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# Control Strategies for HIV/AIDS-Hepatitis B Coinfection using Optimal Control Approach and Cost-Effectiveness Analysis

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**ABSTRACT**

HIV/AIDS and Hepatitis B are infectious diseases caused by viruses, sharing similar transmission mechanisms. This study seeks to determine the most effective and cost-efficient strategies for controlling the spread of these diseases by utilizing a modified HIV/AIDS-Hepatitis B coinfection model with various control variables. The model divides the total population into nine subpopulations, each representing a specific disease state. Based on these classifications, the model incorporates four key control variables, namely Hepatitis B vaccination program, Hepatitis B treatment, HIV/AIDS treatment, and public health education program. The research employs optimal control theory and the Pontryagin Maximum Principle to address the optimal control problem to minimize infection rates and implementation costs over a specific periode. The Hamilton function integritas the dynamic system and cost function. The model is analyzed through simulations using parameter values from previous studies, then optimizing control variables to generate a numerically solved system of differential equations that uses Scilab 2024 software. Simulation result show that the optimal combination of four control strategies reduces HIV/AIDS-Hepatitis B infection by 79,2% in under ten years. Furthermore, the cost-effectiveness of different strategies is evaluated using the Average Cost-Effectiveness Ratio (ACER) and Incremental Cost-Effectiveness Ratio (ICER) indicates that single control strategies are more cost-efficient, while combining all four strategies is more expensive. However, successful implementation depends on financial constraints (limited vaccination and ARV treatment), healthcare infrastructure (availability of testing facilities), and public compliance with health education programs. Consequently, the proposed strategies are recommended for policymakers, with consideration of associated costs to ensure feasibility.

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## A. INTRODUCTION

Infectious diseases remain a significant global health challenge, particularly in developing nations such as Indonesia, where they frequently result in severe complications and fatalities (Endashaw & Mekonnen, 2022). Coinfections, which occur when an individual simultaneously contracts two or more distinct diseases, are a pressing concern globally, particularly in regions with high disease prevalence and limited healthcare resources. These conditions not only complicate prevention and control efforts but also exacerbate the progression and severity of the underlying viral infections, leading to increased morbidity and mortality (Ma et al., 2020). For instance, coinfections such as dengue fever and Zika virus (Bonyah et al., 2019), HIV/AIDS and HBV (Mphahlele, 2015; Weldemhret, 2022), Sifilis and HBV/HCV (Shimelis et al., 2022) are frequently observed in areas with overlapping disease transmission routes. Additionally,

combinations like HIV/AIDS and HPV (Chazuka et al., 2023; Gurmu et al., 2021), measles and dysentery (Berhe et al., 2019), and pneumonia and typhoid fever (Tilahun et al., 2018) highlight the complexity of addressing coinfections across diverse settings. Therefore, it is important to select effective measures to address coinfection issues, based on existing diagnostic capabilities. The selection of treatment and prevention strategies that are integrated and tailored to the specific epidemiological context is also indispensable.

One form of coinfection that is of major concern and the focus of this study is HIV/AIDS and Hepatitis B dual infection. A condition that arises because of the similarity of disease transmission routes, such as blood contact and unprotected sexual activity, thus contributing to the high rate of transmission and death (Echeng, 2020). In 2023, Hepatitis B affected approximately 301 million individuals worldwide, with a substantial risk of progression to severe complications such as liver cirrhosis and liver cancer (WHO 2023a). In Indonesia alone, an estimated 18 million people live with Hepatitis B, with up to 90% of cases potentially advancing to chronic stages (Kemenkes RI, 2023a). Concurrently, global HIV/AIDS cases reached 39.9 million in 2023, with 526.841 cases reported in Indonesia, where these individuals remain a significant source of community transmission (Kemenkes RI 2023b; WHO 2023b). The significant prevalence and associated risks of HIV/AIDS-Hepatitis B coinfection, which account for approximately 11% of all infected individuals, highlight the urgent need for comprehensive and integrated control measures to curb disease transmission effectively (WHO, 2023a). To mitigate the spread of HIV/AIDS-Hepatitis B coinfection, government strategies should focus on preventing mother-to-child transmission during pregnancy, promoting widespread Hepatitis B vaccination, ensuring access to effective treatment for Hepatitis B, and enhancing public awareness through educational campaigns on HIV/AIDS and Hepatitis B. Furthermore, antiretroviral (ARV) therapy plays a critical role in managing HIV/AIDS and reducing its transmission (Omondi et al., 2019). Given the complexity of HIV/AIDS-Hepatitis B coinfection and the urgent need for effective intervention strategies, various studies have explored mathematical modeling as a valuable tool for understanding disease dynamics and optimizing control measures.

Numerous studies have focused on the development and application of mathematical models to address the challenges associated with controlling the spread and treatment of HIV/AIDS and Hepatitis B. Nah et al. (2017) explored the use of mathematical modeling to design and evaluate strategies for testing and treatment, aiming to improve the management of HIV/AIDS. This study highlighted the potential of mathematical approaches in optimizing resource allocation and enhancing the effectiveness of intervention programs. Similarly, Anwarud et al. (2021) introduced a sophisticated model for Hepatitis B infection, demonstrating the applicability of optimal control theory in mitigating the disease's spread. Their findings underscored the importance of integrating mathematical tools with public health strategies to achieve better outcomes. Expanding on these efforts, Endashaw and Mekonnen (2022) developed a mathematical model that examines the impact of vaccination and treatment on the transmission dynamics of HIV/AIDS-Hepatitis B coinfection. This study provided critical insights into how combined interventions could influence disease prevalence and highlighted the role of mathematical modeling in identifying cost-effective and impactful strategies. These studies collectively emphasize the value of mathematical modeling as a powerful tool for

understanding complex disease dynamics and formulating evidence-based control measures (Nainggolan, 2017).

Building on previous research, the mathematical model proposed in this study was developed through modifications and extensions to previous frameworks. The model modification for Hepatitis B was developed from the framework of Wodajo and Mekonnen (2022) and Kamyad et al. (2014). Meanwhile, the model modification for HIV/AIDS and coinfection was developed from the framework of Endashaw & Mekonnen (2022). The model introduced significant modifications from previous studies, including the incorporation of four main control variables, namely Hepatitis B vaccination program, Hepatitis B treatment, HIV/AIDS treatment, and health education program. In addition, a new compartment, Recovered (R), was added to explicitly represent individuals who have recovered from Hepatitis B infection. Furthermore, these structural modifications significantly improved the validation of the model in presenting the dynamics of multiple transmissions by considering various issues, such as the effects of vaccination, treatment, and the impact of health education on at-risk individuals. Thus, the developed model not only adopts the theories from previous studies, but provides a more complex modified model of the HIV/AIDS-Hepatitis B coinfection model.

The study employs optimal control theory and the Pontryagin Maximum Principle as analytical tools to derive and evaluate optimal strategies for disease control. Furthermore, economic considerations are integrated by calculating the Average Cost-Effectiveness Ratio (ACER) and Incremental Cost-Effectiveness Ratio (ICER), allowing for a comprehensive assessment of both the effectiveness and the cost-efficiency of different intervention strategies. The ultimate goal of this study is to identify strategies that not only minimize infection rates but also do so in a manner that is economically sustainable and practical for implementation in real-world strategies. This approach emphasizes the need for mathematical models that are not only robust in their theoretical formulation but also adaptable to varying public health contexts, offering valuable insights for policymakers and healthcare practitioners alike. Future research could further enhance these models by exploring additional variables such as behavioral factors, testing the scalability of interventions in diverse epidemiological contexts (Rose 2015), incorporating real-time data, such as treatment adherence rates and the role of healthcare infrastructure, to enhance its applicability.

## B. METHODS

The section explains the strages of the research so that the predetermined objectives can be achieved. These stages are formulating optimal control problems by adding control variables into the system so that the model can be analyzed according to the objectives, determining objective functions, and forming Hamilton functions. Hamilton functions are used to find the necessary conditions for optimization using Pontryagin's maximum principle. Then, numerical simulations will be carried out using Scilab 2024 software to analyze the application of several predetermined control strategies.

