

Packing Chromatic Numbers of Certain Unicyclic Graphs

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ABSTRACT

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Packing coloring is a distance-constrained vertex coloring in which any two vertices assigned the same color i must be separated by a distance greater than i . The minimum number of colors required for such a coloring is called the packing chromatic number and is denoted by $\chi_\rho(G)$. Exact packing chromatic numbers have been determined for various graph families, including trees, cycles, graph transformations, graph products, and several other structured graph classes. Nevertheless, the packing coloring behavior of unicyclic graphs with pendant attachments remains less completely understood, particularly because the interaction between the unique cycle and the attached structures may alter the relevant distance constraints. This paper determines the packing chromatic numbers of three unicyclic graph families: the Twin Double Pendant Cycle graph $TDPC_n$, the Jellyfish graph J_n , and the Caveman graph $CV_{m,n}$. The graph $TDPC_n$ is constructed from the cycle C_n by attaching four pendant vertices, with two pendant vertices incident with each of two adjacent cycle vertices. The study employs a theoretical graph-based approach in which explicit packing-coloring constructions are used to establish upper bounds, while structural and distance arguments provide the corresponding lower bounds. The results show that $\chi_\rho(TDPC_n) = 4$ for all $n \geq 3$. Moreover, $\chi_\rho(J_3) = 3$, whereas $\chi_\rho(J_n) = 4$ for all $n \geq 4$. For the Caveman graph $CV_{m,n}$, the packing chromatic number is given by $\chi_\rho(CV_{m,n}) = 3$ when $L = m(n-1) \equiv 0 \pmod{4}$ and n is odd, and $\chi_\rho(CV_{m,n}) = 4$ otherwise. These exact characterizations show that packing coloring in unicyclic graphs is governed not only by the length and periodic structure of the underlying cycle, but also by the location and distribution of pendant attachments. The results provide a structural basis for further investigations of packing coloring in broader classes of cycle-based graphs.



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A. INTRODUCTION

Graph theory has become an important branch of discrete mathematics due to its wide range of applications in communication networks, scheduling, optimization, coding theory, and computer science. Alongside the development of classical graph invariants, increasing attention has been devoted to distance-based graph parameters, which provide richer structural information by incorporating the relative positions of vertices within a graph. Recent studies on unicyclic graphs have shown that the arrangement of cycles and attached substructures can significantly influence distance-related invariants, highlighting the importance of graph structure in distance-based investigations (Li & Zhou, 2025; Ilić et al., 2011; Ayna, 2025).

Among the various distance-based concepts, packing coloring has emerged as one of the most active research topics in graph coloring theory. The concept is closely related to frequency

assignment problems in wireless communication networks, where transmitters using the same frequency must be separated by sufficient distances to avoid interference. This idea was initially introduced through broadcast coloring and later developed into the notion of packing coloring, which provides a natural mathematical framework for modeling distance-constrained resource allocation problems (Goddard et al., 2008; Shao & Vesel, 2015; Aardal et al., 2007; Brešar et al., 2017).

Formally, a packing coloring of a graph G is a vertex coloring in which any two vertices assigned the same color i must be at distance at least $i + 1$. The minimum number of colors required in such a coloring is called the packing chromatic number and is denoted by $\chi_\rho(G)$ (Brešar et al., 2020; Fresán-Figueroa et al., 2021; Gastineau et al., 2019). Unlike ordinary vertex coloring, where only adjacency constraints are considered, packing coloring imposes distance requirements that become progressively stronger as the color index increases. Consequently, determining the exact value of $\chi_\rho(G)$ is often considerably more challenging than determining the classical chromatic number.

The difficulty of the problem is further reflected in its computational complexity. Several studies have established that determining the packing chromatic number is NP-complete for general graphs and remains computationally difficult for a variety of restricted graph classes (Kim et al., 2018; Holub et al., 2020; Martin et al., 2017). As a result, much of the existing research has focused on obtaining exact values or sharp bounds for specific graph families whose structural properties can be exploited analytically.

A substantial portion of the literature has concentrated on trees and tree-related graphs. Exact packing chromatic numbers have been determined for several families, including centipede graphs, firecracker graphs, broom graphs, double star graphs, banana tree graphs, and vertex amalgamations of tree structures (Citra et al., 2021; Brešar & Ferme, 2018b; Kristiana et al., 2024). These results reveal that even relatively simple acyclic graphs may exhibit diverse packing coloring behavior, depending on the arrangement of branches and pendant vertices.

Beyond trees, packing coloring has been extensively investigated for graph transformations and derived graph classes. Previous studies have determined packing chromatic numbers for transformation graphs, total graphs, helm-related graphs, fan-related graphs, wheel-related graphs, and several other graph constructions (Srinivasan & Vivekanandan, 2021; Chaluvvaraju & Kumara, 2021; Rajalakshmi & Venkatachalam, 2019). These investigations consistently demonstrate that local structural modifications may substantially alter distance relationships and consequently affect packing coloring patterns.

Cycle-based graphs form another major research direction within the packing coloring literature. The presence of cycles introduces additional distance interactions that are absent in trees, often resulting in more restrictive coloring conditions. Exact values and bounds of packing chromatic numbers have been established for cycles, cycle-related graphs, Cartesian products of cycles, lexicographic products, and several graph operations involving cyclic structures (Jacobs et al., 2013; William et al., 2014; Győri et al., 2016). Collectively, these studies indicate that cycle configurations play a fundamental role in determining packing coloring properties.

The scope of packing coloring research has subsequently expanded to highly structured and infinite graphs. Significant progress has been achieved for square grids, triangular grids, hypercubes, Sierpiński-type graphs, cubic graphs, Moore graphs, and subcubic outerplanar graphs (Subercaseaux & Heule, 2022; Brešar & Ferme, 2018a; Fresán-Figueroa et al., 2021; Brešar & Ferme, 2018b). At the same time, investigations on subcubic graphs and infinite lattices have revealed unexpectedly complex behavior, including the existence of graph families with unbounded packing chromatic numbers (Balogh et al., 2018; Korže et al., 2019).

Motivated by these developments, several generalizations of packing coloring have also been introduced. These include S -packing coloring, (d, n) -packing coloring, quasi-packing coloring, facial packing coloring, list packing, and limited packing, among others (Bai et al., 2020; Korže et al., 2019; Czap et al., 2019; Brešar & Ferme, 2018b). The emergence of these variants underscores the growing importance of distance-based coloring models and highlights the richness of the theoretical questions associated with packing constraints.

