JTAM (Jurnal Teori dan Aplikasi Matematika)
http://journal.ummat.ac.id/index.php/jtam
p-ISSN 2597-7512 | e-ISSN 2614-1175
Vol. 9, No. 4, October 2025, pp. 1218-1231

# Complexity of Graphs with Wheel Graph and Fan Graph as their Blocks

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**Article History:**
Received : 01-07-2025
Revised : 11-08-2025
Accepted : 13-08-2025
Online : 01-10-2025

**Keywords:**
Spanning Tree;
Laplacian Matrix;
Matrix Tree Theorem;
Block of Graph.

**ABSTRACT**

The complexity of a graphs remains an active area of research within graph theory. Let $G$ be an undirected connected graph. Graph $G$ is said as a non-separable graph if it does not have cut-vertex. A maximal non-separable subgraph of graph $G$ is called a block of $G$. Every connected graph $G$ has at least one spanning tree. The complexity of the graph $G$ is the number of spanning trees in $G$, denoted by $\tau(G)$, and can be determined using the matrix tree theorem. The matrix used in this theorem is the Laplacian matrix, which related to the number of spanning trees in a graph. This research aims to formulate the complexity of graphs with wheel graph $W_n$ and fan graph $F_n$ as their blocks. The focus of this research is on graphs whose blocks are wheel graph $W_n$ and fan graph $F_n$, specifically the $m$-wheel graph $W_{n,m}$, the dragonfly graph $Dg_n$, and the generalized dragonfly graph $Dg_n^{(3,3)}$. This study utilizes a qualitative and theoretical approach grounded in mathematical analysis. It involves a thorough review of relevant books and journal articles. The theoretical contributions of this research include deeper understanding about complexity of graph and elucidating the relationship between the complexity of graphs and their blocks. In this research, the complexity of graphs is determined using the matrix tree theorem, which involves calculating the cofactor of the Laplacian matrix. Based on $\tau(W_n)$ and $\tau(F_n)$, the results obtained in this study are $\tau(W_{n,m}) = (\tau(W_n))^m$, $\tau(Dg_n) = (\tau(F_{n+2}))^2$ and $\tau(Dg_n^{(3,3)}) = (\tau(F_{n+2}))^3$. The results of this study show that the complexity of a graph is related to the complexity of its blocks.

Crossref
https://doi.org/10.31764/jtam.v9i4.32776


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## A. INTRODUCTION

Tree is one of the basic structures in graph theory that represent relationships between objects in a structured way. As a connected and undirected graph without cycles, a tree consists of vertices and the edges that connect them (Koutrouli et al., 2020). Tree is widely used in various fields, such as computing, networking, and data, due to their simple yet very flexible nature to represent structures or relationships. One of the important concepts in tree is spanning trees. A spanning tree is a tree subgraph of a graph that has all vertices in the graph without forming a cycle (Daoud & Saleh, 2020). The number of spanning trees in graph $G$ or the complexity of graph G is denoted by $\tau(G)$ (Daoud & Mohamed, 2017). Complexity of graph is one of the topics in graph theory that has a wide application (Deen, 2023).

This is because complexity of graph can be applied to various fields such as investigating the possibility of maser particle transitions using energy analysis, calculating certain chemical isomers, and calculating the number of Eulerian trajectories in a graph (Liu & Daoud, 2019; Mohamed & Amin, 2024). In extension, complexity has a lot to do with networks. The perfect and higher the quality of the network, the more spanning trees it contains, and this increases the potential connections between any two vertices, and this enhances good reliability and rigidity (Deen & Aboamer, 2021). Owing to its extensive applications, it is essential to determine an effective method for calculating its complexity.

