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Determinants of Tridiagonal and Circulant Matrices Special Form by Chebyshev Polynomials

Nurliantika[1], Fransiskus Fran[1*], Yundari[1]
[1]Department Mathematics, Tanjungpura University of Pontianak, Indonesia
fransiskusfran@math.untan.ac.id

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ABSTRACT

Along with the development of science, many researchers have found new methods to determine the determinant of a matrix of more than three orders. Chebyshev polynomial can be used to find and develop a more efficient formula in calculating the determinant of matrices. This research explores the Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$. Both types of Chebyshev polynomials, $T_n(x)$ and $U_n(x)$, can be represented using recurrence relations. This research aims to determine the determinant of tridiagonal and circulant matrices of special form using Chebyshev polynomials $T_n(x)$ and $U_n(x)$. Determining the determinant of a matrix is a fundamental problem in linear algebra that plays an important role in both theoretical and applied mathematics. Its theoretical contributions include a deeper understanding of matrix properties, the development of efficient computational methods, and the explanation of the relationship between matrices and orthogonal polynomials. By utilizing Chebyshev polynomials, this study strengthens determinant theory, particularly for matrices with special shapes. The steps to determine the determinant of tridiagonal and circulant matrices involve the application of elementary row operations. The first step is to perform row operations on the tridiagonal and circulant matrices to obtain a matrix form that conforms to the determinant theorem of the tridiagonal and circulant matrices. After the elementary row operation is applied, if the form of the tridiagonal and circulant matrices each satisfies the form in the determinant theorem of the tridiagonal and circulant matrices, then the determinant of the matrices can be calculated using each of the theorems that satisfy. Then the determinants of the tridiagonal and the circulant matrices are obtained. The results of this study show that the determinant of tridiagonal and circulant matrices of special form can be determined using Chebyshev polynomials $T_n(x)$ and $U_n(x)$.

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A. INTRODUCTION

A matrix is an arrangement of numbers organised by rows and columns in a rectangular shape placed between two brackets (Anton & Rorres, 2013). The arrangement of numbers in a rectangular array is called the entries of the matrix (Anton & Rorres, 2013). The size of the matrix (matrix order) is indicated by the number of rows (horizontal lines) and the number of columns (vertical lines) that the matrix has. There are several types of matrices, one of which is a square matrix. A square matrix is a matrix with the same number of rows and columns and is denoted as an $n \times n$ matrix (Anton & Rorres, 2013; Rosen, 2012). Every square matrix has a single value called determinant (Anton & Rorres, 2013). A German mathematician named Carl Friedrich Gauss first introduced the term determinant in 1801 (Kartika, 2017).

In determining the determinant value of a matrix, there are several methods that can be used including the Sarrus rule, row reduction, and cofactor expansion methods (Ilhamsyah et al., 2017). Along with the development of science, especially in the world of education, many researchers have found new methods to calculate the determinant of a matrix with a greater than three orders (Fitriyani et al., 2018). In this research, the determinant of the matrix with Chebyshev polynomials is studied. However, most of these methods are computationally complex or have no generalization to certain matrix shapes, such as tridiagonal matrices or circular matrices. In this research, matrix determinants are studied using Chebyshev polynomials, which include both the first and second kinds. The characteristics of these polynomials are discussed in (Gradshteyn & Ryzhik, 2007; Bucur et al., 2007; Belbachir & Bencherif, 2008). Both kinds can be applied to derive and develop formulas for the calculation of matrix determinants. Since Chebyshev polynomials follow a recursive pattern, where each polynomial is defined based on the previous polynomial, they are useful for simplifying matrix determinant calculations. This research focuses on special forms of matrices and utilizes the properties of Chebyshev polynomials to obtain an efficient and general determinant formula.

There are several studies that discuss the general form for the determinant of an $n \times n$ matrix. The generalized determinant formula streamlines the calculation of a large matrix's determinant by adjusting its entries and order. In addition, several previous studies focused on determining the determinant of a tridiagonal matrix (Belbachir & Bencherif, 2008; Jiang et al., 2013; Qi et al., 2019; Fahlevi, 2021; Jitman & Sricharoen, 2024). A tridiagonal matrix is a square matrix that has entries $a_{ij} = 0$ for $|i - j| > 1$ (D. Zhang, 2017). A circulant matrix is a square matrix where each row is generated by shifting the previous row one position to the right in a circular manner (Olson et al., 2014). In this way, the entries of the first row are shifted right by one position to form the next row.

In research Du et al. (2021); Seibert & Trojovský (2006) discusses the recurrence relation to determining the determinants of banded matrices. The research discusses an algorithm about the recurrence relation to calculate the determinant of a matrix. In addition Janji´c (2012), Hetmaniok et al. (n.d.); Jakovčević Stor & Slapničar (2024) conducted research on alternative proofs of several formulas to determine the determinant of a matrix. Meanwhile, Elouafi (2014), Jitman (2020); da Fonseca (2020) also discussed the relationship between the determinants of some tridiagonal matrices and Chebyshev polynomial $U_n(x)$.