## 1. Coinfection Control Formulation

The HIV/AIDS-Hepatitis B coinfection model divides the population into nine subpopulations, defined as follows: $s(t)$ represents the proportion of individuals susceptible to infection at time $t$, $v_b(t)$ denotes the proportion of susceptible individuals vaccinated against Hepatitis B at time $t$, $i_h(t)$ and $i_b(t)$ describe the proportion of individuals infected with HIV/AIDS and those with Hepatitis B at time $t$ respectively, $i_{hb}(t)$ indicates the proportion of individuals coinfected with both HIV/AIDS and Hepatitis B at time $t$, $l_h(t)$ is the proportion of individuals undergoing treatment for HIV/AIDS at time $t$, $l_{hb}(t)$ describes the proportion of individuals receiving treatment for HIV/AIDS-Hepatitis B coinfection at time $t$, $r_b(t)$ represents the proportion of individuals recovered from Hepatitis B infection at time $t$, and $s_v(t)$ indicates the proportion of individuals under viral suppression control for HIV/AIDS-Hepatitis B at time $t$. The $s_v(t)$ compartment refers to individuals who have achieved viral suppression through effective medical intervention, reducing the viral load of both HIV/AIDS and Hepatitis B to levels that are undetectable or clinically manageable. This condition minimizes the risk of disease progression, lowers transmission rates, and improves overall health outcomes. These individuals remain in a controlled state but are still monitored regularly to ensure continued viral suppression.

The HIV/AIDS-Hepatitis B coinfection model provides a framework to understand the interaction between these two infections and their impact on population dynamics. In this study, the subjects analyzed include susceptible individuals, HIV/AIDS infected, Hepatitis B infected, coinfected individuals, individuals receiving treatment for HIV/AIDS and Hepatitis B, individuals recovering from Hepatitis B infection, and individuals under viral suppression control for HIV/AIDS-Hepatitis B. The model serves as a valuable tool for analyzing disease progression, evaluating treatment strategies, and assessing the effectiveness of vaccination programs. By incorporating mathematical representations, the model offers insights into the complex relationships between co-infected individuals and the broader population. Before delving into the model's structure, it is essential to outline the basic assumptions that underpin its development and ensure its practical applicability.

a. Each individual can always be categorized into one of the nine compartments based on their status regarding HIV/AIDS and Hepatitis B.
b. The total population is non-constant, with a birth rate of $C$ and a natural mortality rate of $\mu$. The additional mortality rate induced by HIV/AIDS, Hepatitis B, and HIV/AIDS-Hepatitis B infections are denoted by $\delta_h$, $\delta_b$, and $\delta_{hb}$, respectively.
c. Individuals can be infected with both HIV/AIDS and Hepatitis B because the modes of transmission for both diseases are similar. Additionally, it is assumed that the second infection (such as Hepatitis B) may occur before the first infection (HIV/AIDS) is fully established or detectable in the host. This means that an individual can acquire both infections simultaneously or in quick succession, leading to a coinfection scenario (Echeng, 2020).
d. Mixing within the population is homogeneous, meaning all individuals have an equal probability of contact and potential disease transmission. The contact rate for HIV/AIDS is denoted by $\omega_h$ and that for Hepatitis B is denoted by $\omega_b$. Therefore, the transmission rates for HIV/AIDS and Hepatitis B are given by $\lambda_h = \omega_h(i_h + i_{hb})$ and $\lambda_b = \omega_b(i_b +$

$i_{hb}$), respectively. To reduce the complexity of the model, the transmission of HIV/AIDS and Hepatitis B infection without vertical transmission is assumed to be ignored in the model. This means that transmission from mother to child during childbirth or breastfeeding is not considered as a factor in the disease dynamics.

e. Hepatitis B vaccination program with rate $u_1$ and effectiveness $\varepsilon_1$ provides immunity to a portion of the susceptible population, reducing their likelihood of infection. The rate of recovery induced by vaccination is given by $\tau$ and the waning immunity rates from vaccinated and recovered compartments are denoted by $q$ and $\xi$, respectively.
f. The recovery rate for individuals infected with Hepatitis B is $\gamma_b$, which is constant. This rate can be enhanced through the control variable $u_2$ with effectiveness $\varepsilon_2$, allowing for an increase in treatment intensity. This control action also affects individuals who are co-infected with both HIV/AIDS and Hepatitis B.
g. The treatment rate for individuals infected with HIV/AIDS is $\gamma_h$, which is constant, can be intensified through the control variable $u_3$ with effectiveness $\varepsilon_3$. That of HIV/AIDS-Hepatitis B coinfection, $\gamma_{hb}$, can also be boosted through the control variable $u_3$.
h. The viral supression rate of HIV/AIDS treatment is reflected by $\theta_h$, denoting the rate at which individuals undergoing HIV/AIDS treatment achieve viral load suppression and move to $s_v$ compartment. Similarly, that of HIV/AIDS-Hepatitis B coinfection is denoted by $\theta_{hb}$, representing the rate at which individuals receiving treatment for HIV/AIDS-Hepatitis B coinfection achieve viral suppression.
i. The health education program, denoted by $u_4$ with effectiveness $\varepsilon_4$, improves awareness and compliance with preventive measures, reducing transmission rates of both diseases.

Based on the key assumptions outlined, the HIV/AIDS-Hepatitis B coinfection compartmental model is illustrated in Figure 1. The model takes into account the dynamics of disease progression, treatment, and vaccination, as well as the impact of control variables such as Hepatitis B vaccination program, Hepatitis B treatment, HIV/AIDS treatment, and public health education programs. The mathematical formulation of the model is expressed in the system of nonlinear differential equations (1)–(9), which capture the rates of change within each compartment over time. This system provides a comprehensive framework for understanding the interactions between the two infections and the effectiveness of various interventions.

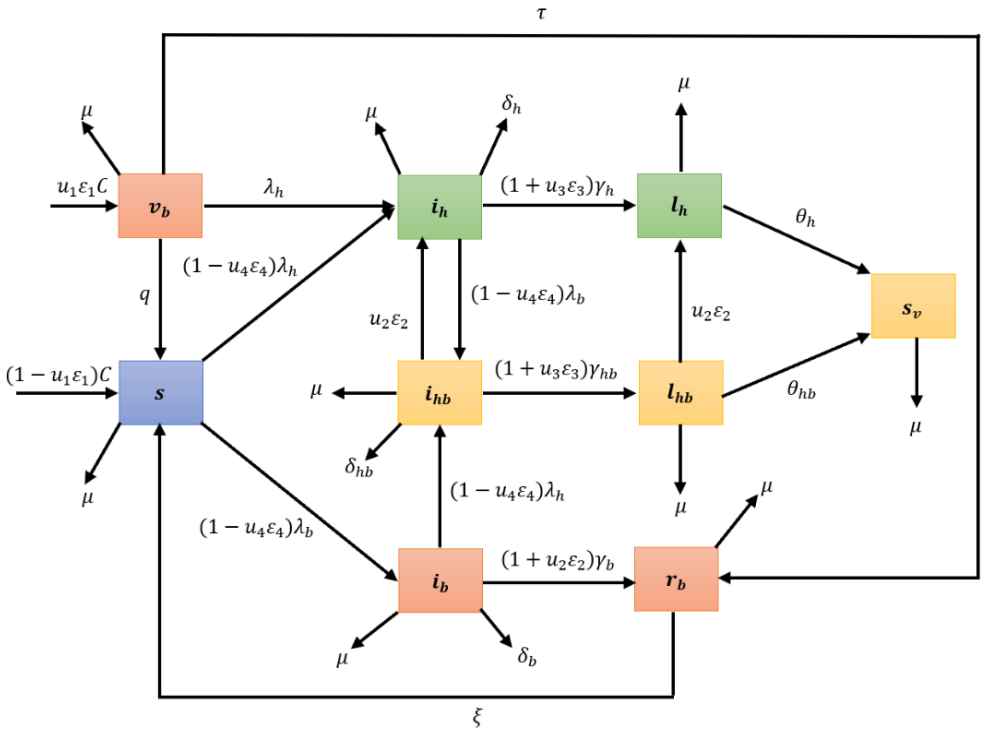



**Figure 1.** Compartmental model of HIV/AIDS-Hepatitis B coinfection

$$\frac{ds}{dt} = (1-\varepsilon_1 u_1)C + q v_b + \xi r_b - \Big(\mu + (1-\varepsilon_4 u_4)\big(\omega_h(i_h + i_{hb}) + \omega_b(i_b + i_{hb})\big)\Big)s, \tag{1}$$

