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Modular Coloring of Comb Graph, Lintang Graph, and Butterfly Graph

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ABSTRACT

Given any graph G that contains no isolated vertices, a labeling c is a mapping from its vertex set to the set of integers modulo k ($c: V(G) \to \mathbb{Z}_k$) for $k \geq 2$, adjacent vertices are allowed to share the same color. The number of color labels of a vertex v $(\sigma(v))$, is the number of color labels of the neighborhood of vertex v $(N(v))$. A labeling c is a modular k-coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all vertices x, y that are neighbors in G. Denoted as $mc(G)$, the modular chromatic number of G is defined as the least integer k that allows for a modular k-coloring of the graph. This research seeks to ascertain the modular chromatic number of the comb graph Cb_n, the lintang graph L_n, and the butterfly graph $BF(n)$. The first step in this research is to define the labeling c, then determine $(N(v))$. Next, determine the number of color labels from the neighborhood at each vertex with $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for x, y being all neighboring vertices. After the condition $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ is satisfied, ascertain $mc(G)$. By performing the same steps on each graph with increasingly larger values of n, a modular coloring pattern will emerge, which is used to formulate the modular coloring formula. This process concludes with the formulation of a modular coloring formula and the determination of the modular chromatic number for comb graph Cb_n, lintang graph L_n, and butterfly graph $BF(n)$. Based on this research, $mc(Cb_n) = 2$, $mc(L_n) = 2$, and $mc(BF(n)) = 3$ are obtained.

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A. INTRODUCTION

Graph coloring is a frequently discussed topic because it can solve various problems in life. Graph coloring can be used to solve problems such as work shift scheduling, networks, and many more (Rajagaspar & Senthil, 2022). One type of modular coloring is vertex coloring. Vertex coloring is the process of assigning colors to graph vertices, with the constraint that any two adjacent vertices must not share the same color (Gross et al., 2018). In contrast to standard vertex coloring, one variation permits neighboring vertices to share the same color, namely modular coloring. Modular coloring is a form of vertex coloring that uses a structured but more permissive set of rules. Modular coloring has rules that take into account the number of color labels on a vertex obtained by summing the color labels of the vertex's neighborhood. This rule distinguishes modular coloring from ordinary vertex coloring. Modular coloring allows neighboring vertices to share the same color label, but the number of color labels from the neighborhood of the neighboring vertices differs in $\mathbb{Z}_k$.

Modular coloring was introduced by Okamoto, Salehi, and Zhang (Okamoto et al., 2010). In their research, modular coloring was applied to chessboards to fulfill conditions that corresponded to initial assumptions on chessboards. These conditions stated that when coins were placed on several squares of a chessboard, for two squares with the same color, the coin counts on adjacent squares had identical parity. In contrast, for two adjacent squares that have different colors, the number of coins in each square has a different parity. By conducting periodic coin placement experiments, a coin placement pattern was discovered, which was then reformulated as a coin placement rule using the concept of modular arithmetic, so that it could meet the conditions in accordance with the initial assumptions of the chessboard.

In another study by Kusumaningrum and Rahadjeng, modular coloring was used to determine the modular chromatic number in star graphs S_n, caterpillar graphs $C_{(m;n_1,n_2,\dots,n_m)}$, fan graphs F_n, helmet graphs H_n, and triangular book graphs Bt_n (Kusumaningrum & Rahadjeng, 2021). The findings obtained from this research are as follows, $mc(S_n) = 2$, $mc\left(C_{(m;n_1,n_2,\dots,n_m)}\right) = 2$, $mc(F_n) = 3$, $mc(H_n) = 3$ if n even, $mc(H_n) = 4$ if n odd, $mc(Bt_n) = 3$. Based on these, there are still many graphs that can be used as research objects in applying modular coloring.

This research uses the comb graph Cb_n, the lintang graph L_n, and the butterfly graph $BF(n)$ as modular coloring objects. The comb graph Cb_n is formed through the corona product of the graph P_n and the graph K_1. (Barik et al., 2018; Akram & Nawaz, 2015; Tarawneh et al., 2016; Basavanagoud et al., 2021), the lintang graph L_n is formed by performing the join operation on the graph K_2 and the graph K_n (Artes & Dignos, 2015; Akram & Nawaz, 2015), and the butterfly graph $BF(n)$ is a combination of several graphs and all other vertices are adjacent to a central vertex. This research seeks to both construct modular colorings and compute the modular chromatic number for comb graphs Cb_n, lintang graphs L_n, and butterfly graphs $BF(n)$ by using modular arithmetic as the main basis for performing modular coloring on the graph used.