The study of calculating the complexity of graph began in 1847 when Gustav Kirchhoff introduced a method to determine the complexity of graph by analyzing the relationship between the number of spanning trees in the graph $G$ and the cofactor of the Laplacian matrix of graph $G$ (Daoud, 2017). This method is known as the matrix tree theorem. The matrix used in this theorem is the Laplacian matrix $L(G)$, which is one of the representation matrices of graph $G$ (Holzer, 2022). In addition, various approaches have been developed to calculate the complexity of graph.

Some previous research have found the general form of the complexity of several types of graphs by using many various techniques, such as the matrix tree theorem, the Laplacian spectrum or eigenvalues of the corresponding adjacency matrix, the deletion-contraction method, and the graph complement approach (Daoud, 2015; Liu & Daoud, 2018; Daoud, 2018, 2019; Deen et al., 2023; Hanssen et al., 2024; Fran et al., 2024, 2025). There are also other research that apply the electrically equivalent transformations for the enumeration of spanning trees (Shang, 2016; Sun et al., 2016; Liu & Daoud, 2019; Daoud & Saleh, 2020). Further research by Wei et al. (2023) discusses the complexity of wheel graphs with double vertices and edges. The results obtained are the complexity of wheel graphs with double vertices and edges, and also assumed that if $DW$ is a double-wheel graph based on the wheel graphs $W_1$ and $W_2$, then the complexity of graph $DW$ is $\tau(DW) = \tau(W_1)\tau(W_2)$ (Wei et al., 2023). The focus of this research is on graphs whose blocks are wheel graph $W_n$ and fan graph $F_n$. A block of $G$ is a maximal non-separable subgraph of graph $G$. A graph is a non-separable graph if it has no cut-vertex (Li et al., 2018). Consequently, a graph $G$ is regarded as a block if it is non-separable, meaning that $G$ itself is a block.

This research aims to determine the complexity of graphs with wheel graph $W_n$ and fan graph $F_n$ as their blocks, specifically the $m$-wheel graph $W_{n,m}$, the dragonfly graph $Dg_n$, and the generalized dragonfly graph $Dg_n^{(3,3)}$ by using the matrix tree theorem. An $m$-wheel graph $W_{n,m}$ is a graph with wheel graph as its blocks. The dragonfly graph $Dg_n$ and the generalized dragonfly graph $Dg_n^{(3,3)}$ are the graphs with fan graph as its blocks. For graphs that contain blocks, it is effective for calculating its complexity by using the matrix tree theorem. The matrix tree theorem provides a direct algebraic formula to calculate the exact number of spanning trees by evaluating the cofactor of the Laplacian matrix.

## B. METHODS

This research uses a literature review, which involves reading literature related to complexity of graphs, particularly the matrix tree theorem. The graphs examined in this study include the wheel graph $W_n$, the $m$-wheel graph $W_{n,m}$, the fan graph $F_n$, the dragonfly graph $Dg_n$, and the generalized dragonfly graph $Dg_n^{(3,3)}$. The research stages to determine the complexity of graph using the matrix tree theorem are as follows.

1. Determine the adjacency matrix $A(G)$ and the degree matrix $D(G)$ of the graph $G$.
2. Find the Laplacian matrix of graph $G$ using $L(G) = D(G) - A(G)$.

3. Compute the number of spanning trees of graph $G$ using $\tau(G) = C_{ij}\big(L(G)\big) = (-1)^{i+j}\det\big(L(G)(i|j)\big)$, where $C_{ij}\big(L(G)\big)$ and $L(G)(i|j)$ are the cofactor of $L(G)$ and the submatrix formed by deleting row $i$ and column $j$ of $L(G)$, respectively.
4. Construct the complexity $\tau(G)$ formula of graph $G$.
5. Prove the complexity $\tau(G)$ formula of graph $G$.

This is important to note that this research does not involve experimental work or empirical data analysis. Instead, all procedures are based on mathematical deduction, utilizing concepts from graph theory, matrix theory, and determinant calculation.