Despite the substantial progress achieved in packing coloring over the past two decades, most existing studies have concentrated on trees, graph transformations, graph products, infinite lattices, and highly regular graph families. Packing coloring has also been investigated for several unicyclic graph families, confirming that the interaction between a unique cycle and its attached structures may produce distinct distance constraints and exact values of the packing chromatic number (Ariningtyas et al., 2020). Nevertheless, unicyclic graphs with varied pendant configurations remain less comprehensively characterized, particularly when the number, form, and placement of the attachments affect the extension of a packing-coloring pattern from the underlying cycle.

The graph families considered in this paper represent three different forms of attachment to a cyclic structure. The Twin Double Pendant Cycle graph $TDPC_n$ is obtained from C_n by attaching two pendant vertices to each of two adjacent cycle vertices, thereby creating a concentrated local arrangement of pendant vertices. The Jellyfish graph J_n contains two pendant paths attached to v_1 and v_n , so that the cycle is extended by symmetric path structures. In contrast, the Caveman graph $CV_{m,n}$ consists of the cycle $C_{m(n-1)}$ together with m pendant vertices placed periodically along the cycle. These three configurations provide distinct settings for examining how localized, elongated, and periodic attachments modify the distance conditions required in a packing coloring.

Motivated by these structural differences, this paper determines the exact packing chromatic numbers of $TDPC_n$, J_n , and $CV_{m,n}$. The proofs combine explicit packing-coloring constructions for the upper bounds with subgraph, parity, and distance arguments for the corresponding lower bounds. The resulting characterizations clarify when the packing-coloring pattern of the underlying cycle can be extended to the attached structures and when an additional color becomes unavoidable. In this way, the study contributes a unified structural perspective on the influence of pendant arrangements in unicyclic graphs and provides a basis for further work on more general cycle-based graph families.

B. METHODS

This study adopts a theoretical and literature-based approach founded on fundamental concepts of graph theory, particularly packing coloring on unicyclic graphs. Established definitions, lemmas, and previous results concerning packing chromatic numbers are employed to investigate the Jellyfish graph J_n , the Caveman graph $CV_{m,n}$, and the proposed Twin Double Pendant Cycle graph $TDPC_n$. The exact packing chromatic number of each graph class is determined through the following stages.

1. Graph Definition and Structural Analysis

Each graph class is formally defined through its vertex and edge sets. Subsequently, its structural characteristics, including cycle components, pendant vertices, attachment vertices, and distance relationships among vertices, are examined to identify the properties relevant to packing coloring.

2. Packing Coloring Construction

A valid packing coloring is constructed by assigning colors to vertices according to the structural configuration and distance constraints of the graph. The construction is designed to satisfy the definition of packing coloring and provides a feasible coloring pattern for the investigated graph.

3. Lower Bound Derivation

The minimum number of required colors is established using structural arguments together with known theoretical results on packing chromatic numbers of cycle graphs and related subgraphs. These arguments demonstrate that no packing coloring with fewer colors can satisfy the required distance constraints.

4. Determination of Exact Values

Finally, the exact packing chromatic number is obtained by combining the valid packing coloring construction with the corresponding lower-bound arguments. This establishes a complete characterization of the packing chromatic number for each investigated graph family, as shown in Figure 1.

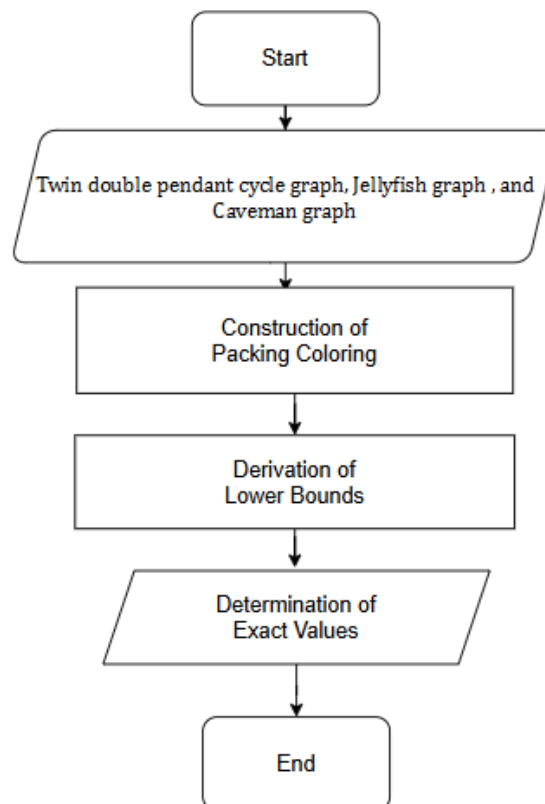


Figure 1. Research procedure for determining the packing chromatic number

Figure 1 illustrates the overall framework of the theoretical analysis employed in this study. The procedure begins with defining each graph and examining its structural properties, followed by the construction of valid packing colorings. Structural arguments are subsequently developed to establish the corresponding lower bounds. By combining these complementary arguments, the exact packing chromatic number is determined for each investigated graph family.

C. RESULT AND DISCUSSION

This section presents the principal findings of the study concerning the packing chromatic number on several classes of unicyclic graphs. The discussion begins with the fundamental definitions related to packing coloring and several preliminary results that are used throughout the proofs. Subsequently, three graph classes are investigated, namely the Twin Double Pendant Cycle graph, the Jellyfish graph, and the Caveman graph. The section concludes with several exact formulas for the packing chromatic number of the considered graph classes, together with constructive coloring patterns and rigorous proofs.

Definition 1 Let G be a connected graph. A packing k -coloring of G is a mapping (Brešar et al., 2020).

$$c: V(G) \rightarrow \{1, 2, 3, \dots, k\}$$

such that every two distinct vertices assigned the same color i are at distance at least $i + 1$.

Definition 2 The minimum number of colors required to produce a packing coloring of a graph G is called the packing chromatic number of G , denoted by $\chi_\rho(G)$ (Kristiana et al., 2024).

Lemma 1 Let H be a subgraph of a graph G . Then, $\chi_\rho(H) \leq \chi_\rho(G)$ (Kristiana et al., 2024).