There is research that uses the determinant of a special form matrix with Chebyshev polynomials to find an uncomplicated formula for calculating the number of spanning trees in certain graphs (Daoud, 2012). Additionally, research Y. Zhang et al. (2005); Baigonakova & Mednykh (2018); Daoud (2019); Deen & Aboamer (2021) has examined the formula for finding the number of spanning trees in graphs, utilizing approaches such as Chebyshev polynomials, linear algebra, and matrix theory. In this study, Chebyshev polynomials are connected to the matrix determinant involved in the calculations. The determinants used are from tridiagonal and circulant matrices of special forms. Tridiagonal and circulant matrices have structured patterns, making them suitable for analysis in this study. These patterns align with the elegant properties of Chebyshev polynomials. Continuing these studies, this research specifically explores the determinants of tridiagonal and circulant matrices with special forms using Chebyshev polynomials. In contrast to previous studies that emphasized more on applications in graph theory, this study explores the mathematical properties of the determinant, by providing a systematic derivation of the determinant formula based on Chebyshev polynomials. This approach not only highlights the importance of determinants in matrix theory, but also provides a stronger theoretical basis that can support future applications, both in graph theory and in other fields.

B. METHODS

This research is a theoretical research that focuses on developing the determinant formula of tridiagonal and circulant matrices with special shapes using Chebyshev polynomials. The subject of this research is tridiagonal and circulant matrices with special forms that can be applied in various fields, such as graph theory and numerical analysis. This research aims to develop mathematical steps in determinant calculation and explore the application of Chebyshev polynomials to simplify the calculation process. The research steps begin with applying elementary row operations to the tridiagonal and circulant matrices to obtain a matrix form that aligns with the determinant theorem for these matrices. After the row operations are applied, if the forms of the tridiagonal and circulant matrices satisfy the conditions of the respective determinant theorems, the determinant of the matrices can be calculated using those theorems. The determinants of the tridiagonal and circulant matrices are then obtained. The results of this study show that the determinant of special form tridiagonal and circulant matrices can be determined using Chebyshev polynomials $T_n(x)$ and $U_n(x)$.

C. RESULT AND DISCUSSION

This section covers Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$. These polynomials are employed to calculate the determinant of tridiagonal and circulant matrices in a special form. Definition 1 is provided below, which explains the Chebyshev polynomial $T_n(x)$.

Definition 1 (Mason & Handscomb, 2003) Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation:

$$T_n(x) = \cos(n\theta) \tag{1}$$

where $x = cos\,\theta;$ for $0 \leq \theta \leq \pi$ and $-1 \leq x \leq 1$. Some of $T_n(x)$ are,

$$\begin{gathered} T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \\ T_4(x) = 8x^4 - 8x^2 + 1, \ldots. \end{gathered} \tag{2}$$

In Figure 1, the graph of $T_n(x)$ is given for $n = 1,2,3,4$.

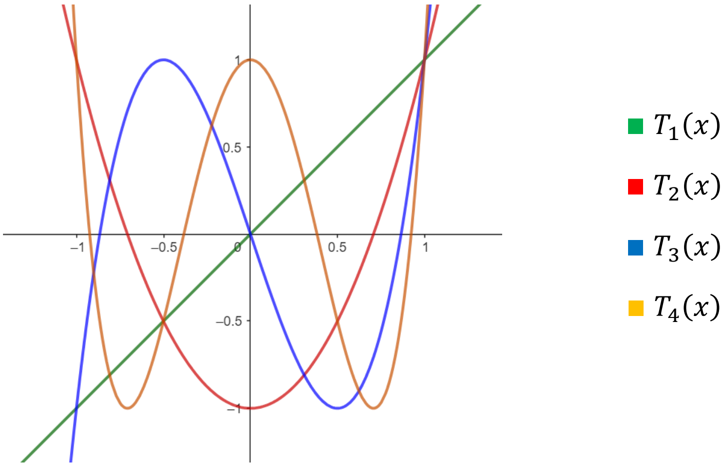

Figure 1. $T_n(x)$ for $n = 1, 2, 3, 4$

In Figure 1, for $n = 1$ a linear curve forms which states a polynomial of degree 1. Then for $n = 2$ the graph is a parabolic curve which states a polynomial of degree 2. Furthermore, for $n = 3$ the graph is a cubic curve which states a polynomial of degree 3 and $n = 4$ the graph is a quartic curve which states a polynomial of degree 4. Therefore, the greater the value of n in the Chebyshev polynomial of first kind, the more waves are formed.first kind, the more waves are formed (Mason & Handscomb, 2003). The Chebyshev polynomial $T_n(x)$ satisfies the recurrence relation presented in Theorem 2.