## C. RESULT AND DISCUSSION

Before determine the complexity of graph, it is necessary to know some theoretical basic such as several definitions of graphs, lemmas, and theorem that define a method to determine the complexity of graph, as follows.

**Definition 1** (Liu et al., 2019) Let $\boldsymbol{n}, \boldsymbol{m}$ be positive integer with $\boldsymbol{n} \geq \mathbf{3}$ and $\boldsymbol{m} \geq \mathbf{1}$. An $\boldsymbol{m}$-wheel graph denoted by $\boldsymbol{W_{n,m}} = (\boldsymbol{V}(\boldsymbol{W_{n,m}}), \boldsymbol{E}(\boldsymbol{W_{n,m}}))$ is a graph that has $\boldsymbol{nm} + \mathbf{1}$ vertices with the vertex and the edge sets as follows.

$$V\big(W_{n,m}\big) = \left\{w_0, w_i^j \,\middle|\, i \in \{1,2,\dots,n\}, j \in \{1,2,\dots,m\}\right\}$$

$$E\big(W_{n,m}\big) = \left\{\big(w_0, w_i^j\big) \,\middle|\, i \in \{1,2,\dots,n\}, j \in \{1,2,\dots,m\}\right\} \cup \left\{\big(w_1^j, w_n^j\big) \,\middle|\, j \in \{1,2,\dots,m\}\right\} \cup \left\{\big(w_i^j, w_{i+1}^j\big) \,\middle|\, i \in \{1,2,\dots,n-1\}, j \in \{1,2,\dots,m\}\right\}.$$

If $m = 1$, then $W_{n,1}$ is called as wheel graph $W_n$ and vertex $w_i^j$ can be written as $w_i$. Figure 1 below illustrates the $m$-wheel graph $W_{n,m}$ and wheel graph $W_n$.

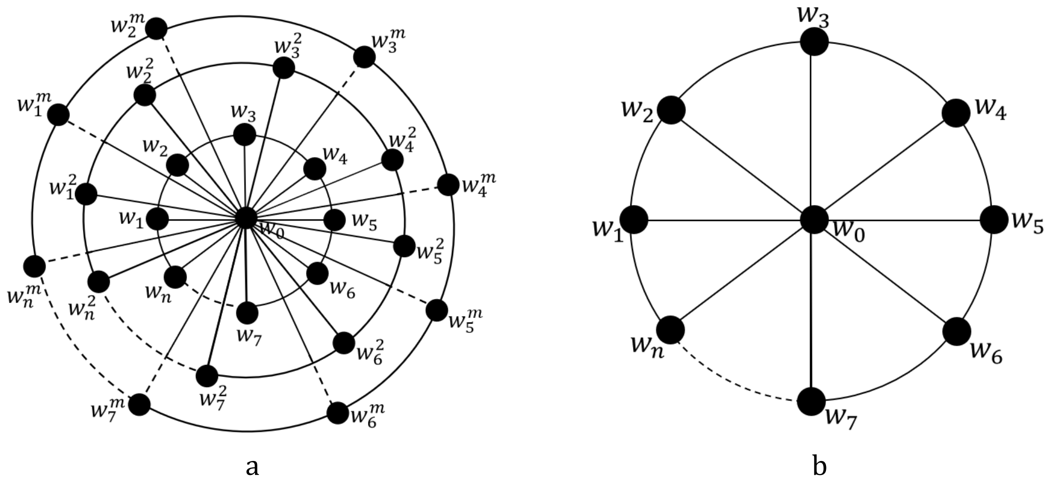


**Figure 1**. (a) $\boldsymbol{m}$-Wheel Graph $\boldsymbol{W_{n,m}}$ and (b) Wheel Graph $\boldsymbol{W_n}$

**Definition 2** Bača et al. (2021) Let $\boldsymbol{n}$ be a positive integer with $\boldsymbol{n} \geq \mathbf{2}$. A fan graph denoted by $\boldsymbol{F_n} = (\boldsymbol{V}(\boldsymbol{F_n}), \boldsymbol{E}(\boldsymbol{F_n}))$ is a graph that has $\boldsymbol{n} + \mathbf{1}$ vertices with the vertex and the edge sets as follows.