Lemma 2 (Kristiana et al., 2024) The packing chromatic number of the cycle graph C_n is given by

$$\chi_\rho(C_n) = \begin{cases} 3, & n = 3 \text{ or } n \equiv 0(\text{mod } 4), \\ 4, & n \not\equiv 0(\text{mod } 4). \end{cases}$$

1. Twin Double Pendant Cycle Graph

The first graph class considered in this study is a new unicyclic graph family called the Twin Double Pendant Cycle graph, denoted by $TDPC_n$. Its construction is motivated by earlier studies on unicyclic graphs and distance-based graph invariants. The graph $TDPC_n$ is obtained from the cycle C_n by attaching exactly four pendant vertices so that each of two adjacent cycle vertices is incident with two pendant vertices. Such a construction produces a unique distance structure between cycle and pendant vertices, which significantly influences the packing coloring pattern of the graph. Consequently, the determination of $\chi_\rho(TDPC_n)$ presents a nontrivial and mathematically interesting problem.

Definition 3 The Twin Double Pendant Cycle graph $TDPC_n$ with $n \geq 3$ is the graph constructed from a cycle graph C_n by adjoining two pendant vertices to the vertex v_1 and two pendant vertices to the adjacent vertex v_2 (Li & Zhou, 2025). Suppose that $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Let u_1, u_2, u_3 , and u_4 be pendant vertices such that u_1 and u_2 are adjacent to v_1 , while u_3 and u_4 are adjacent to v_2 . Accordingly, the vertex set and edge set of $TDPC_n$ are defined by $V(TDPC_n) = \{v_i \mid 1 \leq i \leq n\} \cup \{u_i \mid 1 \leq i \leq 4\}$, and $E(TDPC_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_1 u_1, v_1 u_2, v_2 u_3, v_2 u_4\}$. Thus, the graph $TDPC_n$ may be viewed as a cycle C_n equipped with two twin pendant pairs attached to two consecutive vertices of the cycle, as shown in Figure 2.

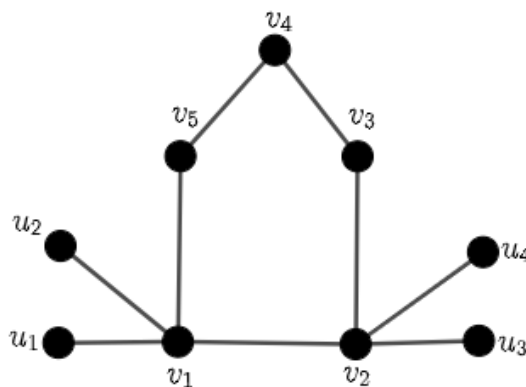


Figure 2. The Twin Double Pendant Cycle graph $TDPC_5$, obtained from the cycle C_5 by attaching two pendant vertices to each of two adjacent cycle vertices v_1 and v_2 .

Figure 2 illustrates the structural configuration of the Twin Double Pendant Cycle graph $TDPC_n$ with $n = 5$. The vertices $v_1, v_2, \dots, v_5$ form the unique cycle C_5 , while u_1 and u_2 are pendant vertices adjacent to v_1 , and u_3 and u_4 are pendant vertices adjacent to v_2 . The two

pairs of pendant vertices create a locally dense distance configuration around the adjacent attachment vertices v_1 and v_2 . In particular, vertices within the same pendant pair are at distance two, whereas pendant vertices attached to different cycle vertices are separated by short paths through the edge v_1v_2 . These distance relationships restrict the reuse of higher colors and distinguish the packing coloring of $TDPC_5$ from that of the underlying cycle C_5 .

Theorem 1 The packing chromatic number of $TDPC_n$ is

$$\chi_\rho(TDPC_n) = \begin{cases} 3, & n = 3, \\ 4, & n \geq 4. \end{cases}$$

Proof. Let Twin Double Pendant Cycle graph $TDPC_n$ with $n \geq 3$. The proof is divided into two cases.

Case 1: $n = 3$

First, we show that $\chi_\rho(TDPC_3) \leq 3$. Define a packing coloring $c: V(TDPC_3) \rightarrow \{1, 2, 3\}$ by $c(v_1) = 3, c(v_2) = 2, c(v_3) = 1$, and $c(u_i) = 1$, for $1 \leq i \leq 4$. The corresponding color classes are $C_1 = \{v_3, u_1, u_2, u_3, u_4\}$, $C_2 = \{v_2\}$, and $C_3 = \{v_1\}$. For every pair of distinct vertices $x, y \in C_i$, it follows that $d(x, y) \geq i + 1$. Therefore, c is a valid packing coloring, implying that $\chi_\rho(TDPC_3) \leq 3$. Next, since C_3 is a subgraph of $TDPC_3$, by Lemma 1 and Lemma 2, $\chi_\rho(TDPC_3) \geq \chi_\rho(C_3) = 3$. Consequently, $3 \leq \chi_\rho(TDPC_3) \leq 3$, and thus, $\chi_\rho(TDPC_3) = 3$.

Case 2: $n \geq 4$

We first establish the upper bound $\chi_\rho(TDPC_n) \leq 4$. Define a packing coloring $c: V(TDPC_n) \rightarrow \{1, 2, 3, 4\}$ with initial assignments $c(v_1) = 2, c(v_2) = 3$, and $c(u_i) = 1$, for $1 \leq i \leq 4$. For the remaining vertices v_i , where $3 \leq i \leq n$, the coloring is defined periodically according to the parity of n . The obtained coloring pattern guarantees that vertices receiving the same color satisfy the required packing distance condition. Hence, every pair of vertices in the same color class satisfies $d(x, y) \geq i + 1$, for each color $i \in \{1, 2, 3, 4\}$. Therefore, $\chi_\rho(TDPC_n) \leq 4$. To prove the lower bound, two subcases are considered.

Subcase 2.1: $n \not\equiv 0 \pmod{4}$

Since C_n is a subgraph of $TDPC_n$, by Lemma 1 and Lemma 2, $\chi_\rho(TDPC_n) \geq \chi_\rho(C_n) = 4$.