Theorem 2 Mason & Handscomb (2003) the Chebyshev polynomial of first kind satisfies the recurrence relation is as follows:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{3}$$

with $n = 2, 3, 4, ...$ and initial conditions,

$$T_0(x) = cos(0\theta) = cos\ 0 = 1$$
$$T_1(x) = cos(1\theta) = cos\ \theta = x.$$

Proof. Based on Definition 1, $T_n(x) = \cos(n\theta)$, then forms of $T_{n-1}(x)$ dan $T_{n-2}(x)$ are as follows:

$$T_{n-1}(x) \quad = \cos(n\theta - \theta) = \cos n\theta \cdot \cos\theta + \sin n\theta \cdot \sin\theta \tag{4}$$
$$T_{n-2}(x) \quad = \cos(n\theta - 2\theta) = \cos n\theta \cdot \cos 2\theta + \sin n\theta \cdot \sin 2\theta \tag{5}$$

Multiply $(2x)$ by Equation (4), obtained:

$$2xT_{n-1}(x) \quad = 2\cos n\theta \cdot \cos^2\theta + \sin n\theta \cdot \sin 2\theta \tag{6}$$

Then, Equation (6) reduced by Equation (5) is obtained:

$$
\begin{aligned}
2xT_{n-1}(x) - T_{n-2}(x) &= 2\cos n\theta \cdot \cos^2\theta + \sin n\theta \cdot \sin 2\theta - (\cos n\theta \cdot \cos 2\theta + \sin n\theta \cdot \\
&\quad \sin 2\theta) \\
&= \cos n\theta\,(\cos^2\theta + \sin^2\theta) \\
&= \cos n\theta \\
&= T_n(x) \quad \blacksquare.
\end{aligned}
$$

The solution of the Chebyshev polynomial of first kind recurrence relation is as follows:

$$T_n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right) \tag{7}$$

Next, we discuss the Chebyshev polynomial of second kind $U_n(x)$. The following Definition 3 explains the Chebyshev polynomial $U_n(x)$.

Definition 3 (Mason & Handscomb, 2003) Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial in x of degree n, defined by the relation:

$$U_n(x) = \frac{\sin(n+1)\,\theta}{\sin\theta} \tag{8}$$

where $x = cos\,\theta$; for $0 < \theta < \pi$ and $-1 < x < 1$. Some of $U_n(x)$ are

$$
\begin{gathered}
U_0(x) = 1, \qquad U_1(x) = 2x, \qquad U_2(x) = 4x^2 - 1, \qquad U_3(x) = 8x^3 - 4x, \\
U_4(x) = 16x^4 - 12x^2 + 1, \dots.
\end{gathered} \tag{9}
$$

In Figure 2, the graph of Chebyshev polynomial $U_n(x)$ is given for $n = 1,2,3,4$.

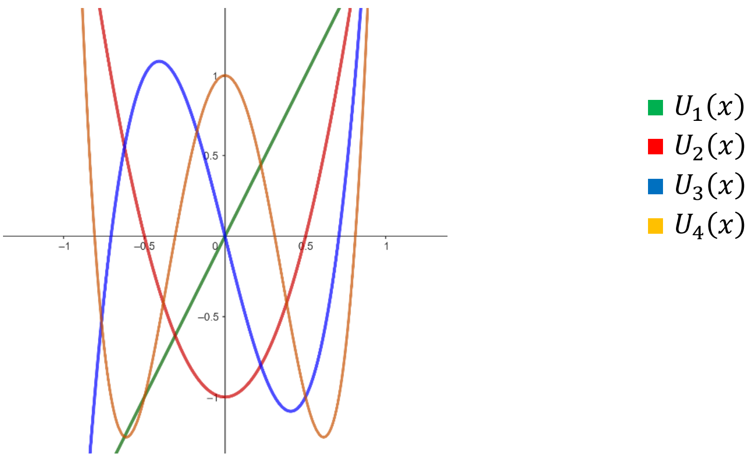

Figure 2 $U_n(x)$ for $n = 1,2,3,4$

In Figure 2, show the graph of $U_1(x), U_2(x), U_3(x)$, and $U_4(x)$. As does in Figure 1, each time the value of n increases, the number of waves increases by one, and the extreme points become more numerous, with their height or low increasing monotonically (Mason & Handscomb, 2003). The Chebyshev polynomial $U_n(x)$ satisfies the recurrence relation outlined in Theorem 4.