$$V(F_n) = \{v_i | i \in \{0,1,2, \dots, n\}\}$$
$$E(F_n) = \{(v_0, v_i) | i \in \{1,2, \dots, n\}\} \cup \{(v_i, v_{i+1}) | i \in \{1,2, \dots, n-1\}\}$$

In Figure 2, we illustrate the fan graph $F_n$.

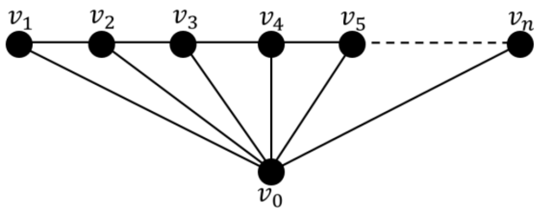


**Figure 2.** Fan Graph $\boldsymbol{F_n}$

**Definition 3** Budi et al. (2021) Let $n$ be a positive integer with $n \geq 1$. A dragonfly graph denoted by $Dg_n = (V(Dg_n), E(Dg_n))$ is a graph that has $2n + 7$ vertices with the vertex and the edge sets as follows.

$$V(Dg_n) = \{u_i, v_j, w_k | i, j \in \{1,2, \dots, n+2\}, k \in \{0,1,2\}\}$$
$$E(Dg_n) = \{(w_0, w_i) | i \in \{1,2\}\} \cup \{(u_i, u_{i+1}) | i \in \{1,2, \dots, n+1\}\} \cup \{(u_i, w_0) | i \in \{1,2, \dots, n+2\}\}$$
$$\cup \{(v_i, v_{i+1}) | i \in \{1,2, \dots, n+1\}\} \cup \{(v_i, w_0) | i \in \{1,2, \dots, n+2\}\}.$$

**Definition 4** (Inayah et al., 2022) Let $n$ be a positive integer with $n \geq 1$. A generalized dragonfly graph denoted by $Dg_n^{(3,3)} = (V(Dg_n^{(3,3)}), E(Dg_n^{(3,3)}))$ is a graph that has $3n + 10$ vertices the vertex and the edge sets as follows.

$$V\left(Dg_n^{(3,3)}\right) = \{v_i^1, v_i^2, v_i^3, w_k | i \in \{1,2,3, \dots, n+2\}, k \in \{0,1,2,3\}\}$$
$$E\left(Dg_n^{(3,3)}\right) = \{(w_0, w_k) | k \in \{1,2,3\}\} \cup \{(v_i^j, v_{i+1}^j) |, i \in \{1,2, \dots, n+1\}, j \in (1,2,3)\} \cup$$
$$\{(v_i^j, w_0) | i \in \{1,2, \dots, n+2\}, j \in (1,2,3)\}.$$

As an illustration, the dragonfly graph $Dg_n$ and the generalized dragonfly graph $Dg_n^{(3,3)}$ are shown in Figure 3.