Subcase 2.2: $n \equiv 0 \pmod{4}$

Assume, for contradiction, that there exists a packing 3-coloring of $TDPC_n$. Since the pendant vertices are mutually at distance at least two, assign $c(u_i) = 1$ for $1 \leq i \leq 4$. Because v_1 is adjacent to u_1 and u_2 , $c(v_1) \neq 1$. Choose $c(v_1) = 2$. Similarly, vertex v_2 is adjacent to v_1, u_3 , and u_4 , implying $c(v_2) \neq 1$ and $c(v_2) \neq 2$. Thus, $c(v_2) = 3$. Since $n \equiv 0 \pmod{4}$, the cycle forces a repeating pattern of the form 2131, 2131, ... around the cycle. Consequently, one of the remaining vertices adjacent or sufficiently close to previously colored vertices cannot receive any color from $\{1, 2, 3\}$, producing a contradiction. Hence, $\chi_\rho(TDPC_n) \geq 4$. Combining the upper and lower bounds yields $\chi_\rho(TDPC_n) = 4$, for all $n \geq 4$. ■

Theorem 1 shows that the packing chromatic number of $TDPC_n$ is determined by the interaction between the cycle C_n and the two pendant pairs attached to consecutive cycle vertices. Since C_n is a subgraph of $TDPC_n$, it follows that $\chi_\rho(C_n) \leq \chi_\rho(TDPC_n)$, so the packing

chromatic number of the underlying cycle provides a natural lower bound. This finding is consistent with previous studies on cycle-related graphs, which emphasize that the periodic structure of an optimal packing coloring of C_n strongly influences the coloring of its extensions. However, the present result further shows that localized pendant attachments may introduce additional distance constraints that prevent an optimal coloring of C_n from extending directly to $TDPC_n$. Consequently, the cycle lower bound is attained only when the coloring pattern on C_n remains compatible with the attachment structure. This result therefore supports earlier findings on the central role of cycles in packing coloring, while extending them by identifying the position and concentration of pendant vertices as additional structural factors. The same lower-bound-and-extension approach may also be applied to other unicyclic graphs formed by attaching leaves, pendant paths, or small rooted trees to selected vertices of a cycle.

2. Jellyfish Graph

The second graph class investigated in this study is the Jellyfish graph, denoted by J_n . The graph is constructed from the cycle C_n by attaching two path graphs $P_{\lfloor \frac{n}{2} \rfloor - 1}$ to the vertices v_1 and v_n , respectively. This construction yields a unicyclic graph with two pendant path structures extending from the cycle. The resulting configuration creates a distinctive distribution of vertex distances, leading to nontrivial packing coloring behavior. Consequently, determining $\chi_\rho(J_n)$ requires a careful examination of the distance relationships between the cycle and the attached paths.

Definition 4 (Arfin, 2025) The Jellyfish graph, denoted by J_n with $n \geq 3$ is defined as a graph obtained from a cycle graph C_n by attaching two path graphs $P_{\lfloor \frac{n}{2} \rfloor - 1}$ to the vertices v_1 and v_n , respectively. Suppose that $V(C_n) = \{v_1, v_2, \dots, v_n\}$. For $i = 1, 2$, let $\{u_{i,j} \mid 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1\}$ denote the vertex set of the i -th path graph, where the initial vertex of the first path is adjacent to v_1 and the initial vertex of the second path is adjacent to v_n . Accordingly, the vertex set of J_n is given by

$$V(J_n) = \{v_i \mid 1 \leq i \leq n\} \cup \{u_{i,j} \mid 1 \leq i \leq 2, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1\},$$

while the edge set is defined by

$$\begin{aligned} E(J_n) = & \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_n v_1, v_1 u_{1,1}, v_n u_{2,1}\} \\ & \cup \{u_{i,j} u_{i,j+1} \mid 1 \leq i \leq 2, 1 \leq j < \lfloor \frac{n}{2} \rfloor - 1\}. \end{aligned}$$

Thus, the graph J_n consists of a cycle C_n with two pendant path structures attached symmetrically at the vertices v_1 and v_n , as shown in Figure 3.

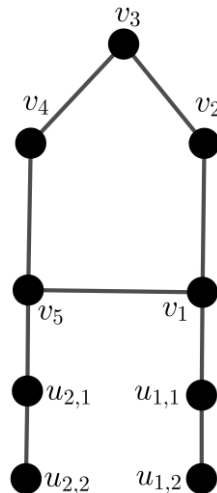


Figure 3. The Jellyfish graph J_5 , obtained from the cycle C_5 by attaching two paths P_2 to the adjacent vertices v_1 and v_5 .

Figure 3 illustrates the Jellyfish graph J_5 . The vertices $v_1, v_2, \dots, v_5$ form the cycle C_5 , while $u_{1,1}u_{1,2}$ and $u_{2,1}u_{2,2}$ form two pendant paths P_2 attached to v_1 and v_5 , respectively. Since v_1 and v_5 are adjacent on C_5 , the two attachments are concentrated around the same cycle edge. This configuration modifies the local distance relationships near v_1v_5 and restricts the reuse of colors on the pendant paths, even though the graph remains unicyclic.

Theorem 2 The packing chromatic number of the Jellyfish graph J_n with $n \geq 3$ is completely determined by $\chi_\rho(J_n) = \begin{cases} 3, & n = 3, \\ 4, & n \geq 4. \end{cases}$

Proof. Suppose that the jellyfish graph J_n with $n \geq 3$. The vertices v_i form the cycle graph C_n , whereas the vertices $u_{i,j}$ form two pendant paths attached to the cycle. To determine the packing chromatic number of J_n , the proof is divided into two cases.

Case 1. For $n = 3$

First, it will be shown that $\chi_\rho(J_3) \leq 3$. Define a packing coloring function $c: V(J_3) \rightarrow \{1, 2, 3\}$ by assigning $c(v_1) = 2, c(v_2) = 1, c(v_3) = 3$, and $c(u_{i,j}) = 1$, for $i = 1, 2$ and $j = 1$. Thus, the corresponding color classes are $C_1 = \{u_{1,1}, u_{2,1}, v_2\}$, $C_2 = \{v_1\}$, and $C_3 = \{v_3\}$. Observe that every pair of distinct vertices $u, v \in C_i$ satisfies $d(u, v) \geq i + 1$. Hence, each color class fulfills the definition of packing coloring, implying that c is a valid packing coloring of J_3 . Therefore, $\chi_\rho(J_3) \leq 3$. Next, it will be shown that $\chi_\rho(J_3) \geq 3$. By Lemma 2, $\chi_\rho(C_3) = 3$. Since C_3 is a subgraph of J_3 , Lemma 1 implies that $\chi_\rho(J_3) \geq \chi_\rho(C_3) = 3$. Combining the upper and lower bounds yields $3 \leq \chi_\rho(J_3) \leq 3$. Therefore, $\chi_\rho(J_3) = 3$.