Theorem 4 (Mason & Handscomb, 2003) The Chebyshev polynomial of second kind satisfies the recurrence relation is as follows:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \tag{10}$$

with $n = 2, 3, 4, \ldots$. and initial conditions,

$$U_0(x) = \frac{\sin(0+1)\theta}{\sin\theta} = 1$$
$$U_1(x) = \frac{\sin(1+1)\theta}{\sin\theta} = 2x$$

Proof. Based on Definition 3, then forms of $U_{n-1}(x)$ dan $U_{n-2}(x)$ are as follows:

$$U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta} \tag{11}$$
$$U_{n-2}(x) = \frac{\sin n\theta \cdot \cos\theta - \cos n\theta \cdot \sin\theta}{\sin\theta} \tag{12}$$

Multiply $(2x)$ by Equation (11), obtained:

$$2xU_{n-1}(x) = \frac{2\sin n\theta \cdot \cos\theta}{\sin\theta} \tag{13}$$

Then, Equation (13) reduced by Equation (12) is obtained:

$$\begin{aligned} 2xU_{n-1}(x) - U_{n-2}(x) &= \frac{2\sin n\theta \cdot \cos\theta}{\sin\theta} - \left(\frac{\sin n\theta \cdot \cos\theta - \cos n\theta \cdot \sin\theta}{\sin\theta}\right) \\ &= \frac{\sin n\theta \cos\theta + \cos n\theta \cdot \sin\theta}{\sin\theta} \\ &= \frac{\sin(n+1)\theta}{\sin\theta} \\ &= U_n(x) \quad \blacksquare. \end{aligned}$$

The solution of the $U_n(x)$ recurrence relation is as follows:

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left(\left(x+\sqrt{x^2-1}\right)^{n+1}\right) - \left(\left(x-\sqrt{x^2-1}\right)^{n+1}\right) \tag{14}$$

Suppose $A_n(x)$ is a tridiagonal matrix as follows:

$$A_n(x) = \begin{bmatrix} 2x & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2x & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2x & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -1 & 2x & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2x \end{bmatrix}$$

Then it can be shown that by finding $|A_n(x)|$, obtained the same form of recurrence relation as Equation (10). Suppose calculated the determinant of the matrix is for $n = 2,3,4,5$, thus

$$\begin{aligned} |A_2(x)| &= 4x^2 - 1 = 2x(2x) - 1 \\ |A_3(x)| &= 8x^3 - 4x = 2x(4x^2 - 1) - 2x \\ |A_4(x)| &= 16x^4 - 12x^2 + 1 = 2x(8x^3 - 4x) - (4x^2 - 1) \\ |A_5(x)| &= 32x^5 - 32x^3 + 6x = 2x(16x^4 - 12x^2 + 1) - (8x^3 - 4x) \end{aligned}$$

Then it can be shown that by performing a cofactor expansion on the matrix $A_n(x)$, obtained the same recurrence relation in Equation (10):

$$|A_n(x)| = 2x|A_{n-1}(x)| - |A_{n-2}|$$

Thus the recurrence relation equation of the determinant value for $n = 2,3,4,5$ is obtained, namely $|A_n(x)| = 2x|A_{n-1}(x)| - |A_{n-2}(x)|$. Based on the determinant of the matrix $A_n(x)$, the following relation is obtained:

$$|A_n(x)| = U_n(x) \tag{15}$$

Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are closely related to determinants, making them useful for calculating larger matrices. The theorem below is applied to compute the determinant of a tridiagonal matrix.

Theorem 5 Let matrix $\mathcal{A}_n(z)$ be a special form tridiagonal matrix of size $n \times n$, with

$$\mathcal{A}_n(z) = \begin{bmatrix} z & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & z+\alpha & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & z+\alpha & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & z+\alpha & -\alpha \\ 0 & \cdots & 0 & 0 & -\alpha & z \end{bmatrix}$$

for all $z, \alpha \in \mathbb{R}, z \neq -3\alpha, z \neq \alpha$ and $n \in \mathbb{N}, n \geq 2$, then the determinant is

$$\det\left(\mathcal{A}_n((z))\right) = (z - \alpha) \cdot \alpha^{n-1} U_{n-1}\left(\frac{z+\alpha}{2\alpha}\right) \tag{16}$$

Proof. The form of the tridiagonal matrix $B_n(r)$ is as follows:

$$B_n(r) = \begin{bmatrix} r & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & r & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & r & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & r & -\alpha \\ 0 & \cdots & 0 & 0 & -\alpha & r \end{bmatrix}$$

Calculating the determinant of matrix $B_n(r)$ using cofactor expansion as follows:

$$\begin{aligned} |B_n(r)| &= \det \begin{bmatrix} r & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & r & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & r & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & r & -\alpha \\ 0 & \cdots & 0 & 0 & -\alpha & r \end{bmatrix} \\ &= r \det \begin{bmatrix} r & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & r & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & r & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & r & -\alpha \\ 0 & \cdots & 0 & 0 & -\alpha & r \end{bmatrix}_{(n-1)\times(n-1)} \\ &\quad -\alpha^2 \det \begin{bmatrix} r & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & r & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & r & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & r & -\alpha \\ 0 & \cdots & 0 & 0 & -\alpha & r \end{bmatrix}_{(n-2)\times(n-2)} \\ &= r|B_{n-1}(r)| - \alpha^2|B_{n-2}(r)| \end{aligned}$$