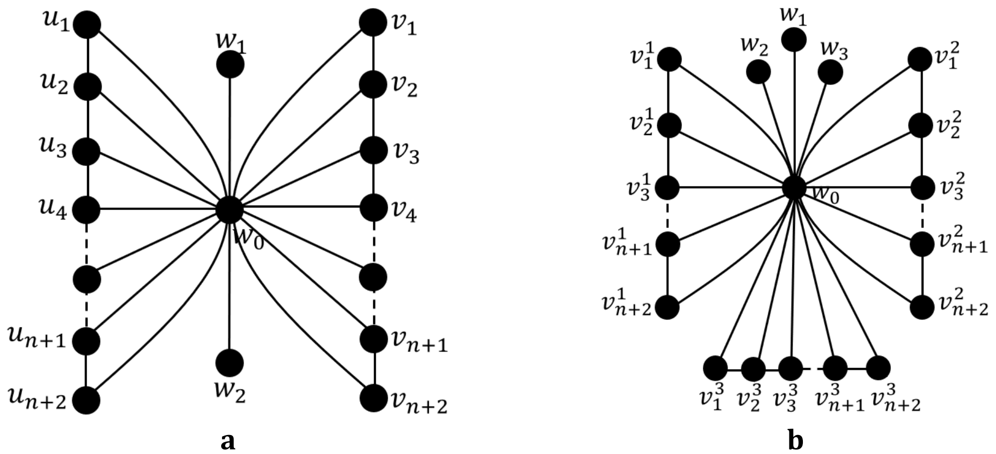


**Figure 3**. (a) Dragonfly Graph $\boldsymbol{Dg_n}$ and (b) Generalized Dragonfly Graph $\boldsymbol{Dg_n^{(3,3)}}$

From Figure 1, it can be figured out that $W_n$ is the blocks of graph $W_{n,m}$. From Figure 2 and Figure 3 It can be figured out that $F_n$ is the blocks of graph $Dg_n$ and $Dg_n^{(3,3)}$. The complexity of some defined graphs can be determined using the matrix tree theorem in Theorem 1. Moreover, we define Lemma 1 and Lemma 2 to help in computing the cofactor or determinant of matrix.

**Theorem 1 (The Matrix Tree Theorem)** Holzer (2022) Let $G$ be an undirected connected graph with $n$ vertices, degree matrix $D(G)$, and adjacency matrix $A(G)$. The number of spanning trees of graph $G$, denoted by $\tau(G)$, is equal to the value of any cofactor of the Laplacian matrix $L(G)$ of graph $G$ where $L(G) = D(G) - A(G)$.

**Lemma 1** Deen & Aboamer (2021) Let matrix $A_n(x)$ be a matrix of size $n \times n$, with

$$A_n(x) = \begin{bmatrix} x & -1 & 0 & \cdots & 0 & -1 \\ -1 & x & -1 & \ddots & \ddots & 0 \\ 0 & -1 & x & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 & x & -1 \\ -1 & 0 & \cdots & 0 & -1 & x \end{bmatrix}$$

for all $x \in \mathbb{R}, x \geq 3$ and $n \in \mathbb{N}, n \geq 3$, then the determinant is

$$\det(A_n(x)) = \left[\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 - 1}\right)^n + \left(\frac{x}{2} - \sqrt{\left(\frac{x}{2}\right)^2 - 1}\right)^n\right] - 2$$

**Lemma 2** Deen & Aboamer (2021) Let matrix $B_n(x)$ be a matrix of size $n \times n$, with

$$B_n(x) = \begin{bmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ -1 & x+1 & -1 & \ddots & \ddots & 0 \\ 0 & -1 & x+1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 & x+1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & x \end{bmatrix}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}, n \geq 2$, then the determinant is

$$\det\big(B_n(x)\big) = \frac{x-1}{2\sqrt{\left(\frac{x+1}{2}\right)^2 - 1}}\left[\left(\frac{x+1}{2} + \sqrt{\left(\frac{x+1}{2}\right)^2 - 1}\right)^n - \left(\frac{x+1}{2} - \sqrt{\left(\frac{x+1}{2}\right)^2 - 1}\right)^n\right]$$

Proof of the matrix tree theorem at Theorem 1 can be found in many articles and one of them is in Fikadila's article at (Fikadila et al., 2024). In addition, the proofs of Lemma 1 and Lemma **2** can be found in Nurliantika's research at (Nurliantika et al., 2025).

