.
5. Obtain the general form of $tr({A_n}^m)$ for m positive integers using the definition of trace matrix and proved through direct proof.
6. Apply the general form of matrix ${A_n}^m$ and $tr({A_n}^m)$ for m positive integers to example problems using MATLAB.

Explanations of the definitions of matrix multiplication and power and theorems related to power matrix rules can be found in (Anđelić & da Fonseca, 2019; Anton & Rorres, 2004; El-mikkawy & Karawia, 2006; Hadi, 2020; Hernita, 2023; Rasmawati et al., 2021; Salkuyeh, 2006). The proof of the general form of matrix power uses the rules of mathematical induction, the description of which is found in (Drs. Sukirman M.Pd, 2006; Munir, 2016). The definitions and theorems related to the rules of power matrix and trace matrix are given in (Rahmawati, Putri, et al., 2019). Proving the general form of the power matrix ${A_n}^m$ for m positive integers using the rules of mathematical induction, as outlined in (Drs. Sukirman M.Pd, 2006; Munir, 2016). The mathematical induction method is highly appropriate for proving mathematical statements in this research, as the mathematical statement holds for all positive integers and is very accurate for the proof.

C. RESULT AND DISCUSSION

The results obtained are based on the steps in the research method. The first step taken was to determine the form of the powers of the $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$, from ${A_3}^2$to ${A_3}^{10}$, followed by ${A_4}^2$ to ${A_4}^{10}$, ${A_5}^2$ to ${A_5}^{10}$ and finally ${A_6}^2$ to ${A_6}^{10}$ up to the point where the pattern becomes apparent. When the matrix powers pattern has been obtained, then we can predict the general form of the powers of the $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$, with the power of a positive integer. This predict was then proven using the method of mathematical induction, which is presented in Section 1 below.

1. General form of the Positive Integer Powers $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$

Theorem 1. Given an $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$ for $n \geq 3$ of the form:

$$A_n = \begin{bmatrix} a & 0 & 0 & \cdots & 0 & 0 & c \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & a \end{bmatrix}, \ \forall\, a, b, c \neq 0, \text{ then}$$

$$
A_n{}^m = \begin{cases}
\begin{bmatrix}
\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i & 0 & 0 & \cdots & 0 & 0 & \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i+1} a^{m-2i-1} b^i c^{i+1} \\
0 & a^m & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a^m & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a^m & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & a^m & 0 \\
\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i+1} a^{m-2i-1} b^{i+1} c^i & 0 & 0 & \cdots & 0 & 0 & \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i
\end{bmatrix}, & m \text{ odd}, m \in \mathbb{Z}^+ \\
\begin{bmatrix}
\sum_{i=0}^{\frac{m}{2}} \binom{m}{2i} a^{m-2i}(bc)^i & 0 & 0 & \cdots & 0 & 0 & \sum_{i=0}^{\frac{m-2}{2}} \binom{m}{2i+1} a^{m-2i-1} b^i c^{i+1} \\
0 & a^m & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a^m & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a^m & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & a^m & 0 \\
\sum_{i=0}^{\frac{m-2}{2}} \binom{m}{2i+1} a^{m-2i-1} b^{i+1} c^i & 0 & 0 & \cdots & 0 & 0 & \sum_{i=0}^{\frac{m}{2}} \binom{m}{2i} a^{m-2i}(bc)^i
\end{bmatrix}, & m \text{ even}, m \in \mathbb{Z}^+
\end{cases} \tag{2}
$$

Proof. The proof is done using mathematical induction. The proof for m odd positive integers is given:

$$
p(m): A_n{}^m = \begin{bmatrix}
\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i+1} a^{m-2i-1} b^i c^{i+1} \\
0 & a^m & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^m & 0 \\
\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i+1} a^{m-2i-1} b^{i+1} c^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i
\end{bmatrix}, m \in \mathbb{Z}^+
$$

a. Will be shown that $p(1)$ is true.

$$
p(1): A_n{}^1 = \begin{bmatrix}
\sum_{i=0}^{\frac{1-1}{2}} \binom{1}{2i} a^{1-2i}(bc)^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{1-1}{2}} \binom{1}{2i+1} a^{1-2i-1} b^i c^{i+1} \\
0 & a^1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^1 & 0 \\
\sum_{i=0}^{\frac{1-1}{2}} \binom{1}{2i+1} a^{1-2i-1} b^{i+1} c^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{1-1}{2}} \binom{1}{2i} a^{1-2i}(bc)^i
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\binom{1}{0} a^{1-0}(bc)^0 & 0 & \cdots & 0 & \binom{1}{0+1} a^{1-0-1} b^0 c^{0+1} \\
0 & a^1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^1 & 0 \\
\binom{1}{0+1} a^{1-0-1} b^{0+1} c^0 & 0 & \cdots & 0 & \binom{1}{0} a^{1-0}(bc)^0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
a & 0 & 0 & \cdots & 0 & 0 & c \\
0 & a & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & a & 0 \\
b & 0 & 0 & \cdots & 0 & 0 & a
\end{bmatrix}.
$$