The recurrence relation for the determinant of matrix $B_n(r)$ is obtained, that is

$$|B_n(r)| = r|B_{n-1}(r)| - \alpha^2|B_{n-2}(r)|.$$

Suppose $r = z + \alpha$, then

$$|B_n(z+\alpha)| = (z+\alpha)|B_{n-1}(z+\alpha)| - \alpha^2|B_{n-2}(z+\alpha)| \tag{17}$$

Multiply $(z-\alpha)$ by Equation (17), obtained:

$$\begin{aligned} (z-\alpha)|B_n(z+\alpha)| &= (z^2-\alpha^2)|B_{n-1}(z+\alpha)| - (z-\alpha)\alpha^2|B_{n-2}(z+\alpha)| \\ (z^2-\alpha^2)|B_{n-1}(z+\alpha)| &- (z-\alpha)\alpha^2|B_{n-2}(z+\alpha)| - (z-\alpha)|B_n(z+\alpha)| = 0 \end{aligned} \tag{18}$$

Equation (17) is equivalent to,

$$|B_{n-1}(z+\alpha)| - (z+\alpha)|B_{n-2}(z+\alpha)| + \alpha^2|B_{n-3}(z+\alpha)| = 0 \quad (19)$$

Multiply (α^2) by Equation (19), obtained:

$$\alpha^2|B_{n-1}(z+\alpha)| - (z+\alpha)\alpha^2|B_{n-2}(z+\alpha)| + \alpha^4|B_{n-3}(z+\alpha)| = 0 \quad (20)$$

The next step is to sum Equation (18) with (20) as follows:

$$\begin{aligned}
&(z^2-\alpha^2)|B_{n-1}(z+\alpha)| - (z-\alpha)\alpha^2|B_{n-2}(z+\alpha)| - (z-\alpha)|B_n(z+\alpha)| \\
&+\alpha^2|B_{n-1}(z+\alpha)| - (z+\alpha)\alpha^2|B_{n-2}(z+\alpha)| + \alpha^4|B_{n-3}(z+\alpha)| = 0 \\
&(z-\alpha)|B_n(z+\alpha)| = z^2|B_{n-1}(z+\alpha)| - 2z\alpha^2|B_{n-2}(z+\alpha)| + \alpha^4|B_{n-3}(z+\alpha)| \\
&(z-\alpha)|B_{n-1}(z+\alpha)| = z^2|B_{n-2}(z+\alpha)| - 2z\alpha^2|B_{n-3}(z+\alpha)| + \alpha^4|B_{n-4}(z+\alpha)| \quad (21)
\end{aligned}$$

Further, obtained the determinant of matrix $\mathcal{A}_n(z)$ using cofactor expansion as follows:

$$|\mathcal{A}_n(z)| = z^2|B_{n-2}(z+\alpha)| - 2z\alpha^2|B_{n-3}(z+\alpha)| + \alpha^4|B_{n-4}(z+\alpha)| \quad (22)$$

Therefore, based on the determinant results of matrices $\mathcal{A}_n(z)$ and $B_n(z+\alpha)$ in Equations (21) and (22), the following relationship is obtained:

$$|\mathcal{A}_n(z)| = (z-\alpha)|B_{n-1}(z+\alpha)|$$

Hence, the determinant of matrix $\mathcal{A}_n(z)$ is

$$|\mathcal{A}_n(z)| \quad = (z-\alpha)|B_{n-1}(z+\alpha)|$$

Based on Equation (15) then obtained:

$$|\mathcal{A}_n(z)| \quad = (z-\alpha)\cdot\alpha^{n-1}U_{n-1}\left(\frac{z+\alpha}{2\alpha}\right) \blacksquare.$$

Based on the solution of the recurrence relation the form $U_n(x)$, the determinant of matrix $\mathcal{A}_n(z)$ as follows:

$$\det(\mathcal{A}_n(z)) = \frac{z-\alpha}{2^n\sqrt{z^2+2z\alpha-3\alpha^2}}\left(\left(z+\alpha+\sqrt{z^2+2z\alpha-3\alpha^2}\right)^n - \left(z+z-\sqrt{z^2+2z\alpha-3\alpha^2}\right)^n\right).$$

Theorem 6 Let matrix $B_n(z)$ be a special form tridiagonal matrix of size $n \times n$, with

$$B_n(z) = \begin{bmatrix} z+\alpha & \alpha & 0 & 0 & \cdots & 0 \\ \alpha & z & \alpha & 0 & \cdots & \vdots \\ 0 & \alpha & z & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & z & \alpha \\ 0 & \cdots & 0 & 0 & \alpha & z+\alpha \end{bmatrix}$$

for all $z, \alpha \in \mathbb{R}$, $z \neq -2\alpha, z \neq 2\alpha$ and $n \in \mathbb{N}, n \geq 2$, then the determinant is

$$\det\big(B_n(z)\big) = (z + 2\alpha) \cdot \alpha^{n-1} U_{n-1}\left(\frac{z}{2\alpha}\right) \tag{23}$$