**Lemma 3** Deen & Aboamer (2021) Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ be matrices of size $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively. Assume $\mathcal{P}$ and $\mathcal{S}$ are nonsingular, then

$$\det\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{R} & \mathcal{S} \end{bmatrix} = \det(\mathcal{P})\det(\mathcal{S} - \mathcal{R}\mathcal{P}^{-1}\mathcal{Q}) = \det(\mathcal{P} - \mathcal{Q}\mathcal{S}^{-1}\mathcal{R})\det(\mathcal{S})$$

After understanding the theoretical basic, the following theorems provides the result of this research.

**Theorem 2** (Daoud, 2017) Given a wheel graph $W_n$, then the complexity of the wheel graph $W_n$ with $n \geq 3$ is

$$\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$$

**Proof.** The adjacency matrix and degree matrix of size $(n+1) \times (n+1)$ for the wheel graph $W_n$ are as follows:

$$A(W_n) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \vdots & 1 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}; D(W_n) = \begin{bmatrix} n & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 3 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 3 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 3 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3 \end{bmatrix}$$

Now, the Laplacian matrix of size $(n+1) \times (n+1)$ can be obtained as follows:

$$L(W_n) = D(W_n) - A(W_n)$$
$$= \begin{bmatrix} n & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & -1 \\ \vdots & -1 & 3 & -1 & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & -1 & 3 & -1 \\ -1 & -1 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}$$

By applying Theorem 1 yields:

$$\tau(W_n) = C_{11}\big(L(W_n)\big) = (-1)^{1+1} \det\big(L(W_n)(1|1)\big)$$
$$= \det \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & \ddots & \ddots & 0 \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 & 3 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}_{n\times n}$$
$$= \left[\left(\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1}\right)^n + \left(\frac{3}{2} - \sqrt{\left(\frac{3}{2}\right)^2 - 1}\right)^n\right] - 2$$
$$= \left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right)^n + \left(\frac{3}{2} - \sqrt{\frac{5}{4}}\right)^n - 2$$
$$= \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2. \blacksquare$$

**Theorem 3** Given an $m$-wheel graph $W_{n,m}$, then the complexity of the $m$-wheel graph $W_{n,m}$ with $n \geq 3$ and $m \geq 1$ is

$$\tau\big(W_{n,m}\big) = \big(\tau(W_n)\big)^m = \left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2\right]^m$$

**Proof.** With the same kind of reasoning from Theorem 2, we obtain the Laplacian matrix of size $(mn+1) \times (mn+1)$ for the $m$-wheel graph $W_{n,m}$ as follows.

$$L\big(W_{n,m}\big) = \begin{bmatrix} mn & R^T & R^T & \cdots & \cdots & R^T \\ R & S & O & \cdots & \cdots & O \\ R & O & S & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & S & O \\ R & O & \cdots & \cdots & O & S \end{bmatrix}$$

where,
$O$ is the $n \times n$ zero matrix;
$R$ is the column matrix of size $n \times 1$, whose all entries are $-1$; and

$$S = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & \ddots & \ddots & 0 \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 & 3 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}_{n\times n} .$$

By applying Theorem 1, it is obtained:

$$\begin{aligned} \tau(W_{n,m}) &= C_{11}\left(L(W_{n,m})\right) = (-1)^{1+1}\det\left(L(W_{n,m})(1|1)\right) \\ &= \det\begin{bmatrix} S & O & \cdots & \cdots & O \\ O & S & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S & O \\ O & \cdots & \cdots & O & S \end{bmatrix}_{m\times m} \\ &= (\det(S))^m \\ &= \left(\det\begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & \ddots & \ddots & 0 \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 & 3 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}_{n\times n}\right)^m \\ &= \left(\tau(W_n)\right)^m \end{aligned}$$

From Theorem 2, we conclude that:

$$\tau(W_{n,m}) = \left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2\right]^m . \blacksquare$$