Case 2. For $n \geq 4$

First, it will be shown that $\chi_\rho(J_n) \leq 4$. A packing coloring $c: V(J_n) \rightarrow \{1, 2, 3, 4\}$ is constructed according to the congruence class of n modulo 4.

Subcase 2.1. $n \equiv 0 \pmod{4}$

Define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4). \end{cases}$$

Furthermore,

$$c(u_{i,j}) = \begin{cases} 1, & i = 1 \text{ and } j \text{ odd, or } i = 2 \text{ and } j \text{ even,} \\ 2, & i = 1 \text{ and } j \equiv 0(\text{mod } 4), \text{ or } i = 2 \text{ and } j \equiv 3(\text{mod } 4), \\ 3, & i = 1 \text{ and } j \equiv 2(\text{mod } 4), \text{ or } i = 2 \text{ and } j \equiv 1(\text{mod } 4), \\ 4, & i = 2 \text{ and } j = 1. \end{cases}$$

The corresponding color classes are

$$\begin{aligned} C_1 &= \{v_2, v_4, \dots, v_n\} \cup \{u_{1,1}, u_{1,3}, u_{1,5}, \dots\} \cup \{u_{2,2}, u_{2,4}, u_{2,6}, \dots\}, \\ C_2 &= \{v_1, v_5, \dots, v_{n-3}\} \cup \{u_{1,4}, u_{1,8}, u_{1,12}, \dots\} \cup \{u_{2,3}, u_{2,7}, u_{2,11}, \dots\}, \\ C_3 &= \{v_3, v_7, v_{11}, \dots, v_{n-1}\} \cup \{u_{1,2}, u_{1,6}, u_{1,10}, \dots\} \cup \{u_{2,5}, u_{2,9}, u_{2,13}, \dots\}, \\ C_4 &= \{u_{2,1}\}. \end{aligned}$$

It can be verified that every pair of distinct vertices $u, v \in C_i$ satisfies $d(u, v) \geq i + 1$. Hence, the coloring function c is a valid packing coloring on J_n for $n \equiv 0(\text{mod } 4)$. Therefore, $\chi_\rho(J_n) \leq 4$.

Subcase 2.2. $n \equiv 1(\text{mod } 2)$

Define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4), \\ 4, & i = n. \end{cases}$$

Furthermore,

$$c(u_{i,j}) = \begin{cases} 1, & j \equiv 1(\text{mod } 4) \text{ or } j \equiv 3(\text{mod } 4), \\ 2, & j \equiv 0(\text{mod } 4), \\ 3, & j \equiv 2(\text{mod } 4). \end{cases}$$

The corresponding color classes are

$$\begin{aligned} C_1 &= \{v_2, v_4, v_6, \dots, v_{n-1}\} \cup \{u_{1,1}, u_{1,3}, u_{1,5}, \dots\} \cup \{u_{2,1}, u_{2,3}, u_{2,5}, \dots\}, \\ C_2 &= \{v_1, v_5, v_9, \dots, v_{n-2}\} \cup \{u_{1,4}, u_{1,8}, u_{1,12}, \dots\} \cup \{u_{2,4}, u_{2,8}, u_{2,12}, \dots\}, \\ C_3 &= \{v_3, v_7, v_{11}, \dots, v_{n-4}\} \cup \{u_{1,2}, u_{1,6}, u_{1,10}, \dots\} \cup \{u_{2,2}, u_{2,6}, u_{2,10}, \dots\}, \\ C_4 &= \{v_n\}. \end{aligned}$$

Again, every pair of distinct vertices $u, v \in C_i$ satisfies $d(u, v) \geq i + 1$. Thus, c defines a valid packing coloring of J_n , implying that $\chi_\rho(J_n) \leq 4$.

Subcase 2.3. $n \equiv 2(\text{mod } 4)$

Define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4), \\ 4, & i = n - 1. \end{cases}$$

Furthermore,

$$c(u_{i,j}) = \begin{cases} 1, & i = 1 \text{ and } j \text{ odd, or } i = 2 \text{ and } j \text{ even,} \\ 2, & i = 1 \text{ and } j \equiv 0(\text{mod } 4), \text{ or } i = 2 \text{ and } j \equiv 3(\text{mod } 4), \\ 3, & i = 1 \text{ and } j \equiv 2(\text{mod } 4), \text{ or } i = 2 \text{ and } j \equiv 1(\text{mod } 4). \end{cases}$$

The corresponding color classes are

$$\begin{aligned} C_1 &= \{v_2, v_4, v_6, \dots, v_n\} \cup \{u_{1,1}, u_{1,3}, u_{1,5}, \dots\} \cup \{u_{2,2}, u_{2,4}, u_{2,6}, \dots\}, \\ C_2 &= \{v_1, v_5, v_9, \dots, v_{n-5}\} \cup \{u_{1,4}, u_{1,8}, u_{1,12}, \dots\} \cup \{u_{2,3}, u_{2,7}, u_{2,11}, \dots\}, \\ C_3 &= \{v_3, v_7, v_{11}, \dots, v_{n-3}\} \cup \{u_{1,2}, u_{1,6}, u_{1,10}, \dots\} \cup \{u_{2,1}, u_{2,5}, u_{2,9}, \dots\}, \\ C_4 &= \{v_{n-1}\}. \end{aligned}$$

Similarly, every pair of distinct vertices $u, v \in C_i$ satisfies $d(u, v) \geq i + 1$. Therefore, c is a valid packing coloring on J_n , and hence $\chi_\rho(J_n) \leq 4$. From Subcases 2.1–2.3, every vertex assigned color i is separated by distance at least $i + 1$. Thus, $\chi_\rho(J_n) \leq 4$, for all $n \geq 4$. Next, it will be shown that $\chi_\rho(J_n) \geq 4$.

Subcase 2.4. $n \not\equiv 0(\text{mod } 4)$

By Lemma 2, $\chi_\rho(C_n) = 4$, for $n \not\equiv 0(\text{mod } 4)$. Since C_n is a subgraph of J_n , Lemma 1 implies that $\chi_\rho(J_n) \geq \chi_\rho(C_n) = 4$.