By looking at Equation (1), $p(1)$ is proved to be true.

b. Assume $p(k)$ is true, that is:

$$p(k): {A_n}^k = \begin{bmatrix} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^i c^{i+1} \\ 0 & a^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^k & 0 \\ \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^{i+1} c^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i \end{bmatrix}, k \in \mathbb{Z}^+$$

For k odd positive integers. Will be proved that $p(k+2)$ is also true, that is:

$$p(k+2): {A_n}^{k+2} = \begin{bmatrix} \sum_{i=0}^{\frac{k+1}{2}} \binom{k+2}{2i} a^{k-2i+2}(bc)^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k+1}{2}} \binom{k+2}{2i+1} a^{k-2i+1} b^i c^{i+1} \\ 0 & a^{k+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{k+2} & 0 \\ \sum_{i=0}^{\frac{k+1}{2}} \binom{k+2}{2i+1} a^{k-2i+1} b^{i+1} c^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k+1}{2}} \binom{k+2}{2i} a^{k-2i+2}(bc)^i \end{bmatrix} \quad (3)$$

With the following proof:

$${A_n}^{k+2} = {A_n}^k \cdot {A_n}^2$$

$$= \begin{bmatrix} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^i c^{i+1} \\ 0 & a^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^k & 0 \\ \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^{i+1} c^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i \end{bmatrix} \cdot \begin{bmatrix} a^2+bc & 0 & \cdots & 0 & 2ac \\ 0 & a^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^2 & 0 \\ 2ab & 0 & \cdots & 0 & a^2+bc \end{bmatrix}$$

The matrix multiplication result is divided into several forms, namely:

1) a_{11}

$$= \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i \right) \cdot (a^2+bc) + 0\cdot 0 + 0 \cdot 0 + \cdots + 0 \cdot 0 + 0 \cdot 0 + \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^i c^{i+1} \right) \cdot (2ab)$$

$$= (a^2+bc)\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i \right) + 2ab \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^i c^{i+1} \right).$$

2) a_{1n}

$$= \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i\right) \cdot (2ac) + 0 \cdot 0 + 0 \cdot 0 + \cdots + 0 \cdot 0 + 0 \cdot 0 + \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^i c^{i+1}\right) \cdot (a^2 + bc)$$

$$= 2ac\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i\right) + (a^2 + bc)\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^i c^{i+1}\right)$$

3) a_{n1}

$$= \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^{i+1} c^i\right) \cdot (a^2 + bc) + 0 \cdot 0 + 0 \cdot 0 + \cdots + 0 \cdot 0 + 0 \cdot 0 + \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i\right) \cdot (2ab)$$

$$= (a^2 + bc)\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^{i+1} c^i\right) + 2ab\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i\right)$$

4) a_{nn}

$$= \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^{i+1} c^i\right) \cdot (2ac) + 0 \cdot 0 + 0 \cdot 0 + \cdots + 0 \cdot 0 + 0 \cdot 0 + \left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i\right) \cdot (a^2 + bc)$$

$$= 2ac\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} a^{k-2i-1} b^{i+1} c^i\right) + (a^2 + bc)\left(\sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i} a^{k-2i}(bc)^i\right)$$

5) a_{ij} with $i = j = 2, 3, 4, 5, \cdots, n-1$

The entries in ${A_n}^k$ in rows $i = 2,3, \cdots, n-1$ (which are a^k only in a_{ij} with $i = j = 2,3,4,5, \cdots, n-1$ otherwise they are 0) are multiplied by the entries in ${A_n}^k$ columns to $j = 2,3, \cdots, n-1$ (which is a^2 only in a_{ij} with $i = j = 2,3,4,5, \cdots, n-1$, the rest is 0) then the result of multiplying the matrix entries will be a^{k+2} only in a_{ij} with $i = j = 2,3,4,5, \cdots, n-1$, the rest is 0. Here is the calculation:

$$a_{ij} = 0 \cdot 0 + \cdots + \underbrace{a^k \cdot a^2}_{\text{On the } i-\text{th term}} + \cdots + 0 \cdot 0 = a^{k+2}$$

6) The other entries will be 0

For the result of multiplying 2 matrices other than the entries above will be 0 because one or both of the corresponding matrix entries are 0 so that if multiplication is done it will be 0.