$$\frac{dv_b}{dt} = C\varepsilon_1 u_1 - (\omega_h(i_h + i_{hb}) + q + \tau + \mu)v_b, \tag{2}$$

$$\frac{di_h}{dt} = (1-\varepsilon_4 u_4)\omega_h(i_h + i_{hb})s + \omega_h(i_h + i_{hb})v_b + \varepsilon_2 u_2 i_{hb} - ((1+u_3\varepsilon_3)\gamma_h + (1-\varepsilon_4 u_4)\omega_b(i_b + i_{hb}) + \delta_h + \mu)i_h, \tag{3}$$

$$\frac{di_b}{dt} = (1-\varepsilon_4 u_4)\omega_b(i_b + i_{hb})s - \big((1-\varepsilon_4 u_4)\omega_h(i_h + i_{hb}) + (1+\varepsilon_2 u_2)\gamma_b + \delta_b + \mu\big)i_b, \tag{4}$$

$$\frac{di_{hb}}{dt} = (1-\varepsilon_4 u_4)\omega_b(i_b + i_{hb})i_h + (1-\varepsilon_4 u_4)\omega_H(i_h + i_{hb})i_b - (\varepsilon_2 u_2 + (1+\varepsilon_3 u_3)\gamma_{hb} + \delta_{hb} + \mu)i_{hb}, \tag{5}$$

$$\frac{dl_h}{dt} = (1+\varepsilon_3 u_3)\gamma_h i_h + \varepsilon_2 u_2 l_{hb} - (\theta_h + \mu)l_h, \tag{6}$$

$$\frac{dl_{hb}}{dt} = (1+\varepsilon_3 u_3)\gamma_{hb} i_{hb} - (\varepsilon_2 u_2 + \theta_{hb} + \mu)l_{hb}, \tag{7}$$

$$\frac{dr_b}{dt} = (1+\varepsilon_2 u_2)\gamma_b i_b + \tau v_b - (\xi + \mu)r_b, \tag{8}$$

$$\frac{ds_v}{dt} = \theta_h l_h + \theta_{hb} l_{hb} - \mu s_v. \tag{9}$$

Initial values are necessary for each subpopulation to ensure the proper implementation and analysis of the model. These initial conditions reflect the starting point of the system at time $t = 0$ and are crucial for solving the system of differential equations. The following initial values are assigned to each subpopulation to begin the model's simulation:

$$s(0) = s^0, v_b(0) = v_b^0, i_h(0) = i_h^0, i_b(0) = i_b^0, i_{hb}(0) = i_{hb}^0, l_h(0) = l_h^0, l_{hb}(0) = l_{hb}^0, \\ r_b(0) = r_b^0, s_v(0) = s_v^0, \tag{10}$$

where all initial values lie in the interval [0,1]. We also assume that, at the end of the control period $T$, the proportion of individuals in each subpopulation is not determined, namely.

$$s(T), v_b(T), i_h(T), i_b(T), i_{hb}(T), l_h(T), l_{hb}(T), r_b(T), s_v(T) \text{ are all free.} \tag{11}$$

## 2. Control Measures

As previously explained, the HIV/AIDS-Hepatitis B coinfection model is equipped with four control variables, namely Hepatitis B vaccination program ($u_1$), treatment for Hepatitis B ($u_2$), treatment for HIV/AIDS ($u_3$), and public health education programs ($u_4$). These control actions are commonly implemented through various activities. For vaccination, activities may include mass vaccination campaigns, school-based immunization programs, and outreach to high-risk populations. Treatment for Hepatitis B typically involves the use of antiviral medications, regular liver function monitoring, and ensuring adherence to treatment regimens. HIV/AIDS treatment involves the administration of antiretroviral therapy (ART), regular check-ups, and addressing co-morbidities associated with HIV. Public health education program includes awareness campaigns, media outreach, community-based programs, and training healthcare providers to improve knowledge about prevention, safe practices, and early detection. These interventions are essential in controlling the spread of both infections and improving overall public health outcomes. In the analysis, we assume bounded controls

$$0 \leq u_j(t) \leq \bar{u}_j \leq 1, \tag{12}$$

for $j = 1,2,3,4$ and $t \in [0, T]$, ensures that the control efforts remain realistic and feasible, reflecting the limited resources and practical implementation capacities. The upper bound $\bar{u}_j$ in (12) represents the maximum achievable intensity of the control intervention, determined by available resources, infrastructure, and logistical limitations.

# C. RESULT AND DISCUSSION

## 1. Control Analysis

The purpose of applying control variables is to minimize the infected population $i_h, i_b$, and $i_{hb}$ with the minimum cost function, which consist of Hepatitis B vaccination program ($u_1$), Hepatitis B treatment ($u_2$), HIV/AIDS treatment ($u_3$), and public health education programs ($u_4$). The cost function will be a nonlinear model with the control function to be chosen in quadratic form $u_j^2$. Mathematically, the performance criterion of model (1)–(9) can be expressed as follows:

$$\min J = \int_0^T \left( A_1 i_h + A_2 i_b + A_3 i_{hb} + \frac{1}{2}(B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2 + B_4 u_4^2) \right) dt, \tag{13}$$

where $T$ is the control period. Weights for infected individuals are defined as $A_1, A_2$ and $A_3$, while those for control implementation are denoted by $B_1, B_2, B_3$ and $B_4$.

The application of Pontryagin's maximum principle requires the definition of the Hamiltonian function $H$, which combines the system dynamics, control variables, and adjoint variables (Tu, 1994). The Hamiltonian plays a central role in characterizing the optimal control strategy by providing necessary conditions for optimality. It integrates the state equations, the objective functional, and the costate variables to guide the optimization process, ensuring that the control variables maximize the system's performance while adhering to the given constraints. Based on model (1)–(9) and objective functional (13), the Hamiltonian $H$ is defined as follows:

$$\begin{aligned} H = {} & A_1 i_h + A_2 i_b + A_3 i_{hb} + \frac{1}{2}(B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2 + B_4 u_4^2) \\ & + p_1 \Big( (1 - \varepsilon_1 u_1) C + q v_b + \xi r_b \\ & - \Big( \mu + (1 - \varepsilon_4 u_4)\big(\omega_h (i_h + i_{hb}) + \omega_b (i_b + i_{hb})\big)\Big) s \Big) \\ & + p_2 (u_1 \varepsilon_1 C - (\omega_h (i_h + i_{hb}) + q + \tau + \mu) v_b) \\ & + p_3 \big( (1 - u_4 \varepsilon_4)(\omega_h (i_h + i_{hb})) s + (\omega_h (i_h + i_{hb})) v_b + u_2 \varepsilon_2 i_{hb} \\ & - ((1 + u_3 \varepsilon_3)\gamma_h + (1 - u_4 \varepsilon_4)\omega_b (i_b + i_{hb}) + \delta_h + \mu) i_h \big) \\ & + p_4 \Big( (1 - u_4 \varepsilon_4)(\omega_b (i_b + i_{hb})) s \\ & - \big( (1 - u_4 \varepsilon_4)(\omega_h (i_h + i_{hb})) + (1 + u_2 \varepsilon_2)\gamma_b + \delta_b + \mu \big) i_b \Big) \\ & + p_5 \Big( (1 - u_4 \varepsilon_4)(\omega_b (i_b + i_{hb})) i_h + \big( (1 - u_4 \varepsilon_4)(\omega_h (i_h + i_{hb})) \big) i_b \\ & - (u_2 \varepsilon_2 + (1 + u_3 \varepsilon_3)\gamma_{hb} + \delta_{hb} + \mu) i_{hb} \Big) \\ & + p_6 \big( (1 + u_3 \varepsilon_3)\gamma_h i_h + u_2 \varepsilon_2 l_{hb} - (\theta_h + \mu) l_h \big) \\ & + p_7 \big( (1 + u_3 \varepsilon_3)\gamma_{hb} i_{hb} - (u_2 \varepsilon_2 + \theta_{hb} + \mu) l_{hb} \big) \\ & + p_8 \big( (1 + u_2 \varepsilon_2)\gamma_b i_b + \tau v_b - (\xi + \mu) r_b \big) + p_9 (\theta_h l_h + \theta_{hb} l_{hb} - \mu s_v), \end{aligned} \tag{14}$$

