

Wiener Index, Harary Index, and Szeged Index of the Total Graph on $\mathbb{Z}_{p^k}$

Wahyu Ulyafandhie Misuki^{1*}, Sugi Guritman¹, Bib Paruhum Silalahi¹, Sita Armi Aulia¹

¹Department of Mathematics, IPB University, Indonesia

wahyumisuki@apps.ipb.ac.id

ABSTRACT

Article History:

Received : 02-05-2026

Revised : 25-06-2026

Accepted : 01-07-2026

Online : 30-10-2026

Keywords:

Wiener Index;

Harary Index;

Szeged Index

Total Graph;

Modulo Integer Ring.



Although total graphs over $\mathbb{Z}_{p^k}$ have been investigated previously, the structural and topological consequences of defining adjacency using the set of nonzero zero divisors have not been explored. Moreover, the Szeged index for this class of total graphs has not yet been established. This study investigates the total graph over $\mathbb{Z}_{p^k}$ with adjacency defined by nonzero zero divisors. Using algebraic and graph-theoretic methods, a coset-based partition induced by the canonical homomorphism onto $\mathbb{Z}_{p^k}$ is employed to characterize the graph structure, determine its distance distribution, and derive its topological indices. The total graph is shown to be disconnected and decomposes into components determined by pairs of cosets, each of which is a modified complete or complete bipartite graph. Closed-form formulas for the Wiener, Harary, and Szeged indices are derived as functions of p and k , and each connected component is shown to have bounded diameter. These results provide a structural characterization and closed-form formulas for the Wiener, Harary, and Szeged indices of this class of total graphs, thereby extending algebraic graph theory and enriching the study of topological graph invariants.



<https://doi.org/10.31764/jtam.v10i4.39846>



This is an open access article under the [CC-BY-SA](#) license

A. INTRODUCTION

Graph theory has been extensively applied in chemical graph theory, where molecular structures are represented as graphs and analyzed using topological indices to predict physicochemical and biological properties (Gutman & Polansky, 2022; Hua et al., 2019; Islam et al., 2021). Among the most prominent distance-based descriptors are the Wiener index, introduced as a path-based molecular measure (Wiener, 1947), the Harary index, defined from the reciprocal distance matrix (Plavšić et al., 1993), and the Szeged index, which incorporates the distribution of vertices relative to graph edges (Khadikar et al., 1995).

The integration of graph theory and algebra has led to the development of numerous graph constructions on groups and rings, including zero-divisor graphs (Anderson & Livingston, 1999), nilpotent and non-nilpotent graphs (Abdollahi & Zarrin, 2010; Das & Nongsang, 2015), commuting and non-commuting graphs (Abdollahi, 2008; Abdollahi et al., 2006), coprime and non-coprime graphs (Ma et al., 2014; Mansoori et al., 2016), and total graphs (Anderson & Badawi, 2008). Previous studies have investigated their structural properties, such as graph parameters, chromatic numbers, and clique numbers (Dorbidi, 2016; Gayatri, Nurhabibah, et

al., 2023; Hamm & Way, 2021), as well as their realizations on algebraic structures including integer modulo groups, dihedral groups, and generalized quaternion groups (Fitriani & Kurniawan, 2024; Juliana et al., 2020; Kurniawan et al., 2025; Misuki et al., 2021; Munandar, 2023; Nurhabibah et al., 2021; Putra et al., 2022).

In parallel, topological indices on algebraically defined graphs have received considerable attention due to their effectiveness in quantitatively characterizing graph structures. Various indices, including the Wiener, Harary, Szeged, Harmonic, Gutman, and the first and second Zagreb indices, have been studied for graphs associated with groups and rings (Alimon et al., 2020; Gayatri, Fadhilah, et al., 2023; Husni et al., 2022; Sarmin et al., 2020; Syarifudin et al., 2023; Zahidah et al., 2021). Nevertheless, studies that simultaneously address the structural characterization of total graphs and explicit formulations of topological indices, particularly over the ring $\mathbb{Z}_{p^k}$, remain limited.

Furthermore, existing studies on total graphs of finite rings generally define adjacency using the zero-divisor set, yielding graph constructions based on the conventional adjacency relation (Kharkongor et al., 2023; Knor et al., 2016; Meinawati et al., 2024; Nikmehr et al., 2014). Although this approach has produced important structural results, it does not address how restricting adjacency to nonzero zero divisors affects the resulting graph. Consequently, the impact of this restriction on graph decomposition, distance distribution, and distance-based topological indices, particularly the Szeged index, remains unknown. Among finite commutative rings, $\mathbb{Z}_{p^k}$ is of particular interest because its nontrivial zero-divisor structure induces intricate adjacency relations and natural coset partitions through the canonical homomorphism onto $\mathbb{Z}_p$. These algebraic properties make the structural analysis of its total graph considerably more challenging than that of simpler residue rings.

Motivated by these observations, this study investigates the total graph over $\mathbb{Z}_{p^k}$ by defining adjacency using the set of nonzero zero divisors. An algebraic coset-based approach is employed to characterize the graph decomposition, determine the distance distribution, and derive closed-form expressions for the Wiener, Harary, and Szeged indices. The obtained results clarify how the choice of the adjacency criterion influences the structural and topological properties of the graph and provide new theoretical insights into total graphs associated with finite commutative rings.

B. METHODS

This study adopts a theoretical and deductive approach integrating ring theory and graph theory to investigate the total graph over $\mathbb{Z}_{p^k}$. The analysis is based on the algebraic structure of the finite commutative ring $\mathbb{Z}_{p^k}$ and is carried out through formal mathematical reasoning using definitions, propositions, lemmas, and theorems. The research begins by constructing the ring $\mathbb{Z}_{p^k}$ and determining its set of nonzero zero divisors, which defines the adjacency relation of the total graph. The canonical homomorphism $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$ is then used to obtain a coset partition of the vertex set, providing the basis for analyzing the graph decomposition into connected components. Subsequently, the distance distribution within each component is determined and used to derive closed-form expressions for the Wiener, Harary, and Szeged indices. All results are established deductively by relating the algebraic properties of $\mathbb{Z}_{p^k}$ to the

corresponding graph-theoretical properties. The overall research procedure is summarized in Figure 1.

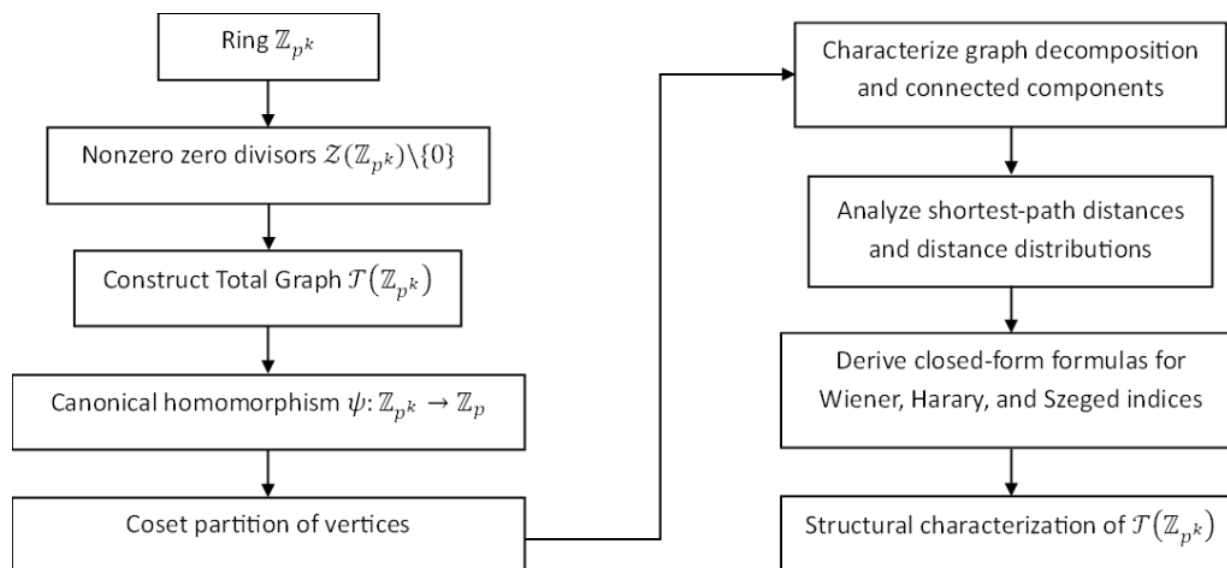


Figure 1. Research framework of the proposed algebraic approach

C. RESULT AND DISCUSSION

This section presents a comprehensive analysis of the total graph on $\mathbb{Z}_{p^k}$ using the set of nonzero zero divisors. By combining algebraic and graph-theoretic approaches, the study characterizes the adjacency structure, graph decomposition, connected components, distance distribution, and several topological indices. The results show that the algebraic structure of the ring, particularly its principal ideals and coset decomposition, fundamentally determines the global and metric properties of the graph.

1. Algebraic Foundation and Partition Structure

This section establishes the algebraic foundation through ring homomorphisms and coset structures on $\mathbb{Z}_{p^k}$. This foundation serves as the primary basis for determining graph partitions and subsequent analysis.

Definition 3.1 (Fraleigh, 2003)

Let p be a prime number and $k \geq 2$. The modulo ring is defined as

$$\mathbb{Z}_{p^k} = \{0, 1, 2, \dots, p^k - 1\}$$

with addition and multiplication performed modulo p^k . The ring $\mathbb{Z}_{p^k}$ is commutative with unity and has a simple ideal structure, particularly the ideal generated by p . For further analysis, a canonical mapping to $\mathbb{Z}_p$ is used, which preserves the ring structure, as formulated in the following proposition.