So the result of the matrix multiplication above is obtained as follows:

$$
\begin{aligned}
&= \begin{bmatrix}
(a^2+bc)\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)+2ab\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^ic^{i+1}\right) & 0 & \cdots & 0 & 2ac\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)+(a^2+bc)\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^ic^{i+1}\right) \\
0 & a^{k+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^{k+2} & 0 \\
(a^2+bc)\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+1}c^i\right)+2ab\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right) & 0 & \cdots & 0 & 2ac\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+1}c^i\right)+(a^2+bc)\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)
\end{bmatrix} \\
&= \begin{bmatrix}
a^2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)+bc\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)+2ab\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^ic^{i+1}\right) & 0 & \cdots & 0 & 2ac\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)+a^2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^ic^{i+1}\right)+bc\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^ic^{i+1}\right) \\
0 & a^{k+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^{k+2} & 0 \\
a^2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+1}c^i\right)+bc\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+1}c^i\right)+2ab\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right) & 0 & \cdots & 0 & 2ac\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+1}c^i\right)+a^2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)+bc\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^i\right)
\end{bmatrix} \\
&= \begin{bmatrix}
\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i+2}(bc)^i+\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^{i+1}+2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i}(bc)^{i+1}\right) & 0 & \cdots & 0 & 2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i+1}b^ic^{i+1}\right)+\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i+1}b^ic^{i+1}+\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+1}c^{i+2} \\
0 & a^{k+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^{k+2} & 0 \\
\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i+1}b^{i+1}c^i+\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i-1}b^{i+2}c^{i+1}+2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i+1}b^{i+1}c^i\right) & 0 & \cdots & 0 & 2\left(\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}a^{k-2i}(bc)^{i+1}\right)+\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i+2}(bc)^i+\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i}a^{k-2i}(bc)^{i+1}
\end{bmatrix} \\
&= \begin{bmatrix}
\sum_{i=1}^{\frac{k-1}{2}}\binom{k+2}{2i}a^{k-2i+2}(bc)^i+a^{k+2}+(k+2)a(bc)^{\frac{k+1}{2}} & 0 & \cdots & 0 & \sum_{i=1}^{\frac{k-1}{2}}\binom{k+2}{2i+1}a^{k-2i+1}b^ic^{i+1}+(k+2)a^{k+1}c+b^{\frac{k+1}{2}}c^{\frac{k+3}{2}} \\
0 & a^{k+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^{k+2} & 0 \\
\sum_{i=1}^{\frac{k-1}{2}}\binom{k+2}{2i+1}a^{k-2i+1}b^{i+1}c^i+(k+2)a^{k+1}b+b^{\frac{k+3}{2}}c^{\frac{k+1}{2}} & 0 & \cdots & 0 & \sum_{i=1}^{\frac{k-1}{2}}\binom{k+2}{2i}a^{k-2i+2}(bc)^i+a^{k+2}+(k+2)a(bc)^{\frac{k+1}{2}}
\end{bmatrix} \\
&= \begin{bmatrix}
\sum_{i=0}^{\frac{k+1}{2}}\binom{k+2}{2i}a^{k-2i+2}(bc)^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k+1}{2}}\binom{k+2}{2i+1}a^{k-2i+1}b^ic^{i+1} \\
0 & a^{k+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a^{k+2} & 0 \\
\sum_{i=0}^{\frac{k+1}{2}}\binom{k+2}{2i+1}a^{k-2i+1}b^{i+1}c^i & 0 & \cdots & 0 & \sum_{i=0}^{\frac{k+1}{2}}\binom{k+2}{2i}a^{k-2i+2}(bc)^i
\end{bmatrix}.
\end{aligned}
$$

Based on Equation (3), it can be said that $p(k+2)$ is true, by looking at steps (1) and (2), the proof for m odd positive integers is proven true. Furthermore, to prove ${A_n}^m$ for m even positive integers using the same steps. Through the steps of proof that have been carried out systematically, Theorem 1 is proven correct.

2. Trace of the Positive Integer Powers $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$

Based on Theorem 1, the trace of the Positive Integer Powers $(n-1)-$Tridiagonal Toeplitz Matrix $n \times n$ is obtained as shown in Corollary 1.