where $p_i = p_i(t)$ for $i = 1, 2, \ldots, 9$ are adjoint variables (costate variables). Pontryagin's maximum principle provides necessary conditions for solving optimal control problems [Tu, 1984]. It states that for an optimal control to minimize or maximize an objective functional, there exists a set of adjoint variables $p_i$ such that the Hamiltonian $H$ is minimized with respect to the control $u_j$ at every point in time, namely $H(x^*, u^*, p^*, t) \leq H(x, u, p, t)$ for state variable vector $x$, control variable vector $u$, and adjoint variable vector $p$, which can be represented by

$$\frac{\partial H}{\partial u_j} = 0, \tag{15}$$

for $j = 1,2,3,4$. Next conditions establishe the dynamic relationship between the state and adjoint variables through the Hamiltonian. It ensures that the system's evolution is consistent with the optimization process, with the state dynamics describing how the system changes over

time and the adjoint dynamics capturing the sensitivity of the objective functional to changes in the state variables:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \tag{16}$$

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \tag{17}$$

for $i = 1,2,\dots,9$ and $x_i \in \{s, v_b, i_h, i_b, i_{hb}, l_h, l_{hb}, r_b, s_v\}$. While condition (17) gives back the dynamical system (1)–(9), condition (15) specifies the optimal controls as given in Theorem 1 and condition (16) characterizes the adjoint system provided in Theorem 2.

**Theorem 1.** The optimal controls $u_1^*$, $u_2^*$, $u_3^*$, and $u_4^*$, which minimize objective functional (13), are given by

$$u_1^* = \min\left\{\bar{u}_1, \max\left\{0, \frac{(p_1 - p_2)\varepsilon_1 C}{B_1}\right\}\right\}, \tag{18}$$

$$u_2^* = \min\left\{\bar{u}_2, \max\left\{0, \frac{-(p_3 - p_5)\varepsilon_2 i_{hb} - (p_8 - p_4)\gamma_b \varepsilon_2 i_b - (p_6 - p_7)\varepsilon_2 l_{hb}}{B_2}\right\}\right\}, \tag{19}$$

$$u_3^* = \min\left\{\bar{u}_3, \max\left\{0, \frac{-(p_6 - p_3)\varepsilon_3 \gamma_h i_h - (p_7 - p_5)\varepsilon_3 \gamma_{hb} i_{hb}}{B_3}\right\}\right\}, \tag{20}$$

$$u_4^* = \min\left\{\bar{u}_4, \max\left\{0, \frac{K_4}{B_4}\right\}\right\}, \tag{21}$$

where

$$\begin{aligned} K_4 = {} & -(p_1 - p_3)\varepsilon_4 \omega_h (i_h + i_{hb}) s - (p_1 - p_4)\varepsilon_4 \omega_b (i_b + i_{hb}) s - (p_3 - p_5)\varepsilon_4 \omega_b (i_h + i_{hb}) i_h \\ & - (p_4 - p_5)\varepsilon_4 \omega_h (i_h + i_{hb}) i_b. \end{aligned}$$

**Proof.** Optimal controls (18)–(21) are obtained by application of (15) and by considering bounded controls (12).

**Theorem 2.** Given the optimal state variable $x = (s, v_b, i_h, i_b, i_{hb}, l_h, l_{hb}, r_b, s_v)^T$ associated with the optimal control pair $u = (u_1, u_2, u_3, u_4)^T$ in Theorem 1, the adjoint variables $p_i$ $(i = 1,2,\dots,9)$ satisfy the following differential equations system:

$$\begin{aligned} \dot{p_1} = {} & p_1\big((1 - u_4\varepsilon_4)(\omega_h(i_h + i_{hb}) + \omega_b(i_b + i_{hb})) + \mu\big) - p_3(1 - u_4\varepsilon_4)\big(\omega_h(i_h + i_{hb})\big) \\ & - p_4(1 - u_4\varepsilon_4)\big(\omega_b(i_b + i_{hb})\big), \end{aligned} \tag{22}$$

$$\dot{p_2} = -p_1 q + p_2(\omega_h(i_h + i_{hb}) + q + \tau + \mu) - p_3\big(\omega_h(i_h + i_{hb})\big) - p_8\tau, \tag{23}$$

$$\begin{aligned}\dot{p_3} = &-A_1 + p_1(1-u_4\varepsilon_4)\omega_h s + p_2\omega_h v_b \\ &+ p_3(-(1-u_4\varepsilon_4)\omega_h s - \omega_h v_b + (1+u_3\varepsilon_3)\gamma_h \\ &+ (1-u_4\varepsilon_4)\omega_b(i_b+i_{hb}) + \delta_h + \mu) + p_4\big((1-u_4\varepsilon_4)\omega_h i_b\big) \\ &- p_5(1-u_4\varepsilon_4)(\omega_b(i_b+i_{hb}) + \omega_h i_b) - p_6(1+u_3\varepsilon_3)\gamma_h,\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{p_4} = &-A_2 + p_1(1-u_4\varepsilon_4)\omega_b s - p_3\big((1-u_4\varepsilon_4)\omega_b i_h\big) \\ &+ p_4(-(1-u_4\varepsilon_4)\omega_b s + (1-u_4\varepsilon_4)\omega_h(i_h+i_{hb}) + (1+u_2\varepsilon_2)\gamma_b + \delta_b \\ &+ \mu) - p_5\big((1-u_4\varepsilon_4)\omega_b i_h + (1-u_4\varepsilon_4)\omega_h(i_h+i_{hb})\big) \\ &- p_8\big((1+u_2\varepsilon_2)\gamma_b\big),\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{p_5} = &-A_3 + p_1\big((1-u_4\varepsilon_4)(\omega_h s + \omega_b s)\big) + p_2\omega_h v_b \\ &+ p_3(-(1-u_4\varepsilon_4)\omega_h s - \omega_h v_b - u_2\varepsilon_2 + (1-u_4\varepsilon_4)\omega_b i_h) \\ &+ p_4(-(1-u_4\varepsilon_4)\omega_h s + (1-u_4\varepsilon_4)\omega_h i_b) \\ &+ p_5(-(1-u_4\varepsilon_4)\omega_b i_h - (1-u_4\varepsilon_4)\omega_h i_b + u_2\varepsilon_2 + (1+u_3\varepsilon_3)\gamma_{hb} + \delta_{hb} \\ &+ \mu) - p_7(1+u_3\varepsilon_3)\gamma_{hb},\end{aligned} \tag{26}$$

$$\dot{p_6} = p_6(\theta_h + \mu) - p_9\theta_h, \tag{27}$$

$$\dot{p_7} = -p_6 u_2\varepsilon_2 + p_7(u_2\varepsilon_2 + \theta_{hb} + \mu) - p_9\theta_{hb}, \tag{28}$$

$$\dot{p_8} = -p_1\xi + p_8(\xi + \mu), \tag{29}$$

$$\dot{p_9} = p_9\mu, \tag{30}$$

with terminal time

$$p_i(T) = 0, \qquad i = 1,2,\dots,9. \tag{31}$$

**Proof.** Adjoint system (22)–(30) is obtained by application (16), while the transversality condition (31) is caused by free terminal times (11).

The optimal control and corresponding optimal trajectories are obtained by numerically solving the dynamical system (1)–(9), the adjoint system (22)–(30), and the control equations (18)–(21). This problem is particularly interesting because the dynamic system is subject to initial conditions (10), while the adjoint system is governed by terminal conditions (31). To address this complexity, a combination of the Runge-Kutta algorithm and the forward-backward sweep method is employed (Lenhart & Workman, 2007). This approach ensures a precise and iterative resolution of the coupled differential equations, providing a robust framework for identifying optimal solutions in control problems.