Proof. The form of the tridiagonal matrix $D_n(z)$ is as follows:

$$D_n(z) = \begin{bmatrix} z & \alpha & 0 & 0 & \cdots & 0 \\ \alpha & z & \alpha & 0 & \cdots & \vdots \\ 0 & \alpha & z & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & z & \alpha \\ 0 & \cdots & 0 & 0 & \alpha & z \end{bmatrix}$$

Calculating the determinant of matrix $D_n(z)$ using cofactor expansion is obtained as follows:

$$|D_n(z)| = \det\begin{bmatrix} z & \alpha & 0 & 0 & \cdots & 0 \\ \alpha & z & \alpha & 0 & \cdots & \vdots \\ 0 & \alpha & z & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & z & \alpha \\ 0 & \cdots & 0 & 0 & \alpha & z \end{bmatrix} = z|D_{n-1}(z)| - \alpha^2|D_{n-2}(z)|$$

The recurrence relation for the determinant of matrix $D_n(z)$ is obtained, that is

$$|D_n(z)| = z|D_{n-1}(z)| - \alpha^2|D_{n-2}(z)| \tag{24}$$

Multiply $(z + 2\alpha)$ by Equation (24), obtained:

$$(z^2 + 2z\alpha)|D_{n-1}(z)| - (z + 2\alpha)\alpha^2|D_{n-2}(z)| - (z + 2\alpha)|D_n(z)| = 0 \tag{25}$$

Equation (24) is equivalent to,

$$|D_{n-1}(z)| - z|D_{n-2}(z)| + \alpha^2|D_{n-3}(z)| \tag{26}$$

Multiply (α^2) by Equation (26), obtained:

$$\alpha^2|D_{n-1}(z)| - z\alpha^2|D_{n-2}(z)| + \alpha^4|D_{n-3}(z)| = 0 \tag{27}$$

The next step is to sum Equation (25) with (27) as follows:

$$(z^2 + 2z\alpha)|D_{n-1}(z)| - (z + 2\alpha)\alpha^2|D_{n-2}(z)| - (z + 2\alpha)|D_n(z)| + \alpha^2|D_{n-1}(z)|$$
$$-z\alpha^2|D_{n-2}(z)| + \alpha^4|D_{n-3}(z)| = 0$$
$$(z^2 + 2z\alpha + \alpha^2)|D_{n-1}(z)| - 2(z\alpha^2 + \alpha^3)|D_{n-2}(z)| - (z + 2\alpha)|D_n(z)| + \alpha^4|D_{n-3}(z)| = 0$$
$$(z + 2\alpha)|D_n(z)| = (z^2 + 2z\alpha + \alpha^2)|D_{n-1}(z)| - 2(z\alpha^2 + \alpha^3)|D_{n-2}(z)| + \alpha^4|D_{n-3}(z)|$$
$$(z + 2\alpha)|D_{n-1}(z)| = (z^2 + 2z\alpha + \alpha^2)|D_{n-2}(z)| - 2(z\alpha^2 + \alpha^3)|D_{n-3}(z)| + \alpha^4|D_{n-4}(z)| \tag{28}$$

Further, obtained the determinant of matrix $B_n(z)$ using cofactor expansion as follows:

$$|B_n(z)| = (z^2 + 2z\alpha + \alpha^2)|D_{n-2}(z)| - 2(z\alpha^2 + \alpha^3)|D_{n-3}(z)| + \alpha^4|D_{n-4}(z)| \tag{29}$$

Therefore, based on the determinant results of matrices $B_n(z)$ and $D_n(z)$ in Equations (28) and (29), the following relationship is obtained:

$$|B_n(z)| = (z + 2\alpha)|D_{n-1}(z)|.$$

Hence, the determinant of matrix $B_n(z)$ is

$$|B_n(z)| \quad = (z + 2\alpha)|D_{n-1}(z)|$$

Based on equation (15) then obtained:

$$|B_n(z)| \quad = (z + 2\alpha) \cdot \alpha^{n-1} U_{n-1}\left(\frac{z}{2\alpha}\right) \blacksquare.$$

Based on the solution of the recurrence relation the form $U_n(x)$, the determinant of matrix $B_n(z)$ as follows:

$$\det\big(B_n(z)\big) = \frac{z + 2\alpha}{2^n\sqrt{z^2 - 4\alpha^2}}\left(\left(z + \sqrt{z^2 - 4\alpha^2}\right)^n - \left(z - \sqrt{z^2 - 4\alpha^2}\right)^n\right).$$

In the following the determinant of a special form circulant matrix with Chebyshev polynomial of the first kind is given.