Proposition 3.1. The canonical mapping $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$, defined by $\psi(x) = x \bmod p$, is a surjective ring homomorphism.

Proof. Let $x, y \in \mathbb{Z}_{p^k}$. Then we have $\psi(x + y) = (x + y) \bmod p = (x \bmod p) + (y \bmod p) = \psi(x) + \psi(y)$ and $\psi(xy) = (xy) \bmod p = (x \bmod p)(y \bmod p) = \psi(x)\psi(y)$. Thus, ψ is a ring

homomorphism. Next, for every $w \in \mathbb{Z}_p$, choose $x = w \in \mathbb{Z}_{p^k}$. Then $\psi(x) = w$, so ψ is surjective. ■

Proposition 3.2. Let $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$ be defined by $\psi(x) = x \bmod p$. Then $\ker(\psi) = \langle p \rangle = \{x \in \mathbb{Z}_{p^k} \mid p \mid x\}$.

Proof. By definition, $\ker(\psi) = \{x \in \mathbb{Z}_{p^k} \mid \psi(x) = 0\}$. Since $\psi(x) = 0 \Leftrightarrow x \equiv 0 \bmod p$, this condition is equivalent to $p \mid x$. Thus, every element in the kernel can be written as $x = mp$ for some integer m . Hence, $\ker(\psi) = \{0, p, 2p, \dots, (p^{k-1} - 1)p\} = \langle p \rangle$. ■

With the kernel and the properties of the homomorphism established, the Ring Homomorphism Theorem can now be applied to the mapping ψ .

Theorem 3.1. Let $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$ be defined by $\psi(x) = x \bmod p$. Then $\mathbb{Z}_{p^k}/\ker(\psi) \cong \text{Im}(\psi) = \mathbb{Z}_p$.

Proof. Since ψ is a surjective ring homomorphism, the Ring Homomorphism Theorem implies that $\mathbb{Z}_{p^k}/\ker(\psi) \cong \text{Im}(\psi)$. Because ψ is surjective, we have $\text{Im}(\psi) = \mathbb{Z}_p$. Therefore, $\mathbb{Z}_{p^k}/\ker(\psi) \cong \mathbb{Z}_p$. ■

The homomorphism result and the characterization of its kernel not only provide the structure of the factor ring but also serve as the main mechanism for constructing the graph partition. Thus, the coset partition induced by the kernel is not merely algebraic in nature, but also plays a direct role in determining the vertex structure and connectivity patterns of the graph.

Corollary 3.1. The ring $\mathbb{Z}_{p^k}$ can be partitioned as $\mathbb{Z}_{p^k} = \bigsqcup_{i=0}^{p-1} \bar{i}$, where $\bar{i} = \{x \in \mathbb{Z}_{p^k} \mid x \equiv i \bmod p\}$, and each coset satisfies $|\bar{i}| = p^{k-1}$.

Proof. From Propositions 3.1 and 3.2, it is known that $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$ is a surjective homomorphism with $\ker(\psi) = \langle p \rangle$. Hence, for each $i \in \mathbb{Z}_p$, the coset is given by $\bar{i} = i + \ker(\psi) = \{i + mp \mid m \in \mathbb{Z}\}$, which is equivalent to $\{x \in \mathbb{Z}_{p^k} \mid x \equiv i \bmod p\}$.

These cosets are pairwise disjoint and cover the entire ring $\mathbb{Z}_{p^k}$. Since each coset has the same cardinality as the kernel, namely $|\ker(\psi)| = p^{k-1}$, it follows that $|\bar{i}| = p^{k-1}$. Therefore, the desired partition is obtained. ■

Example 3.1. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. Then $\mathbb{Z}_9 = \{0, 1, \dots, 8\}$. The canonical homomorphism $\psi: \mathbb{Z}_9 \rightarrow \mathbb{Z}_3$ is given by $\psi(x) = x \bmod 3$. Its kernel is $\ker(\psi) = \{0, 3, 6\}$. Hence $\mathbb{Z}_9 = \bar{0} \cup \bar{1} \cup \bar{2}$ where $\bar{0} = \{0, 3, 6\}$, $\bar{1} = \{1, 4, 7\}$, $\bar{2} = \{2, 5, 8\}$. Each coset contains $3^{2-1} = 3$ elements, confirming Corollary 3.1. Furthermore, $|\mathbb{Z}_9| = 9 = 3 \times 3$, so the partition reduces the analysis from nine individual vertices to three algebraically defined cosets.

The coset partition obtained in Example 3.1 serves as the basis for constructing the total graph. In the subsequent section, adjacency relations are determined between and within these cosets. Instead of analyzing all p^k vertices individually, the coset partition allows the graph structure to be studied through the interactions among only p algebraically defined classes. This reduction provides the key framework for deriving the graph decomposition and the closed-form formulas presented in the subsequent sections.

2. Adjacency, Graph Decomposition, and Graph Structure

Next, the adjacency relation in the total graph is defined along with the characterization of its global structure, which serves as the foundation for analyzing connected components, distances, and topological indices. In this study, zero divisors are defined as nonzero elements, in accordance with the approach commonly used in ring theory.

Definition 3.2 (Pinter, 2010)

Let R be a commutative ring with unity. The set of zero-divisors of R is defined as

$$\mathcal{Z}(R) = \{x \in R \setminus \{0\} \mid \exists y \in R \setminus \{0\}, xy = 0\}.$$

In this context, for the ring $\mathbb{Z}_{p^k}$, we obtain

$$\mathcal{Z}(\mathbb{Z}_{p^k}) = \{x \in \mathbb{Z}_{p^k} \setminus \{0\} \mid \exists y \in \mathbb{Z}_{p^k} \setminus \{0\}, xy \equiv 0 \pmod{p^k}\}.$$

Proposition 3.3. Let p be a prime number and $k \geq 2$. Then $\mathcal{Z}(\mathbb{Z}_{p^k}) = \langle p \rangle \setminus \{0\}$.

Proof. Let $x \in \mathbb{Z}_{p^k} \setminus \{0\}$. If $x \in \mathcal{Z}(\mathbb{Z}_{p^k})$, then there exists $y \neq 0$ such that $xy \equiv 0 \pmod{p^k}$, that is, $p^k \mid xy$. If $p \nmid x$, then x is invertible in $\mathbb{Z}_{p^k}$, so $xy \equiv 0$ implies $y \equiv 0$, a contradiction. Hence, $p \mid x$, and thus $x \in \langle p \rangle$. Conversely, if $x \in \langle p \rangle \setminus \{0\}$, then $x = mp$ with $m \neq 0$. Take $y = p^{k-1} \neq 0$. Then $xy = mp \cdot p^{k-1} = mp^k \equiv 0 \pmod{p^k}$, so $x \in \mathcal{Z}(\mathbb{Z}_{p^k})$. Therefore, $\mathcal{Z}(\mathbb{Z}_{p^k}) = \langle p \rangle \setminus \{0\}$. ■

This result serves as the foundation for defining the total graph based on zero-divisors in the ring.

Definition 3.3 (Anderson & Badawi, 2008)

Let R be a commutative ring with unity. The total graph on R , denoted by $\mathcal{T}(R)$, is a graph with vertex set $V = R$, where two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in \mathcal{Z}(R)$, where $\mathcal{Z}(R)$ denotes the set of zero-divisors of R . In this study, we consider $R = \mathbb{Z}_{p^k}$. Thus, the total graph $\mathcal{T}(\mathbb{Z}_{p^k})$ has vertex set $V = \mathbb{Z}_{p^k}$, with adjacency relation

$$x \sim y \Leftrightarrow x + y \in \mathcal{Z}(\mathbb{Z}_{p^k}).$$

Based on Proposition 3.3, we have $\mathcal{Z}(\mathbb{Z}_{p^k}) = \langle p \rangle \setminus \{0\}$, so that

$$x \sim y \Leftrightarrow x + y \in \langle p \rangle \setminus \{0\}.$$

Next, this adjacency relation is analyzed using the coset structure of $\mathbb{Z}_{p^k}$ modulo p .

Lemma 3.1. If $x \in \bar{i}$ and $y \in \bar{j}$, then $x \sim y \Leftrightarrow i + j \equiv 0 \pmod{p}$ and $x + y \neq 0$.

Proof. Let $x \in \bar{i}$ and $y \in \bar{j}$. By the definition of the total graph, $x \sim y \Leftrightarrow x + y \in \mathcal{Z}(\mathbb{Z}_{p^k}) \Leftrightarrow x + y \in \langle p \rangle \setminus \{0\}$. This is equivalent to $p \mid (x + y)$ and $x + y \neq 0$. Since $x \equiv i \pmod{p}$ and $y \equiv j \pmod{p}$, it follows that $x + y \equiv (i + j) \pmod{p}$. Thus, $p \mid (x + y) \Leftrightarrow i + j \equiv 0 \pmod{p}$, which yields the desired result. ■

Example 3.2. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. Then $\mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, which is partitioned into the cosets $\bar{0} = \{0, 3, 6\}$, $\bar{1} = \{1, 4, 7\}$, $\bar{2} = \{2, 5, 8\}$. Since $1 + 2 \equiv 0 \pmod{3}$,

vertices belonging to $\bar{1}$ and $\bar{2}$ may be adjacent whenever their sum is a nonzero zero divisor. For example, $1 + 5 = 6 \in \mathcal{Z}(\mathbb{Z}_9)$, thus $1 \sim 5$. However, $1 + 8 = 9 \equiv 0 \pmod{9}$, so $1 \not\sim 8$, because zero is not included in the set of zero divisors considered in this study.

This example confirms Lemma 3.1 by illustrating that adjacency depends only on the corresponding cosets together with the condition $x + y \neq 0$. Therefore, the local structure of the graph is entirely determined by arithmetic in $\mathbb{Z}_p$, which drastically reduces the complexity of the analysis.