B. METHODS

This research uses combinatorial exploration analysis techniques, manually testing modular coloring starting with values of $k \geq 2$ on comb graphs Cb_n for $n \geq 2$, lintang graphs L_n for $n \geq 1$, and butterfly graphs $BF(n)$ for $n \geq 2$. Increasing the value of n in each graph will form a modular coloring pattern that will be used to determine whether modular coloring with a certain value of k can be applied to each graph with an increasing value of n. Modular coloring begins by labeling $c: V(G) \to \mathbb{Z}_k$ for $k \geq 2$, where G denotes the comb graph Cb_n, the lintang graph L_n, and the butterfly graph $BF(n)$. Next, determine the neighborhood of each vertex v and calculate the number of color labels from the neighborhood at each vertex v, noting that $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for x, y are all neighboring vertices. If there is $\sigma(x) = \sigma(y)$ in $\mathbb{Z}_k$, then c labeling will be performed again until all possible c labelings have been explored. If there is still $\sigma(x) = \sigma(y)$ in $\mathbb{Z}_k$ in all possible c labelings, then the value of k is increased by 1. After the condition $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ is satisfied, ascertain $mc(Cb_n)$, $mc(L_n)$, and $mc(BF(n)$. The modular chromatic number is defined as the least integer k that allows for a modular k-coloring of the graph.

C. RESULT AND DISCUSSION

Formally, a graph G consists of V which represents a non-empty set of vertices and E which represents a set of edges connecting the vertices (Gross et al., 2018; Koh et al., 2015). The graphs discussed are the comb graph Cb_n, the lintang graph L_n, and the butterfly graph $BF(n)$. A path graph, consisting of n vertices and $n-1$ edges is denoted as P_n for $n \geq 2$. A complete graph K_n with $n \geq 1$ is a simple graph where each vertex is adjacent to every other vertex (Mohideen, 2017; Azizin, 2024; Koh et al., 2015). A comb graph denoted as $P_n \odot K_1$, which has $2n$ vertices and $2n-1$ edges (Zhang et al., 2020; Veeraragavan & Arul, 2024). A comb graph Cb_n is formed from the set of vertices $V(Cb_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i \, | 1 \leq i \leq n\}$ and the set of edges $E(Cb_n) = \{(x_i, x_{i+1}) | 1 \leq i < n\} \cup \{(x_i, y_i) | 1 \leq i < n\}$. Comb graph Cb_n is shown in Figure 1 (a). The complement of a graph G (denoted by $\bar{G}$) is a graph with a set of vertices $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G (Gutman et al., n.d.; Upadhyay et al., 2020; Koh et al., 2015). The lintang graph L_n is defined by $L_n = (\overline{K_2}) + (\overline{K_1})$ for $n \geq 1$ (Fran et al., 2025; Wijayanti et al., 2016). The lintang graph L_n is formed from the set of vertices $V(L_n) = \{x_1, x_2\} \cup \{y_i \, | 1 \leq i \leq n\}$ and the set of edges $E(L_n) = \{(x_1, y_i) | 1 \leq i \leq n\} \cup \{(x_2, y_i) | 1 \leq i \leq n\}$. Lintang graph L_n is shown in Figure 1 (b). A butterfly graph $BF(n)$ with order n does not include apex vertices, has $(2n+3)$ vertices with $4n$ edges (M. Shalaan & A. Omran, 2020; Ponraj et al., 2021). The butterfly graph is formed from the set of vertices $V\big(BF(n)\big) = \{z\} \cup \{x_i | 0 \leq i \leq n\} \cup \{y_i | 0 \leq i \leq n\}$ and the set of edges $E\big(BF(n)\big) = \{(x_0, z), (y_0, z)\} \cup \{(x_i, z) | 1 \leq i \leq n\} \cup \{(x_i, x_{i+1}) | 1 \leq i < n\} \cup \{(y_i, z) | 1 \leq i \leq n\} \cup \{(y_i, y_{i+1}) | 1 \leq i < n\}$. Butterfly graph $BF(n)$ is shown in Figure 1 (c).