Subcase 2.5. $n \equiv 0(\text{mod } 4)$

Assume, for contradiction, that there exists a packing coloring $c: V(J_n) \rightarrow \{1, 2, 3\}$. By Lemma 2, $\chi_\rho(C_n) = 3$, for $n \equiv 0(\text{mod } 4)$. Consequently, the vertices of the cycle must follow the repeating pattern 2, 1, 3, 1, ..., 2, 1, 3, 1. Thus,

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4). \end{cases}$$

Since n is even, it follows that $c(v_n) = 1$. Now consider the vertex $u_{2,1}$. Since $u_{2,1}$ is adjacent to v_n , $d(u_{2,1}, v_n) = 1$, which implies that $c(u_{2,1}) \neq 1$. Furthermore, $d(u_{2,1}, v_{n-1}) = 2$ and $d(u_{2,1}, v_1) = 2$. Hence, $c(u_{2,1}) \neq 2$ and $c(u_{2,1}) \neq 3$. Therefore, $c(u_{2,1}) \notin \{1, 2, 3\}$, which contradicts the assumption that J_n admits a packing coloring using only three colors. Consequently, at least one additional color is required, implying that $\chi_\rho(J_n) \geq 4$. Combining the upper and lower bounds, it follows that $\chi_\rho(J_n) = 4$, for all $n \geq 4$. ■

Theorem 2 shows that the packing chromatic number of J_n is determined by the interaction between the cycle C_n and the two pendant paths $P_{\lfloor n/2 \rfloor - 1}$ attached at v_1 and v_n . Since C_n is a subgraph of J_n , it follows that $\chi_\rho(C_n) \leq \chi_\rho(J_n)$, so the packing chromatic number

of the underlying cycle provides a natural lower bound. This finding is consistent with previous studies on cycle-related graphs, which show that the periodic packing-coloring pattern of C_n strongly influences the coloring of its extensions. However, the present result further demonstrates that the two pendant paths may introduce additional distance restrictions near the adjacent attachment vertices v_1 and v_n , preventing a minimum packing coloring of C_n from being extended directly to the whole graph. Therefore, the exact value of $\chi_\rho(J_n)$ depends not only on the cycle order, but also on the compatibility of the cycle-coloring pattern with the attached paths. This result supports earlier findings on the central role of cycles while extending them to a unicyclic graph with symmetric path attachments, and the same approach may be applied to graphs with pendant paths of unequal lengths or attachments placed at different cycle vertices.

3. Caveman Graph

The final graph class discussed in this paper is the Caveman graph $CV_{m,n}$. This graph combines cyclic structures with clustered pendant configurations, resulting in more intricate distance relationships between vertices.

Definition 5 (Adhi Wijaya & Atmojo Kusmayadi, 2019) The Caveman graph $CV_{m,n}$ is a graph consisting of a cycle and pendant vertices with total order $m \times n$ (Figure 4). The graph is partitioned into m clusters, each containing exactly n vertices. The cycle component contains $m(n-1)$ vertices, while the remaining m vertices are pendant vertices attached periodically along the cycle. The vertex set is

$V(CV_{m,n}) = \{v_i \mid 1 \leq i \leq m(n-1)\} \cup \{u_i \mid 1 \leq i \leq m\}$, and the edge set is

$$E(CV_{m,n}) = \{v_i v_{i+1} \mid 1 \leq i \leq m(n-1) - 1\} \cup \{v_{m(n-1)} v_1\} \cup \{v_{(n-1)i-(n-2)} u_i \mid 1 \leq i \leq m\}.$$

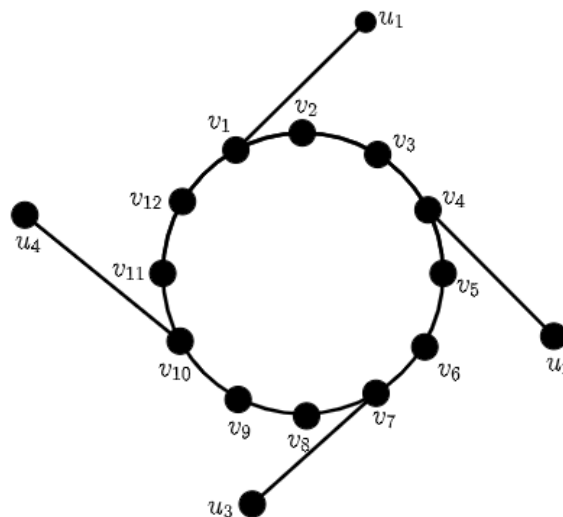


Figure 4. The Caveman graph $CV_{4,4}$, consisting of the cycle C_{12} and four pendant vertices attached periodically at v_1 , v_4 , v_7 , and v_{10} .

Figure 4 illustrates the Caveman graph $CV_{4,4}$, which has total order 16. The vertices $v_1, v_2, \dots, v_{12}$ form the cycle C_{12} , while the pendant vertices u_1, u_2, u_3 , and u_4 are adjacent to v_1, v_4, v_7 , and v_{10} , respectively. These attachment vertices occur at regular intervals of three edges along the cycle, reflecting the partition of $CV_{4,4}$ into four clusters of four vertices. This periodic arrangement preserves the unicyclic structure while modifying the distances near the attachment vertices, which directly affects whether a packing-coloring pattern on C_{12} can be extended to the entire graph.

Theorem 3 Let $CV_{m,n}$ be the caveman graph with parameters $m \geq 3$ and $n \geq 2$. If $L = m(n - 1)$, then the packing chromatic number of $CV_{m,n}$ is given by

$$\chi_\rho(CV_{m,n}) = \begin{cases} 3, & L \equiv 0(\text{mod } 4) \text{ and } n \equiv 1(\text{mod } 2), \\ 4, & L \not\equiv 0(\text{mod } 4), \\ 4, & L \equiv 0(\text{mod } 4) \text{ and } n \equiv 0(\text{mod } 2). \end{cases}$$

Proof. Let the caveman graph $CV_{m,n}$ with $L = m(n - 1)$, $m \geq 3$, and $n \geq 2$. To determine the packing chromatic number of $CV_{m,n}$, the proof is divided into the following two cases.

Case 1: $L \equiv 0(\text{mod } 4)$ and $n \equiv 1(\text{mod } 2)$

Define a packing coloring $c: V(CV_{m,n}) \rightarrow \{1, 2, 3\}$. For every vertex v_i , where $i = 1, 2, \dots, m(n - 1)$, define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4). \end{cases}$$

Furthermore, assign $c(u_i) = 1$, with $i = 1, 2, \dots, m$. Hence, the color classes are given by $c_1 = \{v_2, v_4, v_6, \dots, v_{m(n-1)}\} \cup \{u_1, u_2, \dots, u_m\}$, $c_2 = \{v_1, v_5, v_9, \dots, v_{m(n-1)-3}\}$, and $c_3 = \{v_3, v_7, \dots, v_{m(n-1)-1}\}$. Observe that for every two distinct vertices $u, v \in c_i$, the distance satisfies $d(u, v) \geq i + 1$. Therefore, each color class satisfies the condition of packing coloring. Consequently, c is a valid packing coloring of $CV_{m,n}$, implying that $\chi_\rho(CV_{m,n}) \leq 3$.