Theorem 3.2. Let p be a prime number and $k \geq 2$. For each $i \in \mathbb{Z}_p$, define $V_i = \bar{i} \cup \overline{-i}$. Then

$$\mathcal{T}(\mathbb{Z}_{p^k}) = \mathcal{T}(V_0) \sqcup \mathcal{T}(V_1) \text{ for } p = 2 \text{ and } \mathcal{T}(\mathbb{Z}_{p^k}) = \bigsqcup_{i=0}^{(p-1)/2} \mathcal{T}(V_i) \text{ for odd } p.$$

Proof. From the coset partition, $\mathbb{Z}_{p^k} = \bigsqcup_{i=0}^{p-1} \bar{i}$. By Lemma 3.1, for $x \in \bar{i}$ and $y \in \bar{j}$, we have $x \sim y \Leftrightarrow i + j \equiv 0 \pmod{p}$ and $x + y \neq 0$. Thus, edges occur only between $\bar{i}$ and $\overline{-i}$, so each V_i induces a subgraph, and these subgraphs are pairwise disjoint. Next, each $\mathcal{T}(V_i)$ is connected. For $i = 0$, every pair of vertices is either directly adjacent or connected through an intermediate vertex. For $i \neq 0$, connectivity follows from the fact that vertices in different cosets are adjacent, while vertices within the same coset are connected via their paired coset. Since there are no edges between distinct V_i , each $\mathcal{T}(V_i)$ forms a connected component. The case $p = 2$ yields two components, while for odd p , there are $\frac{p+1}{2}$ components, as stated. ■

To illustrate the decomposition described in Theorem 3.2, Figure 2 presents the total graph $\mathcal{T}(\mathbb{Z}_9)$. The graph consists of two connected components corresponding to $V_0 = \bar{0}$ and $V_1 = \bar{1} \cup \bar{2}$, which agrees with the decomposition theorem. This example demonstrates that the connected components are completely determined by pairs of cosets induced by the canonical homomorphism.

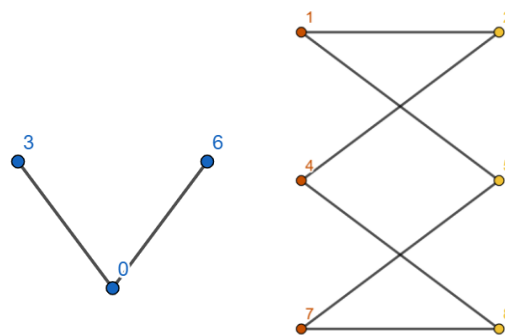


Figure 2. Decomposition of the total graph $\mathcal{T}(\mathbb{Z}_9)$ into the connected components $\mathcal{T}(V_0)$ and $\mathcal{T}(V_1)$.

This decomposition shows that the total graph on $\mathbb{Z}_{p^k}$ is inherently disconnected, and its component structure is entirely determined by pairs of cosets i and $-i$. This phenomenon indicates that graph connectivity is not a random global property, but rather a direct consequence of the factor ring structure $\mathbb{Z}_{p^k}/\langle p \rangle$. In other words, the underlying algebraic structure governs the global topology of the graph. Furthermore, this result allows the reduction of graph analysis to smaller, well-structured components, which is essential for the explicit computation of topological indices.

Corollary 3.2. Let p be a prime number and $k \geq 2$. The number of connected components of the total graph $\mathcal{T}(\mathbb{Z}_{p^k})$ is 2 for $p = 2$ and $\frac{p+1}{2}$ for odd p .

Proof. Based on Theorem 3.2, the connected components correspond to the sets V_i . For $p = 2$, there are two such sets, namely V_0 and V_1 , yielding two components. For odd p , there is one component from V_0 and $\frac{p-1}{2}$ components from the pairs $(\bar{i}, \overline{-i})$, giving a total of $\frac{p+1}{2}$. ■

Proposition 3.4. For $k \geq 2$, each component $\mathcal{T}(V_i)$ of $\mathcal{T}(\mathbb{Z}_{2^k})$ is a complete graph K_n with all edges removed between pairs of vertices $\{u, v\}$ such that $u + v = 0$, where $n = 2^{k-1}$.

Proof. Let $u, v \in V_i$ with $u \neq v$, for $i \in \{0, 1\}$. Since $p = 2$, we have $u \equiv v \equiv i \pmod{2}$, so $u + v \equiv 0 \pmod{2}$. By the definition of adjacency, $u \sim v \Leftrightarrow u + v \in \langle 2 \rangle \setminus \{0\} \Leftrightarrow u + v \neq 0$. Thus, every pair of distinct vertices in V_i is adjacent except when $u + v = 0$. Since the complete graph K_n contains all edges between distinct pairs of vertices, it follows that $\mathcal{T}(V_i)$ is obtained from K_n by removing exactly those edges connecting pairs $\{u, v\}$ with $u + v = 0$. ■

Proposition 3.5. For $k \geq 2$ and an odd prime p , each component $\mathcal{T}(V_i)$ of $\mathcal{T}(\mathbb{Z}_{p^k})$, with $i \neq 0$, is a complete bipartite graph $K_{n,n}$ with partition $(\bar{i}, \overline{-i})$, from which all edges connecting pairs $\{u, v\}$ with $u + v = 0$ are removed, where $n = p^{k-1}$.

Proof. Let $u, v \in V_i = \bar{i} \cup \overline{-i}$ with $u \neq v$. By Lemma 3.1, $u \sim v \Leftrightarrow i + j \equiv 0 \pmod{p}$ and $u + v \neq 0$. Since p is odd and $i \neq 0$, if $u, v \in \bar{i}$ or $u, v \in \overline{-i}$, then $i + i \not\equiv 0 \pmod{p}$, so they are not adjacent. Conversely, if $u \in \bar{i}$ and $v \in \overline{-i}$, then $i + (-i) \equiv 0 \pmod{p}$, so $u \sim v$, except when $u + v = 0$. Thus, all edges occur only between the two partitions, forming a complete bipartite graph $K_{n,n}$, except for the edges connecting pairs $\{u, v\}$ with $u + v = 0$, which are removed. ■

The graph obtained for $p = 3$ and $k = 2$ also illustrates the structural characterization established in Propositions 3.4 and 3.5. The component $\mathcal{T}(V_0)$ is a modified complete graph, whereas $\mathcal{T}(V_1)$ is a modified complete bipartite graph. In both cases, the only missing edges are those joining additive inverse pairs. Consequently, the resulting graphs preserve the overall structure of complete and complete bipartite graphs while exhibiting a regular pattern of deleted edges determined entirely by the algebraic structure of the ring, as shown in Figure 3.

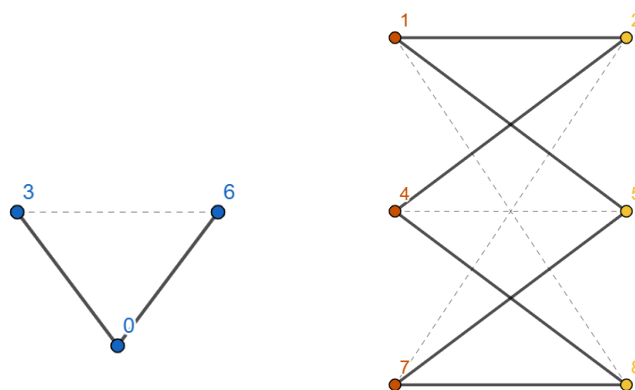


Figure 3. Structural characterization of the connected components of $\mathcal{T}(\mathbb{Z}_9)$: the modified complete graph $\mathcal{T}(V_0)$; the modified complete bipartite graph $\mathcal{T}(V_1)$.

This indicates that the metric properties of the graph are influenced not only by edge density but also by the underlying algebraic structure that governs the presence or absence of specific edges.

3. Distance and Distance Distribution

Distance analysis is carried out to reveal the metric structure of the graph within each component. The resulting distance distribution serves as the main foundation for computing topological indices and provides insight into the efficiency of connectivity in the graph.

Definition 3.4 (Diestel, 2017)

Let G be a simple and connected graph. For every pair of vertices $u, v \in V(G)$, the distance $d(u, v)$ is defined as the length of the shortest path connecting u and v . In the total graph $\mathcal{T}(\mathbb{Z}_{p^k})$, distances are defined within each component $\mathcal{T}(V_i)$, since the graph as a whole is not necessarily connected.

a. Distance Analysis on the Component V_0

Lemma 3.2. Let $V_0 = \bar{0} \subseteq \mathbb{Z}_{p^k}$, where p is a prime number and $k \geq 2$. For every $u \neq v \in V_0$, it holds that

$$d(u, v) \in \{1, 2\}, \text{ with } d(u, v) = \begin{cases} 1, & \text{if } u + v \neq 0, \\ 2, & \text{if } u + v = 0. \end{cases}$$

Proof. For $u, v \in V_0$, we have $u \equiv v \equiv 0 \pmod{p}$, so $u + v \equiv 0 \pmod{p}$. By the definition of adjacency, $u \sim v \Leftrightarrow u + v \in \langle p \rangle \setminus \{0\} \Leftrightarrow u + v \neq 0$. If $u + v \neq 0$, then $d(u, v) = 1$. If $u + v = 0$, then u and v are not adjacent. Since $|V_0| = p^{k-1} \geq 2$, there exists $w \in V_0 \setminus \{u, v\}$ with $w \neq -u$, such that $u \sim w$ and $w \sim v$. Hence, $d(u, v) = 2$. ■

This result indicates that although the graph is not complete, the distances between vertices remain bounded by small values. In particular, the diameter of the graph is bounded by the small constant 2, showing that each component has strong connectivity properties.