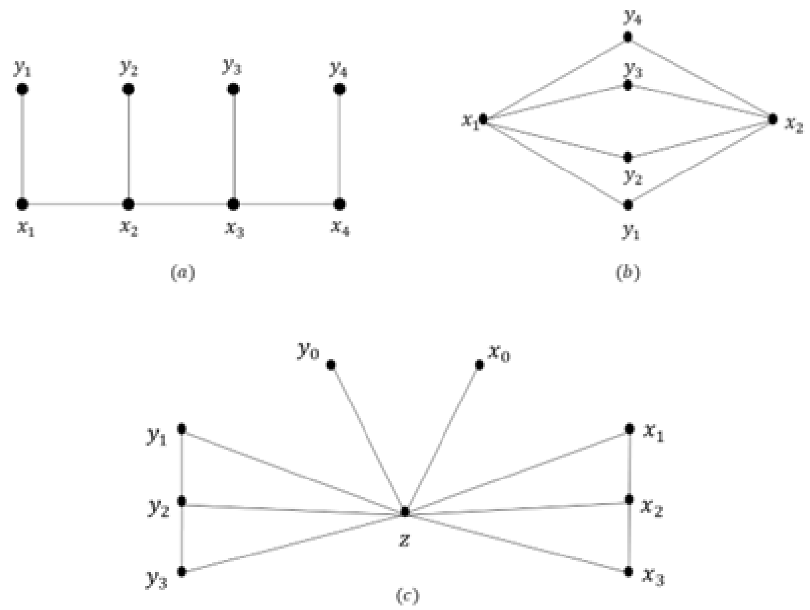

Figure 1. $(\boldsymbol{a})$ Comb Graph $\boldsymbol{Cb_4}$, $(\boldsymbol{b})$ Lintang Graph $\boldsymbol{L_4}$, $(\boldsymbol{c})$ Butterfly Graph $\boldsymbol{BF(3)}$

Definition 1 (Sumathi & Tamilselvi, 2022; Sumathi & Tamilselvi, 2024; Nicholas, 2017; Pamungkas, 2024; Sumathi & Tamilselvi, 2023; Sumathi, 2023) For $\boldsymbol{v}$ is a vertex in graph $\boldsymbol{G}$, the set of vertices adjacent to $\boldsymbol{v}$ is denoted as $\boldsymbol{N(v)}$. Consider a vertex labeling, $\boldsymbol{c}: \boldsymbol{V(G)} \to \mathbb{Z}_{\boldsymbol{k}}$ $(\boldsymbol{k} \geq$

2) for a graph $\boldsymbol{G}$ that has no isolated vertices, where adjacent vertices are permited to have the same color. The number of color label $\boldsymbol{\sigma}(\boldsymbol{v})$ of $\boldsymbol{G}$ is the total number of color labels of the vertices in $\boldsymbol{N}(\boldsymbol{v})$, that is,

$$\sigma(v) = \sum_{u \in N(v)} c(u).$$

A labeling c is a modular k-coloring for $k \geq 2$ of G if $\sigma(x) \neq \sigma(y)$ on $\mathbb{Z}_k$ for all vertices x, y that are adjacent in G. The minimum integer k where G has a modular k-coloring, denoted as $mc(G)$.

From Definition 1, a modular 3-coloring is given on the graph G_5 with the set of vertices

$$V(G_5) = \{v_1, v_2, v_3, v_4, v_5\}$$

and the set of edges

$$E(G_5) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$$

Figure 2 is a modular coloring because the number of colors (red numbers) on each neighboring vertex has a different value in $\mathbb{Z}_3$. The first step is to perform labeling $c: V(G_5) \to \mathbb{Z}_3$ that allows neighboring vertices can share the same color (black number). Then, determine the neighborhood of each vertex in graph G_5 and calculate the number of colors of each vertex by summing the colors of the vertex's neighborhood. Then, check the number of colors of the neighboring vertices in G_5, noting that the number of colors of the neighboring vertices must not have the same value in $\mathbb{Z}_3$. If the neighboring vertices have different numbers of colors, then the modular coloring rule is satisfied, so the labeling c is called a modular 3-coloring. So $mc(G_5) = 3$.