## 2. Numerical Simulation

To analyze the influence of these four controls on the dynamics of the spread of HIV/AIDS-Hepatitis B coinfection, a numerical simulation was carried out using *Scilab* 2024 *software*. The simulation covers five control strategies that are arranged based on a combination of prevention and treatment controls to suppress the dynamics of coinfection spread. This analysis aims to measure the effectiveness of four types of control in reducing the spread of the disease. The simulation is carried out by setting a starting value in each subpopulation, namely $s_0 = 0.3311, v_b^0 = 0.2649, i_h^0 = 0.1325, i_b^0 = 0.1060, i_{hb}^0 = 0.0795,\ l_h^0 = 0.0530,\ l_{hb}^0 = 0.0265,$ $r_b^0 = 0,$ dan $s_v^0 = 0.0066$. It was assumed that the infected individual weight are given by $A_1 =$

$A_2 = A_3 = 100$, while the control cost weight are provided $B_1 = 1$, $B_2 = 0.96$, $B_3 = 0.9$ and $B_4 = 0.15$. The later assumption is based on the fact that the costs of Hepatitis B vaccination program, Hepatitis B treatment, HIV/AIDS treatment are more expensive than that of public health education programs. The parameters used in the numerical simulation refer to the results of previous research on the spread of HIV/AIDS and Hepatitis B. All parameter values used in this simulation can be seen in Table 1 below:

**Table 1.** Parameter Description and Values

| Parameter | Description | Values | Source |
|---|---|---|---|
| $C$ | Individual birth rate | $N^0 * \mu$ | Assumption |
| $\mu$ | Natural mortality rate of individuals | 0.01 | (Teklu & Mekonnen, 2021) |
| $q$ | Waning immunity rates of vaccinated individual | 0.1 | (Bowong & Kurths, 2019) |
| $\delta_h$ | Mortality rate induced by HIV/AIDS | 0.333 | (Teklu & Mekonnen, 2021) |
| $\delta_b$ | Mortality rate induced by Hepatitis B | 0.01 | (Zada et al., 2021) |
| $\delta_{hb}$ | Mortality rate induced by HIV/AIDS-Hepatitis B | 0.001 | (Endashaw & Mekonnen, 2022) |
| $\gamma_{hb}$ | Treatment rate of HIV/AIDS-infected individuals in HIV/AIDS-Hepatitis B coinfection | 0.015 | (Endashaw & Mekonnen, 2022) |
| $\xi$ | Waning immunity rates of recovered individual | 0.03 | (Kamyad et al., 2014) |
| $\omega_b$ | Contact rate for Hepatitis B | 0.04 | Assumption |
| $\omega_h$ | Contact rate for HIV/AIDS | 0.03 | (Endashaw & Mekonnen, 2022) |
| $\gamma_h$ | Treatment rate for individuals infected HIV/AIDS | 0.6 | (Omale, 2020) |
| $\gamma_b$ | Recovery rate of Hepatitis B infected individuals | 0.1 | (Endashaw & Mekonnen, 2022) |
| $\tau$ | Rate of recovery induced by vaccination | 0.336 | Assumption |
| $\theta_h$ | Viral suppression rate of HIV/AIDS treatment | 0.013 | (Endashaw & Mekonnen, 2022) |
| $\theta_{hb}$ | Viral suppression rate of HIV/AIDS-Hepatitis B coinfection treatment | 0.012 | (Endashaw & Mekonnen, 2022) |
| $\varepsilon_1$ | Effectiveness of Hepatitis B vaccination program | 0.95 | Assumption |
| $\varepsilon_2$ | Effectiveness of Hepatitis B treatment | 0.90 | Assumption |
| $\varepsilon_3$ | Effectiveness of HIV/AIDS treatment | 0.85 | Assumption |
| $\varepsilon_4$ | Effectiveness of public health education program | 1 | Assumption |

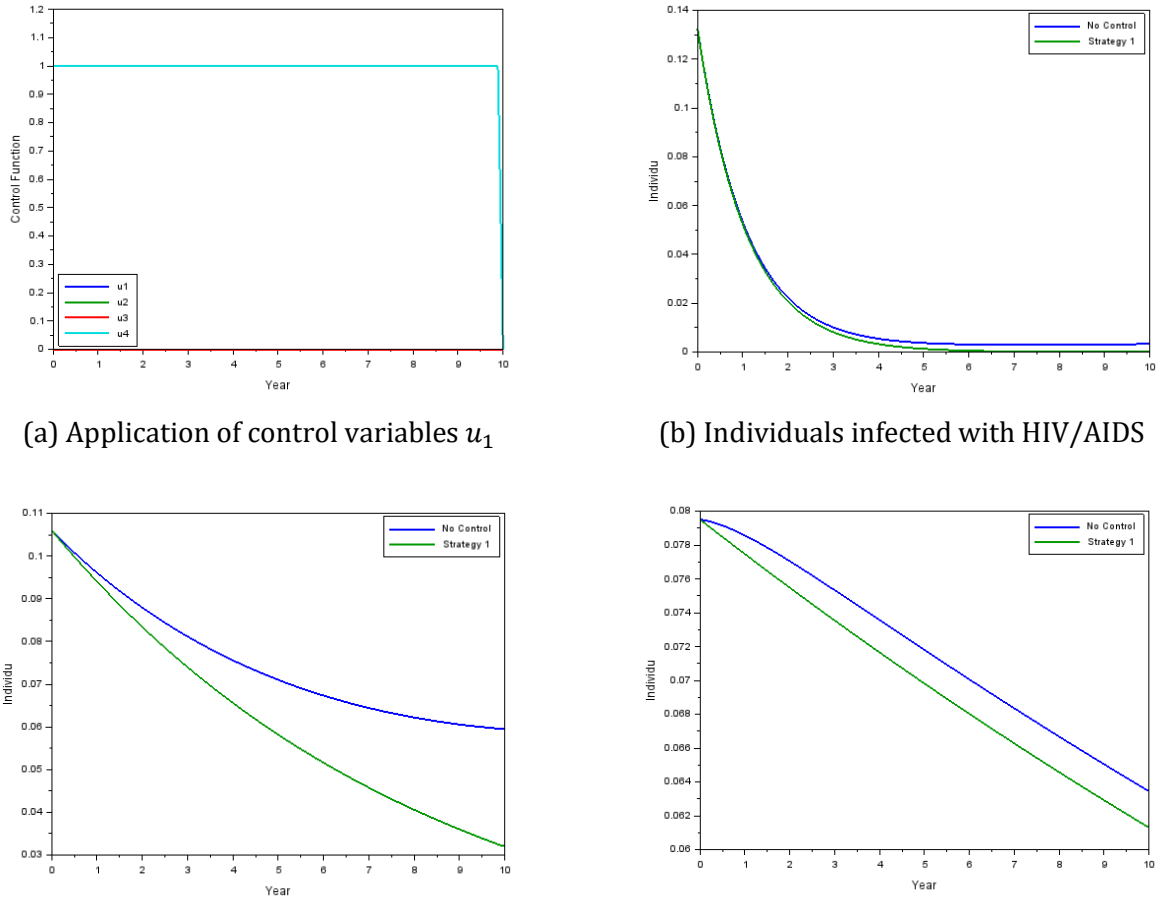
With four control variables $u_1$, $u_2$, $u_3$, and $u_4$, a variety of control strategies can be devised. Single control strategies involve the use of only one control variable at a time, resulting in a total of four possible strategies. Multiple control strategies, on the other hand, involve combinations of two or more control variables, yielding 11 possible strategies. In total, there are fifteen distinct control strategies that can be applied, encompassing both single and multiple control approaches to address the dynamics of HIV/AIDS-Hepatitis B coinfection. However, based on the preliminary studies conducted, this research focuses on five specific control strategies that demonstrate distinct impacts, as outlined in Table 2. Each strategy incorporates the public education program as a key component, highlighting its significance in the control of HIV/AIDS-Hepatitis B coinfection (Marsudi et al., 2019). The implementation of education control combined with Hepatitis B vaccination program, Hepatitis B treatment, and HIV/AIDS treatment, can enhance the effectiveness of control in reducing the number of infected individuals. Futhermore, the public health education program has the potential to raise public awareness, reduce stigma and discrimination towards the disease, encourage positive behavior change, and improve the long-term effectiveness of treatment and prevention (Liu et al., 2020; Nagao et al., 2022).