Lemma 3.3. Let $V_0 = \bar{0} \subseteq \mathbb{Z}_{p^k}$ with $n = p^{k-1}$. The number of pairs $\{u, v\} \subseteq V_0$, with $u \neq v$, satisfying $u + v = 0$ is $\frac{n-1}{2}$ for odd primes p , and $\frac{n-2}{2}$ for $p = 2$.

Proof. Since V_0 is an additive subgroup with $|V_0| = n$, every pair satisfying $u + v = 0$ is of the form $\{u, -u\}$ with $u \neq 0$. For odd p , we have $u \neq -u$ for every $u \neq 0$, so the elements in $V_0 \setminus \{0\}$ can be paired off. Hence, the number of such pairs is $\frac{n-1}{2}$. For $p = 2$, there exists exactly one nonzero element satisfying $u = -u$, namely $u = 2^{k-1}$, which does not form a distinct pair. Thus, the remaining $n - 2$ elements can be paired, giving a total of $\frac{n-2}{2}$ pairs. ■

To analyze the distance distribution between vertices, a function is introduced to count the number of vertex pairs based on their distance.

Definition 3.5. Let G be a simple graph with vertex set $V(G)$. For each $k \geq 1$, define

$$N_k(G) = |\{\{u, v\} \subseteq V(G) \mid u \neq v, d(u, v) = k\}|$$

Thus, $N_k(G)$ denotes the number of distinct vertex pairs whose distance is exactly k . For a component V_i , we write $N_k(V_i) = N_k(\mathcal{T}(V_i))$.

Theorem 3.3. Let p be a prime number, $k \geq 2$, and $|V_0| = n = p^{k-1}$. Then the distance distribution on V_0 is given by

$$N_2(\mathcal{T}(V_0)) = \begin{cases} \frac{n-1}{2}, & \text{if } p \text{ is odd,} \\ \frac{n-2}{2}, & \text{if } p = 2, \end{cases} \text{ and } N_1(\mathcal{T}(V_0)) = \binom{n}{2} - N_2(\mathcal{T}(V_0)).$$

Proof. By Lemma 3.2, for every pair $u \neq v \in V_0$, we have $d(u, v) = 2$ if and only if $u + v = 0$, and otherwise $d(u, v) = 1$. Hence, $N_2(\mathcal{T}(V_0)) = |\{ \{u, v\} \subseteq V_0 \mid u \neq v, u + v = 0 \}|$. From Lemma 3.3, the value of $N_2(\mathcal{T}(V_0))$ follows according to the cases above. Since all pairs have distance either 1 or 2, and the total number of pairs is $\binom{n}{2}$, it follows that $N_1(\mathcal{T}(V_0)) = \binom{n}{2} - N_2(\mathcal{T}(V_0))$. ■

Example 3.3. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. Then $V_0 = \{0, 3, 6\}$, where $n = 3$. The vertex pairs in V_0 are $\{0, 3\}, \{0, 6\}, \{3, 6\}$. Since $3 + 6 \equiv 0 \pmod{9}$, the pair $\{3, 6\}$ has distance $d(3, 6) = 2$. The remaining two pairs satisfy $d(0, 3) = d(0, 6) = 1$. Hence, $N_1 = 2$, and $N_2 = 1$. These values agree with Theorem 3.3 since $N_2 = \frac{n-1}{2} = 1$, and $N_1 = \binom{3}{2} - 1 = 2$.

Lemma 3.4. Let p be a prime number, $k \geq 2$, and $|V_0| = n = p^{k-1}$. Then the number of edges in $\mathcal{T}(V_0)$ is given by $|E(\mathcal{T}(V_0))| = \frac{(n-1)^2}{2}$ for odd p and $|E(\mathcal{T}(V_0))| = \frac{n^2-2n+2}{2}$ for $p = 2$.

Proof. For every $u \neq v \in V_0$, we have $u \sim v \Leftrightarrow u + v \neq 0$. Thus, $\mathcal{T}(V_0)$ is obtained from the complete graph K_n by removing all edges connecting pairs $\{u, v\}$ such that $u + v = 0$. The number of edges in K_n is $\binom{n}{2}$, while the number of pairs satisfying $u + v = 0$ is $N_2(\mathcal{T}(V_0))$, namely $\frac{n-1}{2}$ for odd primes p and $\frac{n-2}{2}$ for $p = 2$. Hence, we obtain for odd primes p , $|E(\mathcal{T}(V_0))| = \binom{n}{2} - N_2(\mathcal{T}(V_0)) = \frac{n(n-1)}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2}$ and for $p = 2$, $|E(\mathcal{T}(V_0))| = \binom{n}{2} - N_2(\mathcal{T}(V_0)) = \frac{n(n-1)}{2} - \frac{n-2}{2} = \frac{n^2-2n+2}{2}$. ■

b. Distance Analysis on the Component $V_i, i \geq 1$

Lemma 3.5. Let p be a prime number, $k \geq 2$, and for each $i \in \{1, 2, \dots, \lfloor p/2 \rfloor\}$, define $V_i = \bar{i} \cup \overline{-i}$. For every $u \neq v \in V_i$, the distance $d(u, v)$ in the graph $\mathcal{T}(\mathbb{Z}_{p^k})$ is given by

For odd p :

$$d(u, v) = \begin{cases} 1, & \text{if } u \in \bar{i}, v \in \overline{-i}, u + v \neq 0, \\ 2, & \text{if } u, v \text{ belong to the same coset,} \\ 3, & \text{if } u \in \bar{i}, v \in \overline{-i}, u + v = 0, \end{cases}$$

and for $p = 2$:

$$d(u, v) = \begin{cases} 1, & \text{if } u + v \neq 0, \\ 2, & \text{if } u + v = 0. \end{cases}$$

Proof. Based on the canonical homomorphism $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$ with $\ker(\psi) = \langle p \rangle$, the ring $\mathbb{Z}_{p^k}$ is partitioned into cosets $\bar{i}$. From Lemma 3.1, for $u \in \bar{i}$ and $v \in \bar{j}$, we have $u \sim v$ if and only if $i + j \equiv 0 \pmod{p}$ and $u + v \neq 0$.

Let $u \neq v \in V_i = \bar{i} \cup \overline{-i}$. For odd p , if $u \in \bar{i}$ and $v \in \overline{-i}$, then $i + (-i) \equiv 0$, so $u \sim v$ if and only if $u + v \neq 0$, which gives $d(u, v) = 1$. If u and v belong to the same coset, then they are not adjacent; however, there exists a vertex w in the paired coset such that $u - w -$

v forms a path, hence $d(u, v) = 2$. If $u \in \bar{i}$, $v \in \overline{-i}$, and $u + v = 0$, then there is no path of length two, but there exists a path $u - x - w - v$, so $d(u, v) = 3$.

For $p = 2$, we have $\bar{i} = \overline{-i}$, so two vertices are adjacent if and only if $u + v \neq 0$. If $u + v = 0$, then there exists a path of length two through another vertex, hence $d(u, v) = 2$. Therefore, the result follows. ■

This result shows that the diameter of the graph on the component V_i is at most 3 for odd p and 2 for $p = 2$, indicating that the graph maintains short distances even though it is not complete.

Lemma 3.6. Let p be a prime number, $k \geq 2$, and $n = p^{k-1}$. The number of unordered pairs $\{u, v\} \subseteq V_i$, with $u \neq v$, satisfying $u + v = 0$ is n for odd p with $i \neq 0$, and $\frac{n}{2}$ for $p = 2$ on V_1 .

Proof. For odd p and $i \neq 0$, every pair satisfying $u + v = 0$ is of the form $\{u, -u\}$. If $u \in \bar{i}$, then $-u \in \overline{-i}$, so each element in $\bar{i}$ produces exactly one such pair. Since $|\bar{i}| = n$ and the two cosets are disjoint, there are exactly n such pairs.

For $p = 2$, we have $V_1 = \bar{1}$ with $|V_1| = n$. The condition $u + v = 0$ is equivalent to $v = -u$. There is no element $u \in V_1$ such that $u = -u$, so each element can be uniquely paired with its additive inverse. Thus, all elements in V_1 can be partitioned into disjoint pairs, giving a total of $\frac{n}{2}$ pairs. ■

Theorem 3.4. Let p be a prime number, $k \geq 2$, and for each $i \in \{1, 2, \dots, \lfloor p/2 \rfloor\}$, define $V_i = \bar{i} \cup \overline{-i}$, with $|\bar{i}| = n = p^{k-1}$. Then the distance distribution on the graph $\mathcal{T}(V_i)$ is given by $N_1 = n(n-1)$, $N_2 = n(n-1)$, $N_3 = n$ for odd p and $N_1 = \frac{n(n-2)}{2}$, $N_2 = \frac{n}{2}$ for $p = 2$.

Proof. Based on the coset structure induced by the canonical homomorphism $\psi: \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_p$, the set V_i consists of two cosets, each of size n . From Lemma 3.5, for odd p , the distances between vertices in V_i can only be 1, 2, or 3. Pairs at distance 1 occur between vertices in $\bar{i}$ and $\overline{-i}$ with $u + v \neq 0$. The total number of cross-coset pairs is n^2 , and exactly n of them satisfy $u + v = 0$. Thus, $N_1 = n^2 - n = n(n-1)$. Pairs within the same coset always have distance 2, and their number is $2\binom{n}{2} = n(n-1)$, so $N_2 = n(n-1)$. Finally, pairs with $u + v = 0$ total n and have distance 3, so $N_3 = n$. For $p = 2$, we have $\bar{i} = \overline{-i}$, so V_1 consists of a single coset with $|V_1| = n$. From Lemma 3.5, distances can only be 1 or 2. The number of pairs with $u + v = 0$ is $\frac{n}{2}$, hence $N_2 = \frac{n}{2}$. Since the total number of pairs is $\binom{n}{2}$, we obtain $N_1 = \binom{n}{2} - \frac{n}{2} = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}$. Thus, the distance distribution follows as stated. ■

Example 3.4. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. The component $V_1 = \bar{1} \cup \bar{2} = \{1, 4, 7, 2, 5, 8\}$ contains six vertices. From the graph shown in Figure 2, three vertex pairs have distance 3, six vertex pairs have distance 2, and six vertex pairs have distance 1. Therefore, $N_1 = 6$, $N_2 = 6$, $N_3 = 3$. These values coincide with Theorem 3.4 for $n = 3$, namely $N_1 = n(n-1) = 6$, $N_2 = n(n-1) = 6$, $N_3 = n = 3$. Thus, the theoretical distance distribution is confirmed by the numerical example.