Theorem 7 Let matrix $C_n(z)$ be a special form circulant matrix of size $n \times n$, with

$$C_n(z) = \begin{bmatrix} z & -\alpha & 0 & 0 & \cdots & -\alpha \\ -\alpha & z & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & z & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & z & -\alpha \\ -\alpha & 0 & \cdots & 0 & -\alpha & z \end{bmatrix}$$

for all $z, \alpha \in \mathbb{R}$ and $n \in \mathbb{N}, n \geq 3$, then the determinant is

$$\det\big(C_n(z)\big) = 2\alpha^n \left(T_n\left(\frac{z}{2\alpha}\right) - 1\right) \tag{30}$$

Proof. The form of the circulant matrix $C_n(z)$ is as follows:

$$C_n(z) = \begin{bmatrix} z & -\alpha & 0 & 0 & \cdots & -\alpha \\ -\alpha & z & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & z & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & z & -\alpha \\ -\alpha & 0 & \cdots & 0 & -\alpha & z \end{bmatrix}$$

The form of the tridiagonal matrix $B_n(r)$ is as follows:

$$B_n(r) = \begin{bmatrix} r & -\alpha & 0 & 0 & \cdots & 0 \\ -\alpha & r & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & r & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & r & -\alpha \\ 0 & 0 & \cdots & 0 & -\alpha & r \end{bmatrix}$$

Based on the cofactor expansion steps of matrix $B_n(r)$ in Theorem 5, the recurrence relation for the determinant of matrix $B_n(r)$ is obtained $|B_n(r)| = r|B_{n-1}(r)| - \alpha^2|B_{n-2}(r)|$. Supposes $r = z$, then

$$|B_n(z)| = z|B_{n-1}(z)| - \alpha^2|B_{n-2}(z)| \tag{31}$$

The next step is to sum Equation (31) with $(-\alpha^2|B_{n-2}(z)| - 2\alpha^n)$ as follows:

$$|B_n(z)| - \alpha^2|B_{n-2}(z)| - 2\alpha^n = z|B_{n-1}(z)| - 2\alpha^2|B_{n-2}(z)| - 2\alpha^n \tag{32}$$

Further, obtained the determinant of matrix $C_n(z)$ using cofactor expansion as follows:

$$|C_n(z)| = z|B_{n-1}(z)| - 2\alpha^2|B_{n-2}(z)| - 2\alpha^n \tag{33}$$

Therefore, based on the determinant results of matrices $C_n(z)$ and $B_n(r)$ in Equations (32) and (33), the following relationship is obtained:

$$\begin{aligned}|C_n(z)| &= |B_n(z)| - \alpha^2|B_{n-2}(z)| - 2\alpha^n \\ &= \alpha^n U_n\left(\frac{z}{2\alpha}\right) - \alpha^2 \cdot \alpha^{n-2} U_{n-2}\left(\frac{z}{2\alpha}\right) - 2\alpha^n \\ &= \alpha^n U_n\left(\frac{z}{2\alpha}\right) - \alpha^n U_{n-2}\left(\frac{z}{2\alpha}\right) - 2\alpha^n \\ &= \alpha^n\left(U_n\left(\frac{z}{2\alpha}\right) - U_{n-2}\left(\frac{z}{2\alpha}\right) - 2\right)\end{aligned}$$

Based on the relationship between Chebyshev polynomials $T_n(x)$ and $U_n(x)$, namely $2T_n(x) = U_n(x) - U_{n-2}(x)$, then

$$\begin{aligned}|C_n(z)| &= \alpha^n\left(2T_n\left(\frac{z}{2\alpha}\right) - 2\right) \\ &= 2\alpha^n\left(T_n\left(\frac{z}{2\alpha}\right) - 1\right) \quad \blacksquare.\end{aligned}$$

Based on the solution of the recurrence relation the form Chebyshev polynomial $T_n(x)$, the determinant of matrix $C_n(z)$ as follows:

$$\det\big(C_n(z)\big) = \frac{1}{2^n}\left(\left(z + \sqrt{z^2 - 4\alpha^2}\right)^n + \left(z - \sqrt{z^2 - 4\alpha^2}\right)^n - 2^{n+1}\alpha^n\right).$$

Theorem 8 Let matrix $D_n(z)$ be a special form circulant matrix of size $n \times n$, with

$$D_n(z) = \begin{bmatrix} z & 0 & \alpha & \alpha & \cdots & 0 \\ 0 & z & 0 & \alpha & \cdots & \alpha \\ \alpha & 0 & z & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 0 & z & 0 \\ 0 & \alpha & \cdots & \alpha & 0 & z \end{bmatrix}$$

for all $z, \alpha \in \mathbb{R}$, $z \neq 3\alpha$ and $n \in \mathbb{N}, n \geq 3$, then the determinant is

$$\det\big(D_n(z)\big) = \frac{2\alpha^n(z + n\alpha - 3\alpha)}{z - 3\alpha}\left(T_n\left(\frac{z-\alpha}{2\alpha}\right) - 1\right) \tag{34}$$