**Table 2.** HIV/AIDS-Hepatitis B control strategies

| Strategy | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|---|---|---|---|
| 1 | - | - | - | on |
| 2 | - | - | on | on |
| 3 | - | on | - | on |
| 4 | - | on | on | on |
| 5 | on | on | on | on |

The dynamics of the HIV/AIDS-Hepatitis B infected population ($i_h, i_b,$ and $i_{hb}$) considering the time of control implementation for ten years is illustrated in the following Figure 2.

**Strategy 1:**




(a) Application of control variables $u_1$

(b) Individuals infected with HIV/AIDS

(c) Individuals infected with Hepatitis B

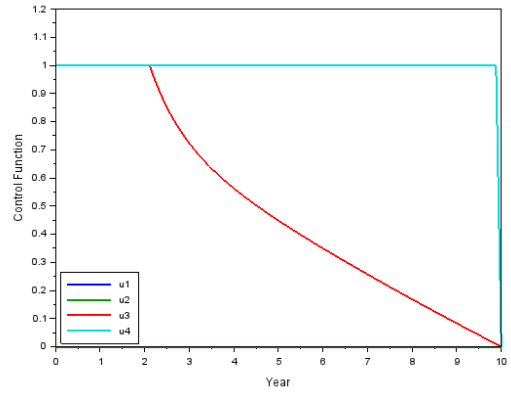
(d) HIV/AIDS-Hepatitis B infected individuals

**Figure 2.** Dynamics of control implementation $u_1$ on strategy 1

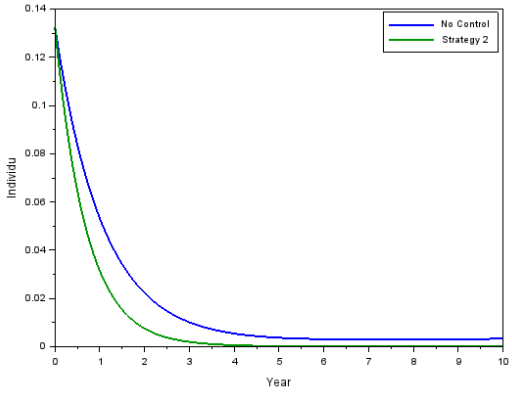
Figure 2 shows the first strategy, which involves the implementation of the public health education program control ($u_4$) for infected individuals. Figure 2(a) illustrates that the proportion of control $u_4$ reaches its maximum value within the time range [0; 9.8] years, after which it can be reduced. The implementation of this control is estimated to reduce the number of infected individuals by 55.5% compared to the pre-control strategy. Figure 2(b) shows that the number of individuals infected with HIV/AIDS gradually decreases by the sixth year after the implementation of the public health education program. Meanwhile, Figures 2(c) and 2(d) show that the number of individuals infected with Hepatitis B and those with the HIV/AIDS-

Hepatitis B coinfection decreases, nearly reaching zero by the tenth year after the control is applied. This indicates that the $u_4$ control, applied to infected individuals, effectively reduces the number of infected individuals and increases the number of individuals who are cured or in a state of viral suppression for both diseases, although this process occurs over a relatively long period of time.

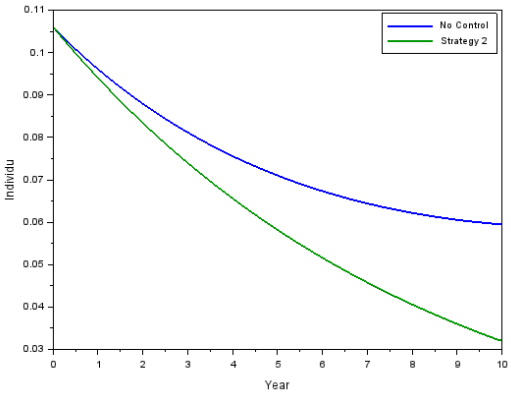
**Strategy 2:**



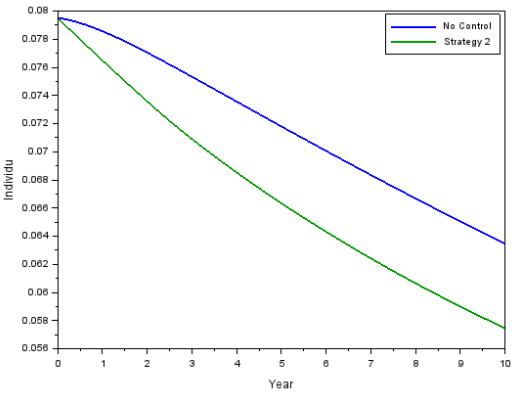

(a) Application of control variables $u_3, u_4$




(b) Individuals infected with HIV/AIDS




(c) Individuals infected with Hepatitis B



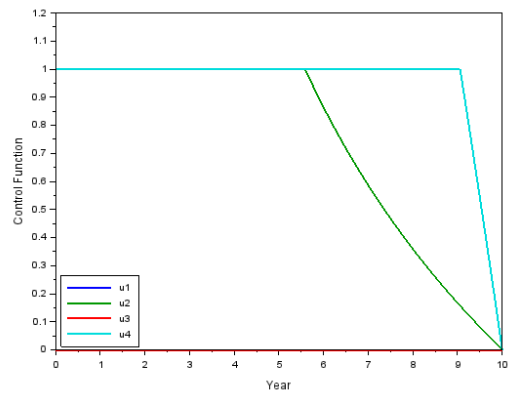

(d) HIV/AIDS-Hepatitis B infected individuals

**Figure 3.** Dinamika penerapan kontrol $u_3, u_4$ pada strategi 2

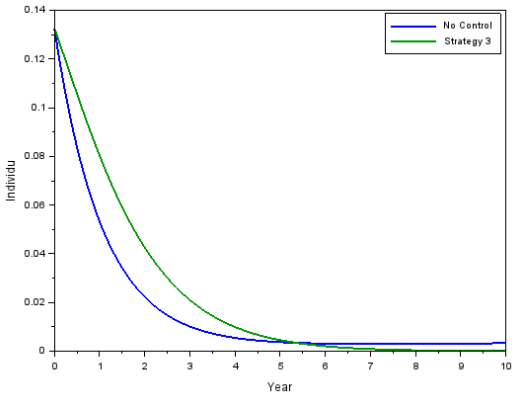
Figure 3 shows the second strategy, which involves the implementation of HIV/AIDS treatment control ($u_3$) and the public health education program control ($u_4$) for infected individuals. In Figure 3(a), it is illustrated that the proportion of control $u_3$ reaches its maximum value within the time range [0; 9.8] years, while control $u_4$ reaches its maximum value within the time range [0; 2.2] years, after which it can be reduced. The implementation of these controls is estimated to reduce the number of infected individuals by 58.2% compared to the pre-control strategy. Figure 3(b) shows that the number of individuals infected with HIV/AIDS gradually decreases by the fourth year after the implementation of HIV/AIDS treatment and the public health education program. Meanwhile, Figures 3(c) and 3(d) show that the number of individuals infected with Hepatitis B and those with the HIV/AIDS-Hepatitis B coinfection decreases, nearly reaching zero by the tenth year after the implementation of the controls. This indicates that the application of controls $u_3$ and $u_4$ for infected individuals

effectively reduces the number of infected individuals and increases the number of individuals who are cured or in a state of viral suppression for both diseases, although this process takes a relatively long time.