Corollary 1. Given an $(n-1)$ −Tridiagonal Toeplitz Matrix $n \times n$ for $n \geq 3$ of the form:

$$
A_n = \begin{bmatrix}
a & 0 & 0 & \cdots & 0 & 0 & c \\
0 & a & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & a & 0 \\
b & 0 & 0 & \cdots & 0 & 0 & a
\end{bmatrix},\ \forall\ a,b,c \neq 0, \text{ then}
$$

$$tr(A_n{}^m) = \begin{cases} (n-2)a^m + 2\left(\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i\right), & m \text{ odd}, m \in \mathbb{Z}^+ \\ (n-2)a^m + 2\left(\sum_{i=0}^{\frac{m}{2}} \binom{m}{2i} a^{m-2i}(bc)^i\right), & m \text{ even}, m \in \mathbb{Z}^+ \end{cases}.$$

Proof. Based on Theorem 1, we will prove the formula $tr(A_n{}^m)$ for m odd positive integers, as follows:

$$tr(A_n{}^m) = \left(\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i\right) + \underbrace{(a^m) + \cdots + (a^m) + \cdots + (a^m)}_{(n-2)\text{ factors}} + \left(\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i\right)$$

$$= (n-2)a^m + 2\left(\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i\right).$$

Hence proved that $tr(A_n{}^m) = (n-2)a^m + 2\left(\sum_{i=0}^{\frac{m-1}{2}} \binom{m}{2i} a^{m-2i}(bc)^i\right)$ for m odd positive integers. The proof for $tr(A_n{}^m)$ for m even positive integers is done in the same step. Corollary 1 is proved by direct proof as described above.

3. Simulation of the Positive Integer Powers $(n-1)-$Tridiagonal Toeplitz Matrix $n \times n$

The following section will give some examples relevant to Theorem 1 and Corollary 1, then the examples will be tested using *Matlab* application.

Example. Given a matrix as follows:

$$B = \begin{bmatrix} 11 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{15} \\ 0 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11 & 0 \\ -20 & 0 & 0 & 0 & 0 & 0 & 0 & 11 \end{bmatrix}$$

a. Determine $B_8{}^{21}$ of the matrix using Theorem 1.
b. Calculate the value of $tr(B_8{}^{12})$ of the matrix using Corollary 1.

Solution.

a. The calculation is based on Theorem 1.

In matrix B with $n = 8, a = 11, b = -20, c = \frac{3}{15}$, the following results are obtained for m odd positive integers:

$$B_8{}^{21} = \begin{bmatrix} \sum_{i=0}^{10}\binom{21}{2i}(11)^{21-2i}(-4)^i & 0 & 0 & 0 & 0 & 0 & 0 & \sum_{i=0}^{10}\binom{21}{2i+1}(11)^{20-2i}(-20)^i\left(\frac{3}{15}\right)^{i+1} \\ 0 & (11)^{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (11)^{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (11)^{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (11)^{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (11)^{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (11)^{21} & 0 \\ \sum_{i=0}^{10}\binom{21}{2i+1}(11)^{20-2i}(-20)^{i+1}\left(\frac{3}{15}\right)^i & 0 & 0 & 0 & 0 & 0 & 0 & \sum_{i=0}^{10}\binom{21}{2i}(11)^{21-2i}(-4)^i \end{bmatrix}$$

$$= \begin{bmatrix} -8380767505363490000000 & 0 & 0 & 0 & 0 & 0 & 0 & -617923565415611000000 \\ 0 & 7400249944258160000000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7400249944258160000000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7400249944258160000000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7400249944258160000000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7400249944258160000000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7400249944258160000000 & 0 \\ 61792356541561000000000 & 0 & 0 & 0 & 0 & 0 & 0 & -8380767505363490000000 \end{bmatrix}$$

To see the correctness of the solution $B_8{}^{21}$, the following solution is given using the *Matlab* application in Figure 1.