Proof. Simplify and perform elementary row operations on matrix $D_n(z)$ to form matrix $C_n(z)$ based on Theorem 7 by applying matrix properties and matrix determinant properties.

$$\det\big(D_n(z)\big) = \det\begin{bmatrix} z & 0 & \alpha & \alpha & \cdots & 0 \\ 0 & z & 0 & \alpha & \cdots & \alpha \\ \alpha & 0 & z & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 0 & z & 0 \\ 0 & \alpha & \cdots & \alpha & 0 & z \end{bmatrix}$$

Add each row $i, i = 2,3, \dots, n$ to row 1, obtained as follows:

$$= \det\begin{bmatrix} z+n\alpha-3\alpha & z+n\alpha-3\alpha & z+n\alpha-3\alpha & z+n\alpha-3\alpha & \cdots & z+n\alpha-3\alpha \\ 0 & z & 0 & \alpha & \cdots & \alpha \\ \alpha & 0 & z & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 0 & z & 0 \\ 0 & \alpha & \cdots & \alpha & 0 & z \end{bmatrix}$$

$$= (z+n\alpha-3\alpha)\det\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & z & 0 & \alpha & \cdots & \alpha \\ \alpha & 0 & z & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 0 & z & 0 \\ 0 & \alpha & \cdots & \alpha & 0 & z \end{bmatrix}$$

Then, by applying the elementary row operations, the following matrix form is obtained:

$$\det(D_n(z)) = \frac{z+n\alpha-3\alpha}{z-3\alpha}\det\begin{bmatrix} z-\alpha & -\alpha & 0 & 0 & \cdots & -\alpha \\ -\alpha & z-\alpha & -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & z-\alpha & -\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha & z-\alpha & -\alpha \\ -\alpha & 0 & \cdots & 0 & -\alpha & z-\alpha \end{bmatrix}$$

Based on Theorem 7, obtained

$$\begin{aligned}\det(D_n(z)) &= \frac{z+n\alpha-3\alpha}{z-3\alpha} \times 2\alpha^n\left(T_n\left(\frac{z-\alpha}{2\alpha}\right)-1\right) \\ &= \frac{2\alpha^n(z+n\alpha-3\alpha)}{z-3\alpha}\left(T_n\left(\frac{z-\alpha}{2\alpha}\right)-1\right) \blacksquare.\end{aligned}$$

Based on the solution of the recurrence relation the form Chebyshev polynomial of first kind, the determinant of matrix $D_n(z)$ as follows:

$$\det(D_n(z)) = \frac{z+n\alpha-3\alpha}{2^n(z-3\alpha)}\left(\left(z-\alpha+\sqrt{z^2-2z\alpha-3\alpha^2}\right)^n + \left(z-\alpha-\sqrt{z^2-2z\alpha-3\alpha^2}\right)^n - 2^{n+1}\cdot\alpha^n\right).$$

D. CONCLUSION AND SUGGESTIONS

By using Chebyshev polynomials of the first and second kinds, this research succeeds in systematically constructing and verifying determinant formulas for tridiagonal and circulant matrices with special forms. Calculating the determinant of the special form of the tridiagonal matrix T_n using the second kind of Chebyshev polynomial $(U_n(x))$, as for the form of the determinant of the tridiagonal matrix $\det(\mathcal{A}_n(z)) = (z-\alpha)\cdot\alpha^{n-1}U_{n-1}\left(\frac{z+\alpha}{2\alpha}\right)$ for all $z, \alpha \in$

$\mathbb{R}, z \neq -3\alpha, z \neq \alpha$ and $n \in \mathbb{N}, n \geq 2$ and $\det(B_n(z)) = (z + 2\alpha) \cdot \alpha^{n-1} U_{n-1}\left(\frac{z}{2\alpha}\right)$ for all $z, \alpha \in \mathbb{R}, z \neq -2\alpha, z \neq 2\alpha$ and $n \in \mathbb{N}, n \geq 2$.

Calculating the determinant of the special form of the circulant matrix C_n using the first type of Chebyshev polynomial $(T_n(x))$, as for the form of the determinant of the circulant matrix $\det(C_n(z)) = 2\alpha^n \left(T_n\left(\frac{z}{2\alpha}\right) - 1\right)$ for all $z, \alpha \in \mathbb{R}$ and $n \in \mathbb{N}, n \geq 3$ and $\det(D_n(z)) = \frac{2\alpha^n(z+n\alpha-3\alpha)}{z-3\alpha}\left(T_n\left(\frac{z-\alpha}{2\alpha}\right) - 1\right)$ for all $z, \alpha \in \mathbb{R}, z \neq 3\alpha$ and $n \in \mathbb{N}, n \geq 3$. These formulas show that Chebyshev polynomials can be directly related to matrix determinants and provide a simpler and more efficient way to calculate determinants.

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