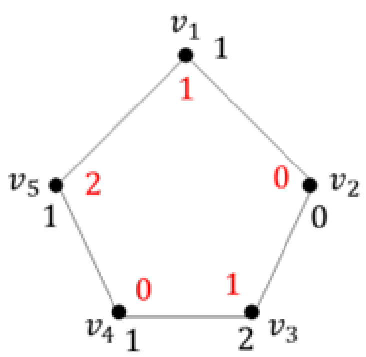

Figure 2 Modular **3**-Coloring On $\boldsymbol{G_5}$ Graph

The following is the modular coloring theorem on comb graphs Cb_n, lintang graphs L_n, and butterfly graphs $BF(n)$.
Theorem 1 Suppose $\boldsymbol{Cb_n}$ is a comb graph with $\boldsymbol{n \geq 2}$, then $\boldsymbol{mc(Cb_n) = 2}$.
Proof. Suppose a comb graph Cb_n with a set of vertices,

$$V(Cb_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\}$$

Case 1. For even n
Define the labeling $c: V(Cb_n) \rightarrow \mathbb{Z}_2$:

$$c(v) = \begin{cases} 0 & \text{when } v = x_i, \quad i \text{ odd} \\ 0 & \text{when } v = y_i, \quad i \text{ even} \\ 0 & \text{when } v = y_i, \quad i = 1 \\ 1 & \text{when } v = x_i, \quad i \text{ even} \\ 1 & \text{when } v = y_i, \quad i \equiv 1 (mod\ 2), \quad i \neq 1 \end{cases}$$

thus obtained

$$\sigma(v) = \begin{cases} 0 & \text{when } v = x_i, \quad i \text{ even} \\ 0 & \text{when } v = y_i, \quad i \text{ odd} \\ 1 & \text{when } v = x_i, \quad i \text{ odd} \\ 1 & \text{when } v = y_i, \quad i \text{ even} \end{cases}$$

obtained condition

$$(x_i, x_{i+1}), (x_i, y_i) \in E(Cb_n), \text{for } i = 1$$
$$(x_i, x_{i-1}), (x_i, y_i), (x_i, x_{i+1}) \in E(Cb_n), \text{for } i \neq 1$$
$$(x_i, x_{i-1}), (x_i, y_i) \in E(Cb_n), \text{for } i = n$$

Under these conditions, it can be seen that $\sigma(x_i) \neq \sigma(y_i)$. Based on Definition 1, for x, y are two adjacent vertices in Cb_n, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_2$, then c is a modular 2-coloring such that $mc(Cb_n) = 2$.

Case 2. For odd n
Define the labeling $c: V(Cb_n) \rightarrow \mathbb{Z}_2$:

$$c(v) = \begin{cases} 0 & \text{when } v = x_i, \quad i \text{ even} \\ 0 & \text{when } v = y_i, \quad i \text{ odd} \\ 1 & \text{when } v = x_i, \quad i \text{ odd} \\ 1 & \text{when } v = y_i, \quad i \text{ even} \end{cases}$$

thus obtained

$$\sigma(v) = \begin{cases} 0 & \text{when } v = x_i, \quad i \text{ odd} \\ 0 & \text{when } v = y_i, \quad i \text{ even} \\ 1 & \text{when } v = x_i, \quad i \text{ even} \\ 1 & \text{when } v = y_i, \quad i \text{ odd} \end{cases}$$

obtained condition

$$(x_i, x_{i+1}), (x_i, y_i) \in E(Cb_n), \text{for } i = 1$$
$$(x_i, x_{i-1}), (x_i, y_i), (x_i, x_{i+1}) \in E(Cb_n), \text{for } i \neq 1$$
$$(x_i, x_{i-1}), (x_i, y_i) \in E(Cb_n), \text{for } i = n$$

Under these conditions, it can be seen that $\sigma(x_i) \neq \sigma(y_i)$. Based on Definition 1, for x, y are two adjacent vertices in Cb_n, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_2$, then c is a modular 2-coloring such that $mc(Cb_n) = 2$.