The graph $CV_{m,n}$ contains the cycle graph $C_{m(n-1)}$ as a subgraph. Since $m(n - 1) \equiv 0(\text{mod } 4)$, Lemma 2 yields $\chi_\rho(C_{m(n-1)}) = 3$. By Lemma 1, $\chi_\rho(CV_{m,n}) \geq \chi_\rho(C_{m(n-1)})$, and hence $\chi_\rho(CV_{m,n}) \geq 3$. Combining the upper and lower bounds gives $3 \leq \chi_\rho(CV_{m,n}) \leq 3$, thus $\chi_\rho(CV_{m,n}) = 3$ for $L \equiv 0(\text{mod } 4)$ and $n \equiv 1(\text{mod } 2)$.

Case 2: $L \not\equiv 0(\text{mod } 4)$ or $(L \equiv 0(\text{mod } 4) \text{ and } n \equiv 0(\text{mod } 2))$

Define a packing coloring $c: V(CV_{m,n}) \rightarrow \{1, 2, 3, 4\}$.

Subcase 2.1 If $n \equiv 1(\text{mod } 2)$, $m \not\equiv 2(\text{mod } 4)$, $m(n - 1) \not\equiv 0(\text{mod } 4)$, then define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4). \end{cases}$$

Assign $c(v_i) = 4$ for $i = m(n-1) - 1$, if $m(n-1) \equiv 2(\text{mod } 4)$, and $i = m(n-1)$, if $m(n-1) \equiv 1(\text{mod } 4)$ or $m(n-1) \equiv 3(\text{mod } 4)$. For the pendant vertices, $c(u_i) = 1, i = 1, 2, \dots, m$. The corresponding color classes are $c_1 = \{v_2, v_4, v_6, \dots\} \cup \{u_1, u_2, \dots, u_m\}$, $c_2 = \{v_1, v_5, v_9, \dots\}$, $c_3 = \{v_3, v_7, v_{11}, \dots\}$, and $c_4 = \{v_{m(n-1)-1}\}$, if $m(n-1) \equiv 2(\text{mod } 4)$, or $c_4 = \{v_{m(n-1)}\}$, if $m(n-1) \equiv 1(\text{mod } 4)$ or $m(n-1) \equiv 3(\text{mod } 4)$.

Subcase 2.2 If $n \equiv 0(\text{mod } 2)$, $n \neq 6$, $m \not\equiv 2(\text{mod } 4)$, $m(n-1) \not\equiv 0(\text{mod } 4)$, define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4). \end{cases}$$

Assign the fourth color as in the previous subcase. For the pendant vertices,

$$c(u_i) = \begin{cases} 1, & i \equiv 1(\text{mod } 2), \\ 4, & i \equiv 0(\text{mod } 2). \end{cases}$$

Thus, $c_1 = \{v_2, v_4, v_6, \dots\} \cup \{u_1, u_3, u_5, \dots, u_{m-1}\}$, $c_2 = \{v_1, v_5, v_9, \dots\}$, $c_3 = \{v_3, v_7, v_{11}, \dots\}$, while $c_4 = \{v_{m(n-1)-1}\} \cup \{u_2, u_4, u_6, \dots\}$, for $m(n-1) \equiv 2(\text{mod } 4)$, or $c_4 = \{v_{m(n-1)}\} \cup \{u_2, u_4, u_6, \dots\}$, for $m(n-1) \equiv 1(\text{mod } 4)$ or $m(n-1) \equiv 3(\text{mod } 4)$.

Subcase 2.3 If $n \equiv 0(\text{mod } 2)$ and $m(n-1) \equiv 0(\text{mod } 4)$, define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4). \end{cases}$$

For the pendant vertices,

$$c(u_i) = \begin{cases} 1, & i \equiv 1(\text{mod } 2), \\ 4, & i \equiv 0(\text{mod } 2). \end{cases}$$

Consequently, $c_1 = \{v_2, v_4, v_6, \dots, v_{m(n-1)}\} \cup \{u_1, u_3, \dots, u_{m-1}\}$, $c_2 = \{v_1, v_5, v_9, \dots, v_{m(n-1)-3}\}$, $c_3 = \{v_3, v_7, \dots, v_{m(n-1)-1}\}$, and $c_4 = \{u_2, u_4, u_6, \dots, u_m\}$.

Subcase 2.4 If $m \equiv 2(\text{mod } 4)$ and $n = 4$, define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\text{mod } 2), \\ 2, & i \equiv 1(\text{mod } 4), \\ 3, & i \equiv 3(\text{mod } 4), \\ 4, & i = m(n-1) - 1. \end{cases}$$

For the pendant vertices,

$$c(u_i) = \begin{cases} 1, & i \equiv 1(\text{mod } 2), \\ 2, & i = m, \\ 4, & i \equiv 0(\text{mod } 2) \text{ and } i \neq m. \end{cases}$$

Subcase 2.5 If $m \equiv 2(\text{mod } 4)$ and $n = 6$, define

$$c(v_i) = \begin{cases} 1, & i \equiv 0(\bmod 2), \\ 2, & i \equiv 1(\bmod 4), \\ 3, & i \equiv 3(\bmod 4) \text{ or } i = m(n-1) - 1, \\ 4, & i = m(n-1) - 3. \end{cases}$$

For the pendant vertices,

$$c(u_i) = \begin{cases} 1, & i \equiv 1(\bmod 2), \\ 3, & i = m, \\ 4, & i \equiv 0(\bmod 2). \end{cases}$$

In each of the above subcases, every pair of distinct vertices $u, v \in c_i$ satisfies $d(u, v) \geq i + 1$. Hence, all color classes satisfy the packing coloring condition. Therefore, $\chi_\rho(CV_{m,n}) \leq 4$.