**Theorem 4** (Daoud, 2015) Given a fan graph $F_n$, then the complexity of the fan graph $F_n$ with $n \geq 2$ is

$$\tau(F_n) = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$$

**Proof.** The adjacency matrix and degree matrix of size $(n+1)\times(n+1)$ for the fan graph $F_n$ are as follows.

$$A(F_n)=\begin{bmatrix} 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \ddots & \ddots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}; D(F_n)=\begin{bmatrix} n & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 3 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 3 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 \end{bmatrix}$$

Now, the Laplacian matrix of size $(n+1)\times(n+1)$ can be obtained as follows.

$$\begin{aligned} L(F_n) &= D(F_n)-A(F_n) \\ &= \begin{bmatrix} n & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & -1 & 3 & -1 & \ddots & \ddots & \vdots \\ \vdots & 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 & 3 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \end{aligned}$$

By applying Theorem 1 yields:

$$\begin{aligned} \tau(F_n) &= C_{11}\big(L(F_n)\big)=(-1)^{1+1}\det\big(L(F_n)(1|1)\big) \\ &= \det\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{n\times n} \\ &= \frac{2-1}{2\sqrt{\left(\frac{2+1}{2}\right)^2-1}}\left[\left(\frac{2+1}{2}+\sqrt{\left(\frac{2+1}{2}\right)^2-1}\right)^n-\left(\frac{2+1}{2}-\sqrt{\left(\frac{2+1}{2}\right)^2-1}\right)^n\right] \\ &= \frac{1}{2\sqrt{\frac{5}{4}}}\left[\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)^n-\left(\frac{3}{2}-\sqrt{\frac{5}{4}}\right)^n\right] \\ &= \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n-\left(\frac{3-\sqrt{5}}{2}\right)^n\right]. \ \blacksquare \end{aligned}$$

**Theorem 5** Given a dragonfly graph $Dg_n$, then the complexity of the dragonfly graph $Dg_n$ with $n\geq 1$ is

$$\tau(Dg_n)=\big(\tau(F_{n+2})\big)^2=\frac{1}{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+2}-\left(\frac{3-\sqrt{5}}{2}\right)^{n+2}\right]^2$$

**Proof.** Observe that the Laplacian matrix of size $(2n+7)\times(2n+7)$ for the dragonfly graph $Dg_n$ is as follows.

$$L(Dg_n)=\begin{bmatrix} 2n+6 & R_2^T & R^T & R^T \\ R_2 & I_2 & O_2^T & O_2^T \\ R & O_2 & S & O \\ R & O_2 & O & S \end{bmatrix}$$

where,
$I_2$ is the $2\times 2$ identity matrix;
$O_2$ is the $(n+2)\times 2$ zero matrix;
$R_2$ is the column matrix of size $2\times 1$, whose all entries are $-1$;
$O$ is the $(n+2)\times(n+2)$ zero matrix;
$R$ is the column matrix of size $(n+2)\times 1$, whose all entries are $-1$; and

$$S=\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{(n+2)\times(n+2)}.$$

From Theorem 1, it follows that:

$$\begin{aligned}\tau(Dg_n) &= C_{11}\big(L(Dg_n)\big)=(-1)^{1+1}\det\big(L(Dg_n)(1|1)\big)\\ &=\det\begin{bmatrix} I_2 & O_2^T & O_2^T \\ O_2 & S & O \\ O_2 & O & S\end{bmatrix}\\ &=\det(I_2)\det\begin{bmatrix} S & O \\ O & S\end{bmatrix}=\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\det(S)\det(S)\\ &=(\det(S))^2\\ &=\left(\det\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{(n+2)\times(n+2)}\right)^2\\ &=\big(\tau(F_{n+2})\big)^2\end{aligned}$$

By applying Theorem 4, we conclude that:

$$\tau(Dg_n) = \left(\frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+2}\right]\right)^2$$
$$= \frac{1}{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+2}\right]^2. \blacksquare$$