Example. Here is a modular coloring on the graph Cb_3. The labeling $c: V(Cb_3) \to \mathbb{Z}_2$ allows neighboring vertices to have the same color (the colors of the labeling c are represented by black numbers). The number of colors on vertex $x_1(\sigma(x_1))$ is obtained by summing the colors from the neighborhood of x_1, namely $c(x_2) + c(y_1) = 0 + 0 = 0$ on $\mathbb{Z}_2$, so $\sigma(x_1) = 0$. $\sigma(x_2) = c(x_1) + c(x_3) + c(y_2) = 1 + 1 + 1 = 1, \sigma(x_3) = c(x_2) + c(y_3) = 0 + 0 = 0, \sigma(y_1) = c(x_1) = 1, \sigma(y_2) = c(x_2) = 0, \sigma(y_3) = c(x_3) = 1$. This result shows that for x, y are two adjacent vertices in Cb_3, then $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_2$ (the number of colors on the vertex is represented by the red number). Thus, c is a modular 2-coloring and $mc(Cb_3) = 2$. The modular coloring on the graph Cb_3 is represented in Figure 3.

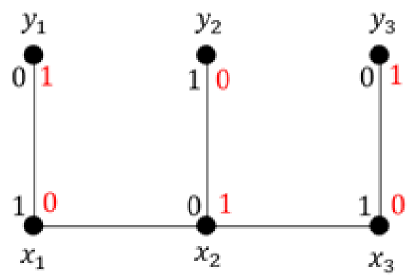

Figure 3. 2-Modular Coloring On $\boldsymbol{Cb_3}$

Theorem 2 Suppose $\boldsymbol{L_n}$ is a lintang graph with $\boldsymbol{n \geq 1}$, then $\boldsymbol{mc(L_n) = 2}$.
Proof. Suppose we have a graph L_n with a set of vertices,

$$V(L_n) = \{x_1, x_2\} \cup \{y_i | 1 \leq i \leq n\}$$

Case 1. For even n

Define the labeling $c: V(L_n) \to \mathbb{Z}_2$:

$$c(v) = \begin{cases} 0 & \text{when } v = x_1 \\ 1 & \text{when } v = x_2 \\ 1 & \text{when } v = y_i, \quad i = 1,2, \dots, n \end{cases}$$

thus obtained

$$\sigma(v) = \begin{cases} 0 & \text{when } v = x_i, \quad i = 1,2 \\ 1 & \text{when } v = y_j, \quad j = 1,2, \dots, n \end{cases}$$

obtained condition

$$(x_1, y_i), (x_2, y_i) \in E(L_n), \text{for } i = 1,2, \dots, n$$

Under these conditions, it can be seen that $\sigma(x_i) \neq \sigma(y_j)$. Based on Definition 1, for x, y are two adjacent vertices in L_n, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_2$, then c is a modular 2-coloring such that $mc(L_n) = 2$.

Case 2. For odd n

Define the labeling $c: V(L_n) \to \mathbb{Z}_2$:

$$c(v) = \begin{cases} 0 & \text{when } v = x_1 \\ 0 & \text{when } v = y_1 \\ 1 & \text{when } v = x_2 \\ 1 & \text{when } v = y_i, \quad i = 2,3,\dots,n \end{cases}$$

thus obtained

$$\sigma(v) = \begin{cases} 0 & \text{when } v = x_i, \quad i = 1,2 \\ 1 & \text{when } v = y_j, \quad j = 1,2,\dots,n \end{cases}$$

obtained condition

$$(x_1, y_i), (x_2, y_i) \in E(L_n), \text{for } i = 1,2,\dots,n$$

Under these conditions, it can be seen that $\sigma(x_i) \neq \sigma(y_j)$. Based on Definition 1, for x, y are two adjacent vertices in L_n, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_2$, then c is a modular 2-coloring such that $mc(L_n) = 2$.

Theorem 3 Suppose $\boldsymbol{BF(n)}$ is a butterfly graph with $\boldsymbol{n \geq 2}$, then $\boldsymbol{mc(BF(n)) = 3}$.