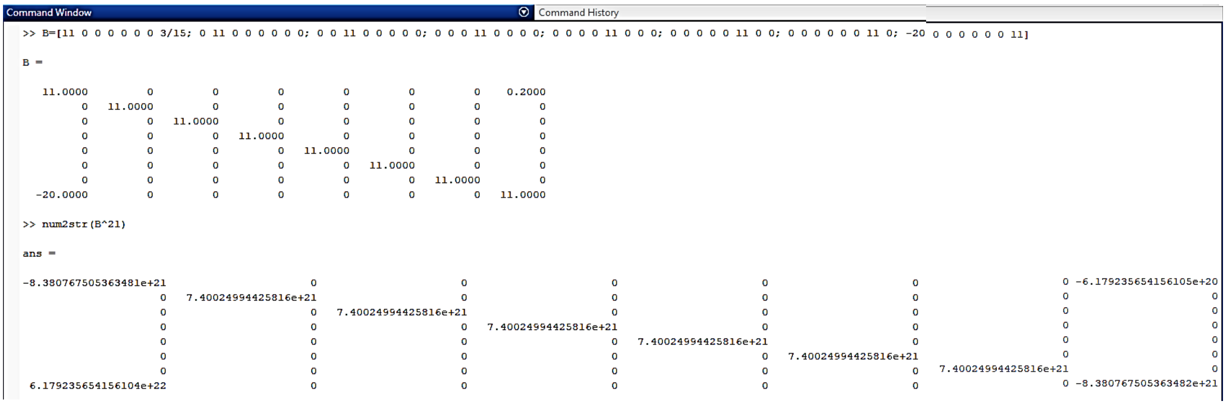

Figure 1. Solving $B_8{}^{21}$ using *Matlab* Application

b. The calculation is done based on Corollary 1. The following results are obtained for m even positive integers:

$$tr\left(B_8{}^{12}\right) = (8-2)(11)^{12} + 2\left(\sum_{i=0}^{6}\binom{12}{2i}(11)^{12-2i}(-4)^i\right)$$
$$= (6)(3138428376721) + 2(-2114245277767)$$
$$= (18830570260326) + (-4228490555534)$$
$$= 14602079704792\,.$$

To see the correctness of the solution of $tr\left(B_8{}^{12}\right)$, the following solution is given using Matlab application in Figure 2.

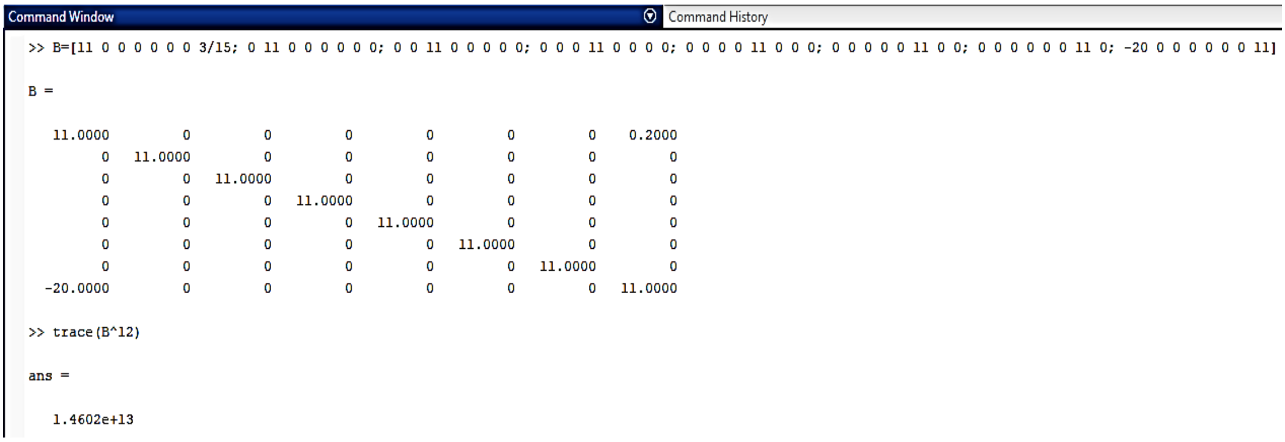

Figure 2. Solving $tr\left({B_8}^{12}\right)$ using *Matlab* Application

D. CONCLUSION

Based on the discussion that has been done, the conclusion of this research is given a positive integer powers $(n-1)$ −tridiagonal toeplitz matrix $n \times n$ for $n \geq 3$ with the form:

$$A_n = \begin{bmatrix} a & 0 & 0 & \cdots & 0 & 0 & c \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & a \end{bmatrix}, \qquad \forall\, a, b, c \neq 0$$

it has two general forms of matrix powers: one for odd positive integer powers and one for even positive integer powers, both of which have been proven using the method of mathematical induction and can be seen in Theorem 1. Meanwhile, the general form of the trace of the matrix of powers is obtained from the general form of the matrix of powers by summing the entries on its main diagonal, based on the definition of the trace of a matrix. The result is two general forms of the trace for the $(n-1)$ −tridiagonal toeplitz matrix $n \times n$: one for odd positive integer powers and one for even positive integer powers, which can be seen in Corollary 1.

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