Lemma 3.7. Let p be a prime number, $k \geq 2$, and $n = p^{k-1}$. For each component V_i with $i \geq 1$, the number of edges satisfies $|E(\mathcal{T}(V_i))| = n^2 - n$ for odd p and $|E(\mathcal{T}(V_i))| = \frac{n(n-2)}{2}$ for $p = 2$.

Proof. For odd p , we have $V_i = \bar{i} \cup \overline{\bar{i}}$ with $|\bar{i}| = |\overline{\bar{i}}| = n$. Edges occur only between the two cosets, giving n^2 possible pairs. From Lemma 3.6, exactly n pairs satisfy $u + v = 0$ and thus do not form edges. Hence, $|E(\mathcal{T}(V_i))| = n^2 - n$.

For $p = 2$, we have $V_i = \bar{i}$ with $|V_i| = n$, and $\mathcal{T}(V_i)$ is obtained from the complete graph K_n by removing all pairs $\{u, v\}$ satisfying $u + v = 0$. From Lemma 3.6, the number of such pairs is $\frac{n}{2}$. Since $|E(K_n)| = \binom{n}{2}$, we obtain $|E(\mathcal{T}(V_i))| = \binom{n}{2} - \frac{n}{2} = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}$. ■

The distance distribution results show that although the graph is not complete, its diameter remains bounded (at most 3), indicating that the graph exhibits small-world properties. This is particularly important in the context of topological indices applications.

4. Wiener, Harary, and Szeged Index

In this section, the Wiener, Harary, and Szeged index are computed for each component $\mathcal{T}(V_i)$ based on the distance distributions obtained previously.

a. Wiener Index

Definition 3.6 (Wiener, 1947)

Let G be a simple and connected graph. The Wiener index is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v).$$

Using the distance distribution, we obtain

$$W(G) = \sum_{k \geq 1} k \cdot N_k(G).$$

Theorem 3.5. Let $p = 2$, $k \geq 2$, and $|V_i| = n = 2^{k-1}$ for $i \in \{0, 1\}$. Then the Wiener index for each component $\mathcal{T}(V_i)$ is given by $W(\mathcal{T}(V_0)) = \frac{n^2-2}{2}$ and $W(\mathcal{T}(V_1)) = \frac{n^2}{2}$.

Proof. From Lemma 3.2 and Lemma 3.5, the distances in V_0 and V_1 take only the values 1 or 2.

For the component V_0 , from Lemma 3.3 we have $N_2(\mathcal{T}(V_0)) = \frac{n-2}{2}$. Thus, $N_1(\mathcal{T}(V_0)) = \binom{n}{2} - N_2(\mathcal{T}(V_0))$. Hence,

$$\begin{aligned}
W(\mathcal{T}(V_0)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_0))} d(u,v) = \sum_{k \geq 1} k \cdot N_k(\mathcal{T}(V_0)) = N_1(\mathcal{T}(V_0)) + 2N_2(\mathcal{T}(V_0)) \\
&= \binom{n}{2} - N_2(\mathcal{T}(V_0)) + 2N_2(\mathcal{T}(V_0)) = \binom{n}{2} + N_2(\mathcal{T}(V_0)) \\
&= \frac{n(n-1)}{2} + \frac{n-2}{2} = \frac{n^2-2}{2}.
\end{aligned}$$

For the component V_1 , from Lemma 3.6 we have $N_2(\mathcal{T}(V_1)) = \frac{n}{2}$, $N_1(\mathcal{T}(V_1)) = \binom{n}{2} - N_2(\mathcal{T}(V_1))$. Thus,

$$\begin{aligned}
W(\mathcal{T}(V_1)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_1))} d(u,v) = \sum_{k \geq 1} k \cdot N_k(\mathcal{T}(V_1)) = N_1(\mathcal{T}(V_1)) + 2N_2(\mathcal{T}(V_1)) \\
&= \binom{n}{2} - N_2(\mathcal{T}(V_1)) + 2N_2(\mathcal{T}(V_1)) = \binom{n}{2} + N_2(\mathcal{T}(V_1)) = \frac{n(n-1)}{2} + \frac{n}{2} \\
&= \frac{n^2}{2}.
\end{aligned}$$

Thus, the theorem is proved. ■

Theorem 3.6. Let p be an odd prime number, $k \geq 2$, and $i \in \{0, 1, 2, \dots, \frac{p-1}{2}\}$ with $|\bar{i}| = n = p^{k-1}$. If $V_0 = \bar{0}$ for $i = 0$ and $V_i = \bar{i} \cup \overline{\bar{i}}$ for $i \geq 1$, then the Wiener index of each component $\mathcal{T}(V_i)$ is given by $W(\mathcal{T}(V_0)) = \frac{n^2-1}{2}$ and $W(\mathcal{T}(V_i)) = 3n^2$ for $i \geq 1$.

Proof. On the component $V_0 = \bar{0}$, for every pair of distinct vertices $u, v \in V_0$, the distance takes only the values 1 or 2, with $d(u, v) = 2 \iff u + v = 0$. The number of pairs at distance 2 is $N_2(\mathcal{T}(V_0)) = \frac{n-1}{2}$, while the total number of pairs is $\binom{n}{2}$. Hence, $N_1(\mathcal{T}(V_0)) = \binom{n}{2} - N_2(\mathcal{T}(V_0))$. Thus,

$$\begin{aligned}
W(\mathcal{T}(V_0)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_0))} d(u,v) = \sum_{k \geq 1} k \cdot N_k(\mathcal{T}(V_0)) = N_1(\mathcal{T}(V_0)) + 2N_2(\mathcal{T}(V_0)) \\
&= \binom{n}{2} - N_2(\mathcal{T}(V_0)) + 2N_2(\mathcal{T}(V_0)) = \binom{n}{2} + N_2(\mathcal{T}(V_0)) \\
&= \frac{n(n-1)}{2} + \frac{n-1}{2} = \frac{n^2-1}{2}.
\end{aligned}$$

For the component $V_i = \bar{i} \cup \overline{\bar{i}}$, the distance distribution is given by $N_1 = n(n-1)$, $N_2 = n(n-1)$, $N_3 = n$. Hence, the Wiener index is

$$\begin{aligned}
W(\mathcal{T}(V_i)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_i))} d(u,v) = \sum_{k \geq 1} k \cdot N_k(\mathcal{T}(V_i)) \\
&= N_1(\mathcal{T}(V_i)) + 2N_2(\mathcal{T}(V_i)) + 3N_3(\mathcal{T}(V_i)) = n(n-1) + 2n(n-1) + 3n \\
&= 3n(n-1) + 3n = 3n^2.
\end{aligned}$$

Thus, the theorem is proved. ■

Theorem 3.7. Let p be a prime number, $k \geq 2$, and let $\mathcal{T}(\mathbb{Z}_{p^k})$ be the total graph, with the convention that $d(u, v) = \infty$ for any pair of vertices that are not connected and $W^*(\mathcal{T}(\mathbb{Z}_{p^k}))$ is defined on a not connected graph. Then $W^*(\mathcal{T}(\mathbb{Z}_{p^k})) = \infty$.

Proof. From Corollary 3.2, the graph $\mathcal{T}(\mathbb{Z}_{p^k})$ is disconnected, so there exist pairs of vertices u, v belonging to different components. For such pairs, $d(u, v) = \infty$. Since $W(\mathcal{T}(\mathbb{Z}_{p^k})) = \sum_{\{u,v\} \subseteq V} d(u, v)$, the presence of even a single term equal to ∞ makes the entire sum equal to ∞ . ■

Example 3.5. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. Then $n = p^{k-1} = 3$. The total graph $\mathcal{T}(\mathbb{Z}_9)$ consists of two connected components, namely $\mathcal{T}(V_0)$ and $\mathcal{T}(V_1)$. For the component $\mathcal{T}(V_0)$, the distance distribution is $N_1 = 2, N_2 = 1$. Hence, $W(\mathcal{T}(V_0)) = 1 \cdot N_1 + 2 \cdot N_2 = 1(2) + 2(1) = 4$ which agrees with Theorem 3.6, $W(\mathcal{T}(V_0)) = \frac{n^2-1}{2} = \frac{9-1}{2} = 4$.

For the component $\mathcal{T}(V_1)$, the distance distribution is $N_1 = 6, N_2 = 6, N_3 = 3$. Therefore, $W(\mathcal{T}(V_1)) = 1(6) + 2(6) + 3(3) = 6 + 12 + 9 = 27$ which is identical to $W(\mathcal{T}(V_1)) = 3n^2 = 3(9) = 27$. Since $\mathcal{T}(\mathbb{Z}_9)$ is disconnected, every pair of vertices belonging to different connected components has distance $d(u, v) = \infty$. Consequently, $W^*(\mathcal{T}(\mathbb{Z}_9)) = \infty$, which confirms the global Wiener index formula given in Theorem 3.7.