Proof. In this proof, modular coloring will be shown on the $BF(n)$ butterfly graph when n is even. For odd n, it can be proven using similar steps. Suppose the $BF(n)$ butterfly graph with a set of vertices,

$$V(BF(n)) = \{z\} \cup \{x_i | 0 \leq i \leq n\} \cup \{y_i | 0 \leq i \leq n\}$$

When n is even:

Case 1. $n = a_k + 1$ with,

$$a_{k+1} = \begin{cases} a_k + 2 \text{ if } k \text{ is odd} \\ a_k + 10 \text{ if } k \text{ is even} \end{cases}$$

with $a_1 = 2$.

Define the labeling $c: V(BF(n)) \to \mathbb{Z}_3$:

$$c(v) = \begin{cases} 2 & \text{when } v = x_0 \\ 2 & \text{when } v = y_0 \\ 1 & \text{when } v = x_i \quad \text{with} \quad i = 4b - 2, \quad b \in \mathbb{N} \\ 1 & \text{when } v = y_i \quad \text{with} \quad i = 4b - 2, \quad b \in \mathbb{N} \\ 1 & \text{when } v = z \\ 0 & \text{when } v = x_i \quad \text{others} \\ 0 & \text{when } v = y_i \quad \text{others} \end{cases}$$

thus obtained

$$\sigma(v) = \begin{cases} 0 & \text{when } v = z \\ 1 & \text{when } v = x_0 \\ 1 & \text{when } v = y_0 \\ 1 & \text{when } v = x_i, \quad i \text{ even} \\ 1 & \text{when } v = y_i, \quad i \text{ even} \\ 2 & \text{when } v = x_i, \quad i \text{ odd} \\ 2 & \text{when } v = y_i, \quad i \text{ odd} \end{cases}$$

obtained condition

$$(x_i, z), (x_i, z) \in E(BF(n)), \text{for } i = 0$$

$$(x_i, x_{i+1}), (x_i, z), (y_i, y_{i+1}), (y_i, z) \in E(BF(n)), \text{for } i = 1$$

$$(x_i, x_{i-1}), (x_i, x_{i+1}), (x_i, z), (y_i, y_{i-1}), (y_i, y_{i+1}), (y_i, z) \in E(BF(n)), \text{for } 1 < i < n$$

$(x_i, x_{i-1}), (x_i, z), (y_i, y_{i-1}), (y_i, z) \in E(BF(n))$, for $i = n$

This shows that $\sigma(z) \neq \sigma(x_0), \sigma(z) \neq \sigma(y_0), \sigma(z) \neq \sigma(x_i)$, and $\sigma(z) \neq \sigma(y_i)$. Based on Definition 1, for x, y are two adjacent vertices in $BF(n)$, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_3$, then c is a modular 3-coloring such that $mc(BF(n)) = 3$.

Case 2. $n = a_{k+1}$ with,

$$a_{k+1} = \begin{cases} a_k + 2 \text{ if } k \text{ is odd} \\ a_k + 10 \text{ if } k \text{ is even} \end{cases}$$

with $a_1 = 6$.
Define the labeling $c: V(BF(n)) \to \mathbb{Z}_3$:

$$c(v) = \begin{cases} 2 & \text{when } v = x_0 \\ 1 & \text{when } v = x_i \quad \text{with} \quad i = 4b - 2, \quad b \in \mathbb{N} \\ 1 & \text{when } v = y_i \quad \text{with} \quad i = 4b - 2, \quad b \in \mathbb{N} \\ 1 & \text{when } v = z \\ 0 & \text{when } v = y_0 \\ 0 & \text{when } v = x_i \quad \text{others} \\ 0 & \text{when } v = y_i \quad \text{others} \end{cases}$$

thus obtained

$$\sigma(v) = \begin{cases} 0 & \text{when } v = z \\ 1 & \text{when } v = x_0 \\ 1 & \text{when } v = y_0 \\ 1 & \text{when } v = x_i, \quad i \text{ even} \\ 1 & \text{when } v = y_i, \quad i \text{ even} \\ 2 & \text{when } v = x_i, \quad i \text{ odd} \\ 2 & \text{when } v = y_i, \quad i \text{ odd} \end{cases}$$

obtained condition

$(x_i, z), (x_i, z) \in E(BF(n))$, for $i = 0$
$(x_i, x_{i+1}), (x_i, z), (y_i, y_{i+1}), (y_i, z) \in E(BF(n))$, for $i = 1$
$(x_i, x_{i-1}), (x_i, x_{i+1}), (x_i, z), (y_i, y_{i-1}), (y_i, y_{i+1}), (y_i, z) \in E(BF(n))$, for $1 < i < n$
$(x_i, x_{i-1}), (x_i, z), (y_i, y_{i-1}), (y_i, z) \in E(BF(n))$, for $i = n$