Subcase 2.6: $L \not\equiv 0(\bmod 4)$

Assume, for contradiction, that $\chi_\rho(CV_{m,n}) < 4$. Then there exists a packing coloring $c: V(CV_{m,n}) \rightarrow \{1, 2, 3\}$. Since the cycle graph C_L is a subgraph of $CV_{m,n}$, and $L \not\equiv 0(\bmod 4)$, Lemma 2 implies that $\chi_\rho(C_L) = 4$. By Lemma 1, $\chi_\rho(C_L) \leq \chi_\rho(CV_{m,n})$, which yields $4 \leq \chi_\rho(CV_{m,n})$. This contradicts the assumption that only three colors are sufficient. Therefore, $\chi_\rho(CV_{m,n}) \geq 4$.

Subcase 2.7: $L \equiv 0(\bmod 4)$ and n is even

Assume that there exists a packing coloring $c: V(CV_{m,n}) \rightarrow \{1, 2, 3\}$. Since $L = m(n-1) \equiv 0(\bmod 4)$, Lemma 2 gives $\chi_\rho(C_L) = 3$. Thus, the vertices on the cycle may be colored periodically by the pattern 2131, ..., 2131. Because n is even, each pendant vertex u_i is adjacent alternately to vertices of odd and even indices on the cycle. In particular, some pendant vertices become adjacent to vertices colored 1. Hence, such pendant vertices cannot receive color 1. Consider the pendant vertex u_2 . Since u_2 is adjacent to a vertex colored 1, it follows that $c(u_2) \neq 1$. Moreover, $d(v_{n-1}, u_2) = 2$, which implies $c(u_2) \neq 2$. Similarly, $d(v_{n+1}, u_2) \leq 3$, and therefore $c(u_2) \neq 3$. Consequently, $c(u_2) \notin \{1, 2, 3\}$, a contradiction. Hence, at least four colors are required, implying that $\chi_\rho(CV_{m,n}) \geq 4$ for $L \equiv 0(\bmod 4)$ and $n \equiv 0(\bmod 2)$. Combining the upper and lower bounds, we conclude that $\chi_\rho(CV_{m,n}) = 4$ for $L \not\equiv 0(\bmod 4)$, or $L \equiv 0(\bmod 4)$ and $n \equiv 0(\bmod 2)$. ■

Theorem 3 shows that the packing chromatic number of $CV_{m,n}$ is governed by the cycle length $L = m(n-1)$ together with the parity of the cluster order n . Since C_L is a subgraph of $CV_{m,n}$, it follows that $\chi_\rho(C_L) \leq \chi_\rho(CV_{m,n})$, so the underlying cycle provides a natural lower bound. When $L \equiv 0(\bmod 4)$ and $n \equiv 1(\bmod 2)$, the periodic packing-coloring pattern of C_L remains compatible with the regularly distributed pendant vertices, and therefore $\chi_\rho(CV_{m,n}) = 3$. In contrast, if $L \not\equiv 0(\bmod 4)$, the cycle itself requires at least four colors; moreover, when $L \equiv 0(\bmod 4)$ and $n \equiv 0(\bmod 2)$, the pendant attachments obstruct the extension of a three-color packing pattern from C_L , yielding $\chi_\rho(CV_{m,n}) = 4$. This finding is consistent with previous studies on cycle-related graphs, which emphasize the influence of cycle length and periodic coloring patterns, while extending them by showing that the parity-dependent spacing of pendant attachments is equally decisive. The result also suggests that the same lower-bound-and-extension approach may be applied to other unicyclic graphs with nonuniform cluster sizes, different attachment intervals, or multiple pendant structures attached periodically along a cycle.

Taken together, the results for $TDPC_n$, J_n , and $CV_{m,n}$ show that the unique cycle contained in each graph provides the fundamental lower bound for its packing chromatic number, namely $\chi_\rho(C_k) \leq \chi_\rho(G)$, where C_k denotes the cycle subgraph of the corresponding unicyclic graph G . This finding is consistent with previous studies on cycles and cycle-related graphs, which identify the periodic packing-coloring pattern of the underlying cycle as a principal factor in determining $\chi_\rho(G)$. However, the present results refine this perspective by showing that the sharpness of the cycle lower bound depends on the type, position, and spacing of the attached structures. In $TDPC_n$, the decisive feature is the concentration of two pendant pairs at consecutive cycle vertices; in J_n , it is the compatibility of the cycle-coloring pattern with two symmetric pendant paths; and in $CV_{m,n}$, it is the periodic placement of pendant vertices determined by the parameters m and n . Therefore, these results not only support earlier findings on the central role of cycles in packing coloring, but also extend them by identifying structural conditions under which an optimal packing coloring of a cycle can or cannot be extended to the entire unicyclic graph. This viewpoint suggests a broader framework for future studies, in which the packing chromatic number of the unique cycle is first used as a lower bound and the extension of an optimal cycle coloring is then examined across more general attachments, such as pendant paths of unequal lengths, multiple leaves incident with the same cycle vertex, rooted trees placed at regular or irregular intervals, and mixed attachment structures. Such generalizations may lead to structural criteria expressed in terms of cycle length, distances between attachment vertices, and the local form of the attached subgraphs.

D. CONCLUSION AND SUGGESTIONS

This study establishes the exact packing chromatic numbers of the Jellyfish graph J_n , the Caveman graph $CV_{m,n}$, and the newly introduced Twin Double Pendant Cycle graph $TDPC_n$. The results show that $\chi_\rho(J_n) = 3$ for $n = 3$ and $\chi_\rho(J_n) = 4$ for $n \geq 4$, while $\chi_\rho(TDPC_n) = 4$ for every $n \geq 3$. For $CV_{m,n}$, the value of $\chi_\rho(CV_{m,n})$ is determined by the divisibility of $m(n - 1)$ by four and the parity of n . The main theoretical contribution of this work is the identification of how cycle structure and pendant arrangements jointly govern packing-color feasibility in unicyclic graphs. In particular, the introduction of $TDPC_n$ enlarges the class of unicyclic graphs with known exact packing chromatic numbers, while the characterizations of J_n and $CV_{m,n}$ reveal distinct effects produced by pendant paths and periodically distributed pendant vertices. These findings provide a structural basis for further investigations rather than merely adding isolated exact values. Future studies may consider graphs containing more than one cycle, including bicyclic, cactus, and other multi-cyclic families, as well as graphs produced by operations such as corona products, Cartesian products, amalgamations, subdivisions, and graph powers. Packing coloring on derived constructions, particularly token graphs, line graphs, and total graphs, also warrants investigation because these constructions substantially alter the relevant distance relations. Another promising direction is the development of exact, parameterized, or computational algorithms for determining $\chi_\rho(G)$ in broader graph classes.

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