The obtained Wiener index formulas differ from those reported for conventional total graphs in previous studies, where adjacency is defined using the zero-divisor set including zero. Under the nonzero zero-divisor definition adopted in this study, edges joining additive inverse pairs are removed, increasing the shortest-path distances for selected vertex pairs. Consequently, the Wiener index explicitly reflects the structural modification introduced by the revised adjacency relation while preserving a quadratic dependence on the coset size n .

The quadratic growth of the Wiener index is a direct consequence of the coset partition. Although the graph diameter remains bounded, the number of vertex pairs increases quadratically with the coset size. Consequently, the total distance is governed primarily by the algebraic multiplicity of each coset rather than by the graph diameter itself.

b. Harary Index

Definition 3.7. (Plavšić et al., 1993)

Let G be a simple graph. The Harary index is defined as

$$H(G) = \sum_{\{u,v\} \subseteq V(G), u \neq v} \frac{1}{d(u, v)}.$$

Using the distance distribution, we obtain

$$H(G) = \sum_{k \geq 1} \frac{1}{k} N_k(G).$$

Theorem 3.8. Let $p = 2$, $k \geq 2$, and $|V_i| = n = 2^{k-1}$ for $i \in \{0, 1\}$. The Harary index of each component is given by $H(\mathcal{T}(V_0)) = \frac{2n^2 - 3n + 2}{4}$ and $H(\mathcal{T}(V_1)) = \frac{2n^2 - 3n}{4}$.

Proof. From Lemma 3.2, the distances in V_0 and V_1 take only the values 1 or 2.

For the component V_0 , from Lemma 3.3 we have $N_2(\mathcal{T}(V_0)) = \frac{n-2}{2}$, $N_1(\mathcal{T}(V_0)) = \binom{n}{2} - N_2(\mathcal{T}(V_0))$. Thus,

$$\begin{aligned} H(\mathcal{T}(V_0)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_0))} \frac{1}{d(u,v)} = \sum_{k \geq 1} \frac{1}{k} N_k(\mathcal{T}(V_0)) = N_1(\mathcal{T}(V_0)) + \frac{1}{2} N_2(\mathcal{T}(V_0)) \\ &= \binom{n}{2} - N_2(\mathcal{T}(V_0)) + \frac{1}{2} N_2(\mathcal{T}(V_0)) = \binom{n}{2} - \frac{1}{2} N_2(\mathcal{T}(V_0)) \\ &= \frac{n(n-1)}{2} - \frac{n-2}{4} = \frac{2n(n-1) - (n-2)}{4} = \frac{2n^2 - 3n + 2}{4}. \end{aligned}$$

For the component V_1 , from Lemma 3.6 we have $N_2(\mathcal{T}(V_1)) = \frac{n}{2}$, $N_1(\mathcal{T}(V_1)) = \binom{n}{2} - N_2(\mathcal{T}(V_1))$. Thus,

$$\begin{aligned} H(\mathcal{T}(V_1)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_1))} \frac{1}{d(u,v)} = \sum_{k \geq 1} \frac{1}{k} N_k(\mathcal{T}(V_1)) = N_1(\mathcal{T}(V_1)) + \frac{1}{2} N_2(\mathcal{T}(V_1)) \\ &= \binom{n}{2} - N_2(\mathcal{T}(V_1)) + \frac{1}{2} N_2(\mathcal{T}(V_1)) = \binom{n}{2} - \frac{1}{2} N_2(\mathcal{T}(V_1)) = \frac{n(n-1)}{2} - \frac{n}{4} \\ &= \frac{2n(n-1) - n}{4} = \frac{2n^2 - 3n}{4}. \end{aligned}$$

Thus, the theorem is proved. ■

Theorem 3.9. Let p be an odd prime, $k \geq 2$, and $i \in \{0, 1, 2, \dots, \frac{p-1}{2}\}$ with $|\bar{i}| = n = p^{k-1}$. If $V_0 = \bar{0}$ and $V_i = \bar{i} \cup \overline{\bar{i}}$ for $i \geq 1$, then the Harary index of each component is given by $H(\mathcal{T}(V_0)) = \frac{(n-1)(2n-1)}{4}$ and $H(\mathcal{T}(V_i)) = \frac{9n^2 - 7n}{6}$ for $i \geq 1$.

Proof. On the component V_0 , distances take only the values 1 and 2, with $N_2 = \frac{n-1}{2}$, $N_1 = \binom{n}{2} - N_2$. Hence,

$$\begin{aligned}
 H(\mathcal{T}(V_0)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_0))} \frac{1}{d(u,v)} = \sum_{k \geq 1} \frac{1}{k} N_k(\mathcal{T}(V_0)) = N_1(\mathcal{T}(V_0)) + \frac{1}{2} N_2(\mathcal{T}(V_0)) \\
 &= \binom{n}{2} - N_2(\mathcal{T}(V_0)) + \frac{1}{2} N_2(\mathcal{T}(V_0)) = \binom{n}{2} - \frac{1}{2} N_2(\mathcal{T}(V_0)) \\
 &= \frac{n(n-1)}{2} - \frac{1}{2} \cdot \frac{n-1}{2} = \frac{n(n-1)}{2} - \frac{n-1}{4} = \frac{(n-1)(2n-1)}{4} \\
 &= \frac{3(n^2-1)}{4}
 \end{aligned}$$

For the component V_i , the distance distribution is $N_1 = n(n-1)$, $N_2 = n(n-1)$, $N_3 = n$. Thus,

$$\begin{aligned}
 H(\mathcal{T}(V_i)) &= \sum_{\{u,v\} \subseteq V(\mathcal{T}(V_i))} \frac{1}{d(u,v)} = \sum_{k \geq 1} \frac{1}{k} N_k(\mathcal{T}(V_i)) \\
 &= N_1(\mathcal{T}(V_i)) + \frac{1}{2} N_2(\mathcal{T}(V_i)) + \frac{1}{3} N_3(\mathcal{T}(V_i)) \\
 &= n(n-1) + \frac{1}{2} n(n-1) + \frac{1}{3} n = \frac{3}{2} n(n-1) + \frac{1}{3} n = \frac{9n(n-1) + 2n}{6} \\
 &= \frac{9n^2 - 9n + 2n}{6} = \frac{9n^2 - 7n}{6}
 \end{aligned}$$

Thus, the theorem is proved. ■

Theorem 3.10. Let p be a prime number, $k \geq 2$, and $n = p^{k-1}$. Assuming $d(u, v) = \infty$ for disconnected pairs so that $\frac{1}{d(u,v)} = 0$, then $\frac{2n^2-3n+1}{2}$ for $p = 2$ and $\frac{(n-1)(2n-1)}{4} + \frac{p-1}{2} \cdot \frac{9n^2-7n}{6}$ for odd p .

Proof. Since disconnected vertex pairs contribute zero, the global Harary index is the sum over all connected components $H(\mathcal{T}(\mathbb{Z}_{p^k})) = \sum_i H(\mathcal{T}(V_i))$.

For $p = 2$, the graph has two components V_0 and V_1 . From previous results,

$$H(\mathcal{T}(V_0)) = \frac{2n^2 - 3n + 2}{4}, H(\mathcal{T}(V_1)) = \frac{2n^2 - 3n}{4}.$$

Thus,

$$H(\mathcal{T}(\mathbb{Z}_{2^k})) = \frac{4n^2 - 6n + 2}{4} = \frac{2n^2 - 3n + 1}{2}.$$

For odd p , the graph consists of one component V_0 and $\frac{p-1}{2}$ identical components V_i for $i \geq 1$. Using previous results,

$$H(\mathcal{T}(V_0)) = \frac{(n-1)(2n-1)}{4}, H(\mathcal{T}(V_i)) = \frac{9n^2 - 7n}{6}.$$

Hence,

$$H(\mathcal{T}(\mathbb{Z}_{p^k})) = \frac{(n-1)(2n-1)}{4} + \frac{p-1}{2} \cdot \frac{9n^2-7n}{6}.$$

Thus, the theorem is proved. ■

Example 3.6. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. Then $n = p^{k-1} = 3$. For the component $\mathcal{T}(V_0)$, $N_1 = 2, N_2 = 1$. Thus, $H(\mathcal{T}(V_0)) = \frac{N_1}{1} + \frac{N_2}{2} = 2 + \frac{1}{2} = \frac{5}{2}$. This agrees with $H(\mathcal{T}(V_0)) = \frac{(n-1)(2n-1)}{4} = \frac{(2)(5)}{4} = \frac{5}{2}$. For the component $\mathcal{T}(V_1)$, $N_1 = 6, N_2 = 6, N_3 = 3$. Hence, $H(\mathcal{T}(V_1)) = \frac{6}{1} + \frac{6}{2} + \frac{3}{3} = 6 + 3 + 1 = 10$, which is equal to $H(\mathcal{T}(V_1)) = \frac{9n^2-7n}{6} = \frac{81-21}{6} = 10$. Therefore, $H(\mathcal{T}(\mathbb{Z}_9)) = H(\mathcal{T}(V_0)) + H(\mathcal{T}(V_1)) = \frac{5}{2} + 10 = \frac{25}{2}$, confirming the global formula given in Theorem 3.10.

Since the Harary index is inversely proportional to vertex distances, the removal of additive inverse edges produces smaller reciprocal-distance contributions than in conventional total graphs. Nevertheless, the explicit formulas show that the graph remains highly compact because most vertex pairs are still connected by paths of length at most three.