This shows that $\sigma(z) \neq \sigma(x_0), \sigma(z) \neq \sigma(y_0), \sigma(z) \neq \sigma(x_i)$, and $\sigma(z) \neq \sigma(y_i)$. Based on Definition 1, for x, y are two adjacent vertices in $BF(n)$, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_3$, then c is a modular 3-coloring such that $mc(BF(n)) = 3$.

Case 3. $n = a_{k+1}$ with,

$$a_{k+1} = \begin{cases} a_k + 2 \text{ if } k \text{ is odd} \\ a_k + 10 \text{ if } k \text{ is even} \end{cases}$$

with $a_1 = 10$.
Define the labeling $c: V(BF(n)) \to \mathbb{Z}_3$:

$$c(v)=\begin{cases}2 & \text{when } v=x_0\\ 1 & \text{when } v=y_0\\ 1 & \text{when } v=x_i \quad \text{with} \quad i=4b-2, \quad b\in\mathbb{N}\\ 1 & \text{when } v=y_j \quad \text{with} \quad j=4b-2, \quad b\in\mathbb{N}\\ 1 & \text{when } v=z\\ 0 & \text{when } v=x_i \quad \text{others}\\ 0 & \text{when } v=y_j \quad \text{others}\end{cases}$$

thus obtained

$$\sigma(v)=\begin{cases}0 & \text{when } v=z\\ 1 & \text{when } v=x_0\\ 1 & \text{when } v=y_0\\ 1 & \text{when } v=x_i, \quad i \text{ even}\\ 1 & \text{when } v=y_j, \quad j \text{ even}\\ 2 & \text{when } v=x_i, \quad i \text{ odd}\\ 2 & \text{when } v=y_j, \quad j \text{ odd}\end{cases}$$

obtained condition

$(x_i,z),(x_i,z)\in E\big(BF(n)\big)$, for $i=0$

$(x_i,x_{i+1}),(x_i,z),(y_i,y_{i+1}),(y_i,z)\in E\big(BF(n)\big)$, for $i=1$

$(x_i,x_{i-1}),(x_i,x_{i+1}),(x_i,z),(y_i,y_{i-1}),(y_i,y_{i+1}),(y_i,z)\in E\big(BF(n)\big)$, for $1<i<n$

$(x_i,x_{i-1}),(x_i,z),(y_i,y_{i-1}),(y_i,z)\in E\big(BF(n)\big)$, for $i=n$

This shows that $\sigma(z)\neq\sigma(x_0),\sigma(z)\neq\sigma(y_0),\sigma(z)\neq\sigma(x_i)$, and $\sigma(z)\neq\sigma(y_i)$. Based on Definition 1, for x, y are two adjacent vertices in $BF(n)$, if $\sigma(x)\neq\sigma(y)$ in $\mathbb{Z}_3$, then c is a modular 3-coloring such that $mc\big(BF(n)\big)=3$.

D. CONCLUSION AND SUGGESTIONS

Building upon the results of the modular coloring performed on comb graph Cb_n, lintang graph L_n, and butterfly graph $BF(n)$. The modular chromatic number of the comb graph $mc(Cb_n)=2$ for $n\geq 2$, the *lintang* graph $mc(L_n)=2$ for $n\geq 1$, and the butterfly graph $mc\big(BF(n)\big)=3$ for $n\geq 2$. Modular coloring can be done with various k values, but to determine the modular chromatic number, the minimum k value must be used. This is because the minimum k value is the most efficient solution for applying modular coloring. This research also shows that for some graphs, such as the $BF(n)$ butterfly graph, which has a cycle graph C_3 in its structure, the modular chromatic number will be $mc\geq 3$. In this study, one of the graphs used is the $BF(n)$ butterfly graph with equal wing sizes in the butterfly graph. Readers who are interested in developing this research can discuss modular coloring on $BF(m,n)$ butterfly graphs with different butterfly wing sizes.

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