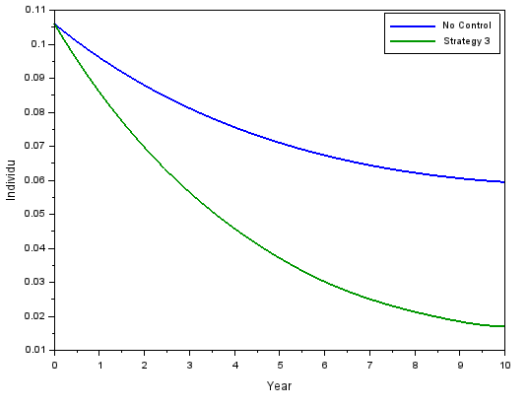
**Strategy 3:**



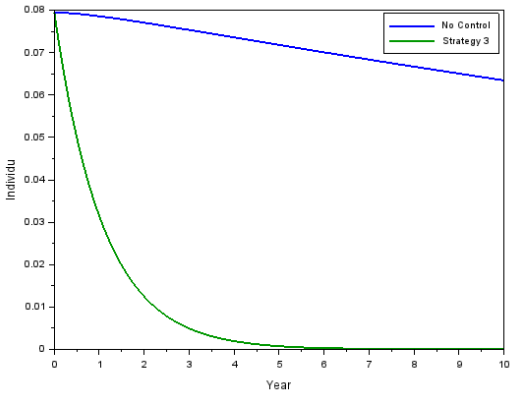

(a) Application of control variables $u_2, u_4$




(b) Individuals infected with HIV/AIDS




(c) Individuals infected with Hepatitis B



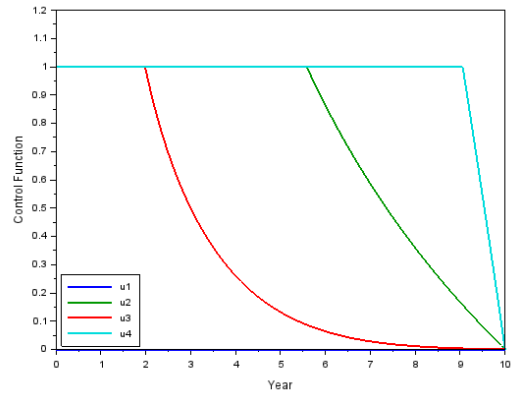

(d) HIV/AIDS-Hepatitis B infected individuals

**Figure 4.** Dynamics of control implementation $u_2, u_4$ on strategy 3

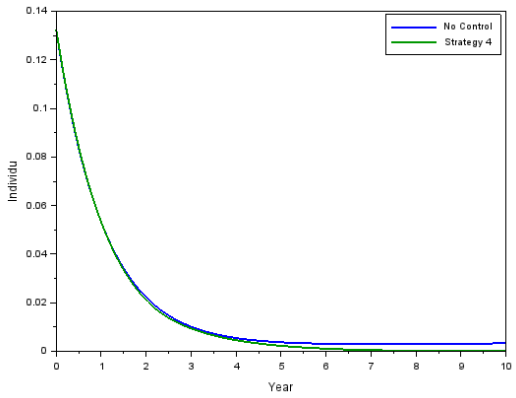
Figure 4 defines the third strategy, which combines the implementation of Hepatitis B treatment control ($u_2$) and the public health education program control ($u_4$) for infected individuals. In Figure 4(a), it can be seen that the proportion of control $u_2$ reaches its maximum value within the time range [0; 5.8] years, while control $u_4$ reaches its maximum value within the time range [0; 9.2] years, after which it can be reduced. The implementation of both controls is estimated to reduce the number of infected individuals by 76.9% compared to the pre-control strategy. Figure 4(b) shows that the number of individuals infected with HIV/AIDS gradually decreases by the seventh year after the implementation of Hepatitis B treatment and the public health education program. This is because Hepatitis B treatment cannot be applied to individuals infected with HIV/AIDS, so the public health education program plays a key role in controlling this infection. Figure 4(c) illustrates that the number of individuals infected with Hepatitis B decreases, nearly reaching zero by the tenth year after the implementation of both controls. Meanwhile, Figure 4(d) shows a gradual decrease in the number of individuals infected with the HIV/AIDS-Hepatitis B coinfection by the sixth year after the implementation

of both controls. These results indicate that the application of controls $u_2$ and $u_4$ for infected individuals effectively reduces the number of infected individuals and increases the number of individuals who are cured or in a state of viral suppression for both diseases, although this process requires a relatively long period of time.

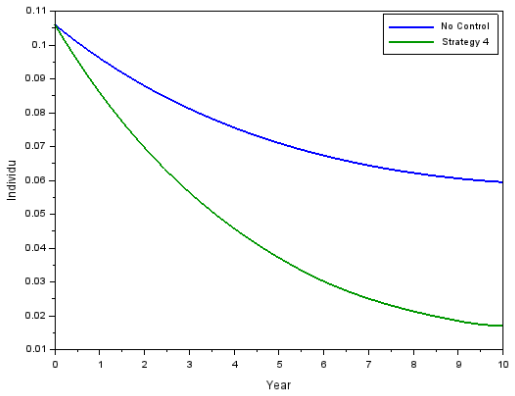
**Strategy 4:**



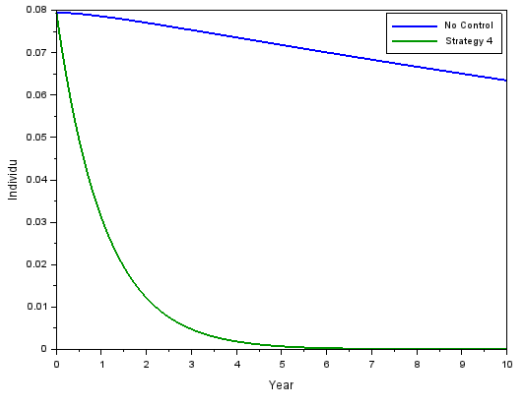

(a) Application of control variables $u_2, u_3, u_4$




(b) Individuals infected with HIV/AIDS




(c) Individuals infected with Hepatitis B



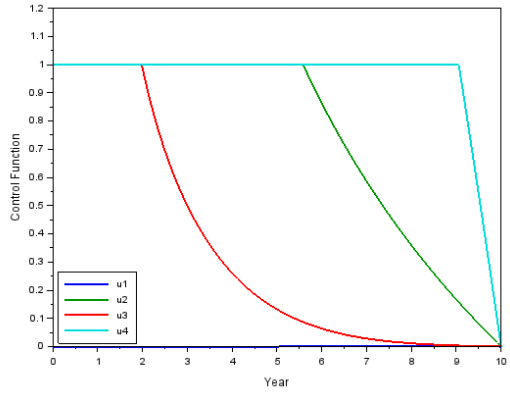

(d) HIV/AIDS-Hepatitis B infected individuals

**Figure 5.** Dynamics of control implementation $u_2, u_3, u_4$ on strategy 4

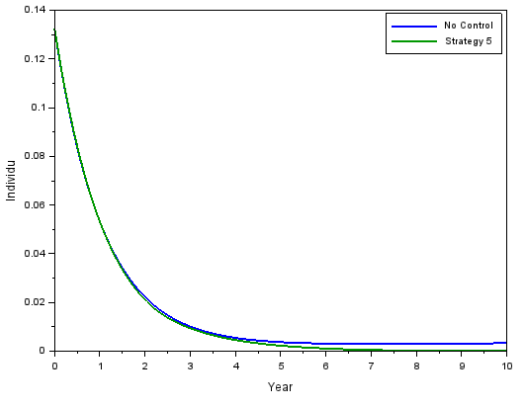
Figure 5 illustrates the fourth strategy, which combines the implementation of Hepatitis B treatment control ($u_2$), HIV/AIDS treatment control ($u_3$), and the public health education program control ($u_4$) for infected individuals. In Figure 5(a), it can be seen that the proportion of control $u_2$ reaches its maximum value within the time range [0; 5.8] years, control $u_3$ reaches its maximum value within the time range [0; 2] years, and control $u_4$ reaches its maximum value within the time range [0; 9.8] years, after which they can be reduced. The implementation of these three controls is estimated to reduce the number of infected individuals by 79% compared to the pre-control strategy. Figure 5(b) shows that the number of individuals infected with HIV/AIDS gradually decreases by the sixth year after the implementation of Hepatitis B treatment, HIV/AIDS treatment, and the public health education program. Figure 5(c) illustrates that the number of individuals infected with Hepatitis B decreases, nearly reaching zero by the tenth year after the implementation of all three controls. Meanwhile, Figure 5(d) shows a gradual decrease in the number of individuals infected with the HIV/AIDS-Hepatitis B coinfection by the sixth and a half years after the implementation of all three controls. This

indicates that the application of controls $u_2$, $u_3$, and $u_4$ for infected individuals can effectively reduce the number of infections and increase the number of individuals who are cured or in a state of viral suppression for both diseases, although this process takes a relatively long time.