The Harary index confirms that the graph preserves strong metric cohesion despite the removal of specific edges. Since reciprocal distances assign greater weight to nearby vertices, the dominance of distances one and two results in relatively large Harary values for every connected component.

c. Szeged Index

Definition 3.8. (Khadikar et al., 1995)

Let G be a simple and connected graph. For each edge $uv \in E(G)$, define $n_u(uv) = |\{w \in V(G) \mid d(w, u) < d(w, v)\}|$ dan $n_v(uv) = |\{w \in V(G) \mid d(w, v) < d(w, u)\}|$. The Szeged index is defined as

$$Sz(G) = \sum_{uv \in E(G)} n_u(uv) n_v(uv).$$

Theorem 3.11. Let p be a prime number, $k \geq 2$, and $n = p^{k-1}$. Let $V_0 = \bar{0} \subset \mathbb{Z}_{p^k}$, and let $\mathcal{T}(V_0)$ be the subgraph induced by V_0 . Then the Szeged index of $\mathcal{T}(V_0)$ is given by $Sz(\mathcal{T}(V_0)) = 2n^2 - 6n + 1$ for $p = 2$ and $Sz(\mathcal{T}(V_0)) = 2(n-1)(n-2)$ for odd p .

Proof. From Proposition 3.4, the graph $\mathcal{T}(V_0)$ is obtained from the complete graph K_n by removing all edges connecting pairs $\{u, -u\}$. For each edge $uv \in E(\mathcal{T}(V_0))$, its contribution to the Szeged index depends on the number of vertices closer to u than to v , and vice versa. Due to the non-uniform structure, edges are classified as follows:

- 1) Let $u, v \neq 0$ with $u + v \neq 0$. For every vertex $w \neq u, v$, only two vertices contribute asymmetrically, namely $w = -u$ and $w = -v$. In this case, $n_u = 2, n_v = 2$, so each such edge contributes 4.

- 2) Without loss of generality, let $u = 0$ and $v \neq 0$. The distance analysis shows that exactly one vertex is closer to each endpoint (besides the endpoints themselves), giving $n_u = 2, n_v = 1$, so each such edge contributes 2.
- 3) For $p = 2$, there exists exactly one nonzero element satisfying $u = -u$. For edges involving this element, all other vertices are equidistant from both endpoints, so $n_u = n_v = 1$, and the contribution is 1.

From Lemma 3.4, the number of edges in $\mathcal{T}(V_0)$ is $|E(\mathcal{T}(V_0))| = \frac{(n-1)^2}{2}$ for p odd and $|E(\mathcal{T}(V_0))| = \frac{n^2-2n+2}{2}$ for $p = 2$.

Summing contributions for each edge type.

For $p = 2$,

$$\begin{aligned} Sz(\mathcal{T}(V_0)) &= \sum_{uv \in E(\mathcal{T}(V_0))} n_u(uv) n_v(uv) \\ &= \left(\frac{n^2 - 2n + 2}{2} - (n-1) - 1 \right) \cdot 4 + (n-2) \cdot 2 + 1 = 2n^2 - 6n + 1 \end{aligned}$$

For odd p ,

$$\begin{aligned} Sz(\mathcal{T}(V_0)) &= \sum_{uv \in E(\mathcal{T}(V_0))} n_u(uv) n_v(uv) = \left(\frac{(n-1)^2}{2} - (n-1) \right) \cdot 4 + (n-1) \cdot 2 \\ &= 2(n-1)(n-2). \end{aligned}$$

Thus, the closed form is obtained. ■

Theorem 3.12. Let p be a prime number, $k \geq 2$, and $n = p^{k-1}$. For each $i \in \{1, 2, \dots, \lfloor p/2 \rfloor\}$, let $V_i = \bar{i} \cup \overline{-i}$, and let $\mathcal{T}(V_i)$ be the induced subgraph. Then $Sz(\mathcal{T}(V_i)) = \frac{n(n-2)}{2}$ for $p = 2$ and $Sz(\mathcal{T}(V_i)) = (n^2 - n)(n-1)^2$ for odd p .

Proof. For $p = 2$, V_i consists of a single coset with $|V_i| = n$. From Proposition 3.4 and Lemma 3.7, $\mathcal{T}(V_i)$ is obtained from K_n by removing all pairs satisfying $u + v = 0$, giving $|E(\mathcal{T}(V_i))| = \binom{n}{2} - \frac{n}{2} = \frac{n(n-2)}{2}$. For any edge uv , every other vertex satisfies $d(w, u) = d(w, v) = 1$, so only u and v contribute, giving $n_u = 1, n_v = 1$. Each edge contributes 1, hence $Sz(\mathcal{T}(V_i)) = \frac{n(n-2)}{2}$. For odd p , from Proposition 3.5 and Lemma 3.7, $\mathcal{T}(V_i)$ is a complete bipartite graph $K_{n,n}$ with n edges removed, so $|E(\mathcal{T}(V_i))| = n^2 - n$. For any edge uv with $u \in \bar{i}, v \in \overline{-i}$, and $u + v \neq 0$:

- 1) If $w \in \bar{i} \setminus \{u\}$, then $d(w, u) = 2, d(w, v) = 1$, so w is closer to v .
- 2) If $w \in \overline{-i} \setminus \{v\}$, then $d(w, u) = 1, d(w, v) = 2$, so w is closer to u .

Thus, $n_u = n-1, n_v = n-1$, and each edge contributes $(n-1)^2$. Therefore, $Sz(\mathcal{T}(V_i)) = (n^2 - n)(n-1)^2$. ■

Theorem 3.13. Let p be a prime number, $k \geq 2$, $n = p^{k-1}$, and let $\mathcal{T}(\mathbb{Z}_{p^k})$ be the total graph. Assuming disconnected pairs do not contribute to the Szeged index, then $Sz(\mathcal{T}(\mathbb{Z}_{p^k})) = (2n^2 - 6n + 1) + \frac{n(n-2)}{2}$ for $p = 2$ and $Sz(\mathcal{T}(\mathbb{Z}_{p^k})) = 2(n-1)(n-2) + \frac{p-1}{2}(n^2 - n)(n-1)^2$ for odd p .

Proof. The graph $\mathcal{T}(\mathbb{Z}_{p^k})$ is disconnected and decomposes into disjoint components:

- 1) For $p = 2$: two components $\mathcal{T}(V_0)$ and $\mathcal{T}(V_1)$,
- 2) For odd p : one component $\mathcal{T}(V_0)$ and $\frac{p-1}{2}$ identical components $\mathcal{T}(V_i), i \geq 1$.

Since the Szeged index is additive over disconnected components, $Sz(\mathcal{T}(\mathbb{Z}_{p^k})) = Sz(\mathcal{T}(V_0)) + \sum_{i \geq 1} Sz(\mathcal{T}(V_i))$. Substituting the previously derived formulas gives the desired result. ■

Example 3.7. Consider the ring $\mathbb{Z}_9$, where $p = 3$ and $k = 2$. Then $n = p^{k-1} = 3$. For the component $\mathcal{T}(V_0)$, the graph contains two edges, namely $(0,3)$ and $(0,6)$. For each edge, $n_u = 2, n_v = 1$, so the contribution of each edge is $n_u n_v = 2$. Hence, $Sz(\mathcal{T}(V_0)) = 2 + 2 = 4$, which agrees with $Sz(\mathcal{T}(V_0)) = 2(n-1)(n-2) = 2(2)(1) = 4$. For the component $\mathcal{T}(V_1)$, there are six edges: $(1,2), (1,5), (4,2), (4,8), (7,5), (7,8)$. For each edge, direct computation from Definition 3.8 gives $n_u = 2, n_v = 2$, so each edge contributes $n_u n_v = 4$. Therefore, $Sz(\mathcal{T}(V_1)) = 6 \times 4 = 24$, which agrees with Theorem 3.12, $Sz(\mathcal{T}(V_1)) = (n^2 - n)(n-1)^2 = (9-3)(2^2) = 6 \times 4 = 24$. Finally, $Sz(\mathcal{T}(\mathbb{Z}_9)) = Sz(\mathcal{T}(V_0)) + Sz(\mathcal{T}(V_1)) = 4 + 24 = 28$, which is exactly the value obtained from the global formula in Theorem 3.13.

Unlike the Wiener and Harary indices, the Szeged index captures the relative position of vertices with respect to every edge. Therefore, it is particularly sensitive to the structural changes produced by deleting edges between additive inverse pairs. To the best of our knowledge, these closed-form formulas constitute the first explicit Szeged index for total graphs over $\mathbb{Z}_p^k$ constructed from the set of nonzero zero divisors, thereby extending the family of algebraically defined graphs whose Szeged index is known.

The Szeged index reveals structural information that cannot be captured solely by shortest-path distances. Because it depends on the partition of vertices relative to every edge, the index reflects the symmetry induced by the coset decomposition and the systematic removal of additive inverse edges. Consequently, the obtained formulas characterize not only the metric properties but also the global positional organization of the graph.

D. CONCLUSION AND SUGGESTIONS

In this study, the total graph over $\mathbb{Z}_p^k$ defined by the set of nonzero zero divisors has been characterized using an algebraic approach based on canonical homomorphisms and coset decomposition. The resulting graph decomposition, distance distribution, and closed-form formulas for the Wiener, Harary, and Szeged indices show that the algebraic structure of $\mathbb{Z}_p^k$ directly determines the graph's metric and topological properties. Unlike conventional total graph constructions that include zero as a zero divisor, the proposed definition yields different graph decompositions, distance distributions, and topological index formulas. In particular, this study establishes the first closed-form formula for the Szeged index of this class of total graphs.

These results extend existing studies on algebraic graphs and provide a foundation for investigating other finite rings and topological graph indices.

ACKNOWLEDGEMENT

The author would like to express sincere gratitude to all parties who have provided support in the completion of this research. Special appreciation is extended to the institution that has provided facilities and a conducive academic environment. The author also thanks colleagues for their valuable discussions and constructive feedback, which have contributed to improving the quality of this work.