**Theorem 6** Given a generalized dragonfly graph $Dg_n^{(3,3)}$, then the complexity of the generalized dragonfly graph $Dg_n^{(3,3)}$ with $n \geq 1$ is

$$\tau\left(Dg_n^{(3,3)}\right) = \left(\tau(F_{n+2})\right)^3 = \frac{1}{25}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+2}\right]^3$$

**Proof.** The generalized dragonfly graph $Dg_n^{(3,3)}$ has $3n + 10$ vertices. Thus, we obtain the Laplacian matrix of size $(3n + 10) \times (3n + 10)$ as follows.

$$L\left(Dg_n^{(3,3)}\right) = \begin{bmatrix} 3n+9 & R_3^T & R^T & R^T & R^T \\ R_3 & I_3 & O_3^T & O_3^T & O_3^T \\ R & O_3 & S & O & O \\ R & O_3 & O & S & O \\ R & O_3 & O & O & S \end{bmatrix}$$

where,
$I_3$ is the $3 \times 3$ identity matrix;
$O_3$ is the $(n + 2) \times 3$ zero matrix;
$R_3$ is the column matrix of size $3 \times 1$, whose all entries are $-1$;
$O$ is the $(n + 2) \times (n + 2)$ zero matrix;
$R$ is the column matrix of size $(n + 2) \times 1$, whose all entries are $-1$; and

$$S = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{(n+2)\times(n+2)}.$$

By applying Theorem 1 yields:

$$\tau\left(Dg_n^{(3,3)}\right) = C_{11}\left(L\left(Dg_n^{(3,3)}\right)\right) = (-1)^{1+1}\det\left(L\left(Dg_n^{(3,3)}\right)(1|1)\right)$$

$$= \det\left[\begin{array}{c|ccc} I_3 & O_3^T & O_3^T & O_3^T \\ \hline O_3 & S & O & O \\ O_3 & O & S & O \\ O_3 & O & O & S \end{array}\right]$$

$$= \det(I_3)\det\begin{bmatrix} S & O & O \\ O & S & O \\ O & O & S \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\det(S)\det(S)\det(S)$$

$$= (\det(S))^3$$

$$= \left(\det\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{(n+2)\times(n+2)}\right)^3$$

$$= \left(\tau(F_{n+2})\right)^3$$

Using the result in Theorem 4, it follows that:

$$\tau\left(Dg_n^{(3,3)}\right) = \left(\frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+2}\right]\right)^3$$

$$= \frac{1}{25}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+2}\right]^3. \blacksquare$$

Based on the results of this research, it is obtained $\tau(W_n)$, $\tau(W_{n,m})$, $\tau(F_n)$, $\tau(Dg_n)$, and $\tau\left(Dg_n^{(3,3)}\right)$. The complexity of wheel graph $W_n$ and fan graph $F_n$ studied in this research is supported by previous studies that have determined $\tau(W_n)$ and $\tau(F_n)$ using the recurrence relation and deletion-contraction method (Daoud, 2015, 2017).

## D. CONCLUSION AND SUGGESTIONS

By using the matrix tree theorem, this research has obtained the complexity formulas of graphs generated by wheel graph and fan graph as their blocks. Based on $\tau(W_n)$, the complexity of $m$-wheel graph $W_{n,m}$ for $n \geq 3$ and $m \geq 1$ is $\tau(W_{n,m}) = \left(\tau(W_n)\right)^m$. Based on $\tau(F_n)$, the complexity of dragonfly graph $Dg_n$ for $n \geq 1$ is $\tau(Dg_n) = \left(\tau(F_{n+2})\right)^2$ and the complexity of generalized dragonfly graph $Dg_n^{(3,3)}$ for $n \geq 1$ is $\tau\left(Dg_n^{(3,3)}\right) = \left(\tau(F_{n+2})\right)^3$. These formulas show that the complexity of a graph related to the complexity of its blocks. Additionally, this research can be extended by investigating other graphs that contain different types of blocks.

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