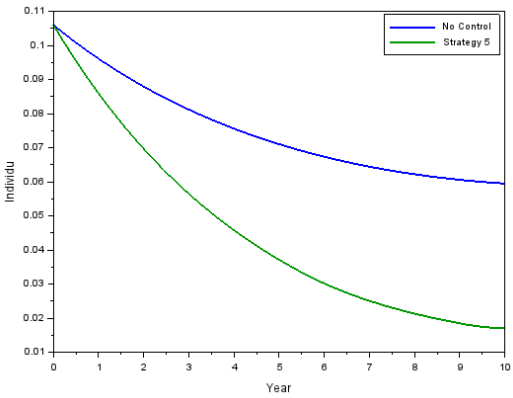
**Strategy 5:**



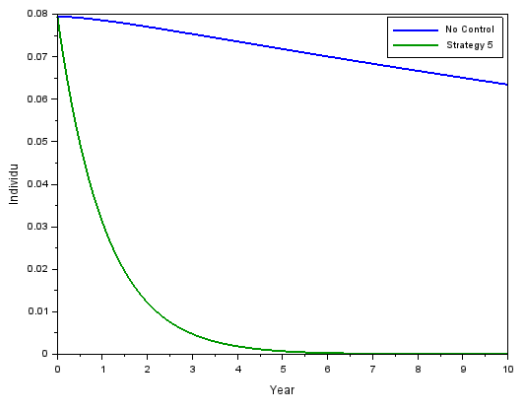

(a) Application of all controls




(b) Individuals infected with HIV/AIDS




(c) Individuals infected with Hepatitis B




(d) HIV/AIDS-Hepatitis B infected individuals

**Figure 6.** Dynamics of control implementation $u_1, u_2, u_3, u_4$ on strategy 5

Figure 6 illustrates the fifth strategy, which involves the implementation of Hepatitis B vaccination program control ($u_1$), Hepatitis B treatment control ($u_2$), HIV/AIDS treatment control ($u_3$), and the public health education program control ($u_4$) for infected individuals. In Figure 6(a), it can be seen that the proportion of control $u_1$ reaches its maximum value at 0.01 years, control $u_2$ reaches its maximum value within the time range [0; 5.8] years, control $u_3$ reaches its maximum value within the time range [0; 2.2] years, and control $u_4$ reaches its maximum value within the time range [0; 9.2] years, after which all controls can be reduced. The implementation of these four controls is estimated to reduce the number of infected individuals by 79.2% compared to the pre-control stratefy. Figure 6(b) shows that the number of individuals infected with HIV/AIDS gradually decreases by the seventh year after the implementation of all controls. Figure 6(c) illustrates that the number of individuals infected with Hepatitis B decreases, nearly reaching zero by the tenth year after the implementation of all controls. Meanwhile, Figure 6(d) shows a gradual decrease in the number of individuals infected with the HIV/AIDS-Hepatitis B coinfection by the sixth year after the implementation

of these controls. Overall, these results indicate that the application of all controls to infected individuals effectively reduces the number of infections and increases the number of individuals who are cured or in a phase of viral suppression for both diseases, although this process takes a relatively long time.

## 3. Cost Effectiveness Analysis

In this section, the Average Cost-Effectiveness Ratio (ACER) and Incremental Cost-Effectiveness (ICER) are used to analyze the cost-effectiveness of each strategy. By conducting a cost-effectiveness analysis, the most cost-effective strategy to control the spread of HIV/AIDS-Hepatitis B coinfection among various control combinations can be determined. This analysis is done by applying the following ACER and ICER formulas (Yuan & Li, 2022):

$$\text{ACER} = \frac{\text{Total cost incurred}}{\text{Benefit}}, \tag{32}$$

$$\text{ICER} = \frac{\text{Change in total cost for strategies } A \text{ and } B}{\text{Change in control benefits in strategies } A \text{ and } B}. \tag{33}$$

ACER evaluates the total cost per unit of health benefit achieved by a specific intervention, providing an overall assessment of its cost-effectiveness. In contrast, ICER compares the additional cost and additional health benefits of one intervention relative to another, focusing on the incremental value of adopting a new or alternative strategy. Together, these measures assist decision-makers in prioritizing interventions based on their economic and health impacts. The calculations of cost, benefit, ACER, and ICER are presented in Table 3.

**Table 3.** Cost, benefit, ACER, and ICER

| Strategy | Cost | Benefit | ACER | ICER |
|---|---|---|---|---|
| **No Control** | **0** | **0** | **-** | **-** |
| 1 | 0.745 | 1.840 | 0.405 | 0.405 |
| 2 | 2.420 | 1.920 | 1.260 | 20.938 |
| 3 | 3.960 | 2.539 | 1.560 | 2.488 |
| 4 | 5.200 | 2.615 | 1.989 | 16.316 |
| 5 | 5.180 | 2.615 | 1.981 | 1.981 |

The cost and benefit analysis of the five strategies, including the baseline scenario without any control, reveals distinct trade-offs between expenditures and health outcomes. The "No Control" strategy incurs no cost and provides no benefit, serving as the baseline for comparison. Strategy 1 shows a low cost of 0.745 with a benefit of 1.84, representing an improvement over the baseline. Strategy 2, at a cost of 2.42, achieves a slightly higher benefit of 1.92, indicating modest efficiency gains. Strategy 3 incurs a cost of 3.96 and delivers a significant benefit increase to 2.539. Strategies 4 and 5 both achieve the highest benefit of 2.615, with costs of 5.2 and 5.18, respectively, reflecting near-identical cost-effectiveness. In fact, we can remove Strategy 4 from the selection process as it contributes same amount of benefits but higher cost than Strategy 5.

The ACER represents the cost per unit of benefit for each strategy. Strategies with lower ACER values are generally more cost-effective. Strategy 1 has the lowest ACER (0.405), indicating that it offers the most benefit relative to its cost. The ICER reflects the additional cost required for each additional unit of benefit when moving from one strategy to the next. For Strategy 2, the ICER is exceptionally high (20.938), suggesting a significant increase in cost for only a marginal gain in benefit compared to Strategy 1. Strategy 3 shows a much lower ICER (2.488), indicating a more reasonable trade-off between cost and benefit compared to Strategy 2. The ICER for Strategy 4 rises sharply to 16.316 and it is dominated by Strategy 5. Strategies with the lowest ACER and ICER values are preferable for cost-effectiveness. Strategy 1 emerges as the most cost-effective option overall, offering a reasonable balance between cost and benefit. Strategies with high ICER values, such as Strategy 2 and Strategy 4, should be carefully evaluated as they provide diminishing returns for the additional cost.

## D. CONCLUSION AND SUGGESTION

This research focuses on mathematical modeling, namely the formulation of optimal control models for the coinfection of HIV/AIDS and Hepatitis B, utilizing a compartmental approach that integrates various control strategies. The model incorporates four key control variables: vaccination, Hepatitis B treatment, HIV/AIDS treatment, and public health education. Based on preliminary analysis, five distinct control strategies were considered, each involving different combinations of these variables, with the goal of minimizing the prevalence of the coinfection while optimizing healthcare costs.

Numerical simulations for optimal control problems are carried out by implementing five strategies. The results of the analysis show that the implementation of a public health education program alone, without the support of other controls, does not provide optimal results. Therefore, support from additional controls, such as the Hepatitis B vaccination program, Hepatitis B treatment, and HIV/AIDS treatment, is needed to effectively reduce the number of infected individuals. Of the five strategies tested, the fifth strategy, which covers all forms of control, has the most significant impact on controlling the spread of the disease. Furthermore, to determine the most cost-effective strategy, an analysis was carried out using ACER and ICER. Based on the calculation results, the first strategy proved to be the most economical option with an ICER value of 0.405. However, if there is a larger budget, the fifth strategy is the most effective option in reducing the number of infected populations, with an ICER value of 1.981.

Further research is recommended to use real, up-to-date data to implement the modified model. Additionally, consideration could be given to developing a new single model by modifying the HIV/AIDS model to separate HIV and AIDS infections and differentiating Hepatitis B into acute and chronic types. The addition of new controls, such as the use of contraceptive methods (e.g., condoms) during sexual intercourse, could also enhance the model's effectiveness and produce better results. Then, it could add exploring the various socio-economic factors and behavioral patterns influence the success of these strategies, ensuring that the recommendations are both effective and feasible within real-world settings.

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