REFERENCES

- Abdollahi, A. (2008). Commuting graphs of full matrix rings over finite fields. *Linear Algebra and Its Applications*, 428(11–12), 2947–2954. <https://doi.org/10.1016/j.laa.2008.01.036>
- Abdollahi, A., Akbari, S., & Maimani, H. R. (2006). Non-commuting graph of a group. *Journal of Algebra*, 298(2), 468–492. <https://doi.org/10.1016/j.jalgebra.2006.02.015>
- Abdollahi, A., & Zarrin, M. (2010). Non-Nilpotent Graph of a Group. *Communications in Algebra*, 38(12), 4390–4403. <https://doi.org/10.1080/00927870903386460>
- Alimon, N. I., Sarmin, N. H., & Erfanian, A. (2020). The topological indices of the non-commuting graph for symmetric groups. *ASM Science Journal*, 13(1), 1–5. [https://doi.org/10.32802/asmscj.2020.sm26\(1.28\)](https://doi.org/10.32802/asmscj.2020.sm26(1.28))
- Anderson, D. F., & Badawi, A. (2008). The total graph of a commutative ring. *Journal of Algebra*, 320(7), 2706–2719. <https://doi.org/10.1016/j.jalgebra.2008.06.028>
- Anderson, D. F., & Livingston, P. S. (1999). Article ID jabr. *Journal of Algebra*, 217, 434–447. <https://doi.org/10.1006/JABR.1998.7840>
- Das, A. K., & Nongsang, D. (2015). On the Genus of the Nilpotent Graphs of Finite Groups. *Communications in Algebra*, 43(12), 5282–5290. <https://doi.org/10.1080/00927872.2014.974256>
- Diestel, R. (2017). *Graph Theory, 5th edition 2017*.
- Dorbidi, H. R. (2016). A note on the coprime graph of a group. *International Journal of Group Theory*, 5(4), 17–22. <https://doi.org/10.22108/IJGT.2016.9125>
- Fitriani, I. N., & Kurniawan, V. Y. (2024). Algorithm For Constructing The Triple Unit Graph Of Type II of Ring $\mathbb{Z}_n$ Using Python. *Barekeng*, 18(3), 1639–1648. <https://doi.org/10.30598/barekengvol18iss3pp1639-1648>
- Fraleigh, J. B. (2003). A First Course in Abstract Algebra. In *Pearson Education India*. Pearson Education India.
- Gayatri, M. R., Fadhillah, R., Lestari, S. T., Pratiwi, L. F., Abdurahim, A., & Wardhana, I. G. A. W. (2023). Indeks Topologi dari Graf Koprime untuk Grup Dihedral dengan Orde Pangkat Prima. *Jurnal Diferensial*, 5(2), 126–134. <https://doi.org/10.35508/jd.v5i2.12462>
- Gayatri, M. R., Nurhabibah, N., Aini, Q., Awanis, Z. Y., Salwa, S., & Wardhana, I. G. A. W. (2023). The Clique Number and The Chromatics Number Of The Coprime Graph for The Generalized Quaternion Group. *JTAM (Jurnal Teori Dan Aplikasi Matematika)*, 7(2), 409. <https://doi.org/10.31764/jtam.v7i2.13099>
- Gutman, I., & Polansky, O. E. (2022). Mathematical Concepts in Organic Chemistry. In *Library of Congress Cataloging*. <https://doi.org/10.1515/9783112570180>
- Hamm, J., & Way, A. (2021). Parameters of the Coprime Graph of a Group. *International Journal of Group Theory*, 10(3), 137–147. <https://doi.org/10.22108/ijgt.2020.112121.1489>
- Hua, H., Das, K. C., & Wang, H. (2019). On atom-bond connectivity index of graphs. *Journal of Mathematical Analysis and Applications*, 479(1), 1099–1114. <https://doi.org/10.1016/j.jmaa.2019.06.069>
- Husni, M. N., Syafitri, H., Siboro, A. M., Syarifudin, A. G., Aini, Q., & Wardhana, I. G. A. W. (2022). the Harmonic Index and the Gutman Index of Coprime Graph of Integer Group Modulo With Order of Prime Power. *Barekeng*, 16(3), 961–966.

<https://doi.org/10.30598/barekengvol16iss3pp961-966>

- Islam, T. U., Mufti, Z. S., Ameen, A., Aslam, M. N., & Tabraiz, A. (2021). On Certain Aspects of Topological Indices. *Journal of Mathematics*, 2021(1), 9913529. <https://doi.org/10.1155/2021/9913529>
- Juliana, R., Masriani, M., Wardhana, I. G. A. W., Switrayni, N. W., & Irwansyah, I. (2020). COPRIME GRAPH OF INTEGERS MODULO n GROUP AND ITS SUBGROUPS. *Journal of Fundamental Mathematics and Applications (JFMA)*, 3(1), 15–18. <https://doi.org/10.14710/jfma.v3i1.7412>
- Khadikar, P. V., Deshpande, N. V., Kale, P. P., Dobrynin, A., Gutman, I., & Domotor, G. (1995). The Szeged Index and an Analogy with the Wiener Index. *Journal of Chemical Information and Computer Sciences*, 35(3), 547–550. <https://doi.org/10.1021/ci00025a024>
- Kharkongor, D., Boro, L., Singh, M. M., & Dutta, S. (2023). Topological indices of total graph of the ring $\mathbb{Z}_n$. *TWMS Journal Of Applied And Engineering Mathematics.*, 13(4), 1434–1445. <http://jaem.isikun.edu.tr/web/index.php/current/122-vol13no4/1126>
- Knor, M., Skrekovski, R., & Tepeh, A. (2016). Mathematical aspects of wiener index. *Ars Mathematica Contemporanea*, 11(2), 327–352. <https://doi.org/10.26493/1855-3974.795.ebf>
- Kurniawan, V. Y., Ekasiw, C. F., & Wiyono, S. B. (2025). The Triple Identity Graph of the Ring $\mathbb{Z}_n$. *Barekeng*, 19(4), 2521–2530. <https://doi.org/10.30598/barekengvol19iss4pp2521-2530>
- Ma, X., Wei, H., & Yang, L. (2014). The Coprime graph of a group. *International Journal of Group Theory*, 3(3), 13–23. <https://doi.org/10.22108/ijgt.2014.4363>
- Mansoori, F., Erfanian, A., & Tolue, B. (2016). Non-coprime graph of a finite group. *AIP Conference Proceedings*, 1750(1), 050017. <https://doi.org/10.1063/1.4954605>
- Meinawati, R., Kurniawan, V. Y., & Kurdhi, N. A. (2024). Algorithm for Constructing Total Graph of Commutative Ring. *JTAM (Jurnal Teori Dan Aplikasi Matematika)*, 8(2), 351. <https://doi.org/10.31764/jtam.v8i2.19850>
- Misuki, W. U., Wardhana, I. G. A. W., Switrayni, N. W., & Irwansyah. (2021). Some results of non-coprime graph of the dihedral group D_{2n} for n a prime power. *AIP Conference Proceedings*, 2329(1), 020005. <https://doi.org/10.1063/5.0042587>
- Munandar, A. (2023). Some Properties on Coprime Graph of Generalized Quaternion Groups. *Barekeng*, 17(3), 1373–1380. <https://doi.org/10.30598/barekengvol17iss3pp1373-1380>
- Nikmehr, M., Heidarzadeh, L., & Soleimani, N. (2014). Calculating different topological indices of total graph of $\mathbb{Z}_n$. *Studia Scientiarum Mathematicarum Hungarica*, 51(1), 133–140. <https://doi.org/10.1556/SScMath.51.2014.1.1270>
- Nurhabibah, N., Syarifudin, A. G., & Wardhana, I. G. A. W. (2021). Some Results of The Coprime Graph of a Generalized Quaternion Group Q_{4n} . *InPrime: Indonesian Journal of Pure and Applied Mathematics*, 3(1), 29–33. <https://doi.org/10.15408/inprime.v3i1.19670>
- Pinter, C. C. (2010). A Book of Abstract Algebra, Second Edition. In *Dover Publications, Inc., Mineola, New York*. Dover Publications.
- Plavšić, D., Nikolić, S., Trinajstić, N., & Mihalić, Z. (1993). On the Harary index for the characterization of chemical graphs. *Journal of Mathematical Chemistry*, 12(1), 235–250. <https://doi.org/10.1007/BF01164638>
- Putra, L. R. W., Awanis, Z. Y., Salwa, Aini, Q., & Wardhana, I. G. A. W. (2022). Power Graph Representation Groups of Integer Modulo With the Order of Power of Prime Number. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, 17(3), 1393–1400. <https://doi.org/10.30598/barekengvol17iss3pp1393-1400>
- Sarmin, N. H., Alimon, N. I., & Erfanian, A. (2020). Topological Indices of the Non-commuting Graph for Generalised Quaternion Group. *Bulletin of the Malaysian Mathematical Sciences Society*, 43(5), 3361–3367. <https://doi.org/10.1007/s40840-019-00872-z>
- Syarifudin, A. G., Santi, L. M., Faradiyah, A. R., Wijaya, V. R., & Suwastika, E. (2023). Topological Indices of the Relative Coprime Graph of the Dihedral Group. *JTAM (Jurnal Teori Dan Aplikasi Matematika)*, 7(3), 698. <https://doi.org/10.31764/jtam.v7i3.14913>
- Wiener, H. (1947). Structural Determination Of Paraffin Boiling Points [Contribution From Department Of Chemistry, Brooklyn College] Structural Determination. *Journal of the American Chemical Society*, 69(1), 17-20, 730(1940), 1945–1948. <https://doi.org/10.1021/ja01193a005>

- Zahidah, S., Mahanani, D. M., & Oktaviana, K. L. (2021). Connectivity Indices of Coprime Graph of Generalized Quaternion Group. *Journal of the Indonesian Mathematical Society*, 27(3), 285–296. <https://doi.org/10.22342/jims.27.3.1043.285-296>