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Game Chromatic Number of Shackle Graphs

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ABSTRACT

Coloring vertices on graph is one of the topics of discrete mathematics that are still developing until now. Exploration Coloring vertices develops in the form of a game known as a coloring game. Let G graph. The smallest number k such that the graph G can be colored in a coloring game is called game chromatic number. Notated as $\chi_g(G)$. The main objective of this research is to prove game chromatic numbers from graphs $\mathrm{shack}(K_n, v_i, t)$,$\mathrm{shack}(S_n, v_i, t)$, and $\mathrm{shack}(K_{n,n}, v_i, t)$. The research method used in this research is qualitative. The result show that $\chi_g(\mathrm{shack}(K_n, v_i, t)) = n$, and $\chi_g(\mathrm{shack}(S_n, v_i, t)) = \chi_g\left(\mathrm{shack}(K_{n,n}, v_i, t)\right) = 3$. The game chromatic number of the shackle graph depends on the subgraph and linkage vertices. Therefore, it is necessary to make sure the vertex linkage is colored first.

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A. INTRODUCTION

Vertice coloring game on a graph is one of the topics of discrete mathematics. This topic continues to develop and be implemented in various fields. The application of coloring vertices is widely used in everyday life such as scheduling problems (Chierichetti, Kleinberg, & Panconesi, 2012; Maarif, 2017), setting traffic lights (Diana, Suryaningtyas, & Suprapti, 2016), even in the games (Bodlaender, 1991; Mycielski, 1992; Dunn, Larsen, Lindke, Retter, & Toci, 2015; Mujib, 2019).

Vertices coloring game was first introduced by Bodlaender (Bodlaender, 1991) who studied the vertices coloring on a Cartesian product graph. Research on these vertices coloring game continues to develop today. Various classes of graphs have been studied such as planar graph(Xuding, 1999), tree graph (Zhu, 2000), Cartesian product graph class (Bartnicki et al., 2008; Mujib & Assiyatun, 2011), Cactus graph (Firmansyah & Mujib, 2013), Dense random graphs (Heckel, 2014), random hypergraph (Bohman, Frieze, & Sudakov, 2019), Caterpillars graph (Furtado, Dantas, De Figueiredo, & Gravier, 2019), and Pot Bunga and Palm tree graphs (Mujib, 2019). In addition, there are many other graph classes that become the subject of research on vertices coloring games. The interesting thing in this game is how to determine the smallest color so that all vertices on the graph are colored. This number is called the game chromatic number. Research on chromatic numbers continues to develop today. Starting from graph chromatic numbers, game chromatic numbers (Bodlaender, 1991), Local irregularity chromatic numbers (Umilasari, Susilowati, & Slamin, 2020), to quantum chromatic numbers (Paulsen & Todorov, 2015).

Let graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ where V is the set of vertices on the graph and E is the set of edges on the graph G. The set $\boldsymbol{C} = \{\mathbf{1}, \mathbf{2}, \mathbf{3} \ldots, \boldsymbol{k}\}$is the set of colors provided in the vertex coloring game. Consider two people playing coloring on a G graph. The first person is A (Alice) and the second person B (Bob). A is the first person to color the vertices on G. Whereas B, the second person to color the vertices on G. A aims to color all the points on G so that each neighboring vertex has a different color. Meanwhile, B aims to color the vertices so that at least one vertex cannot be colored with a different color from its neighbors. k, the smallest color so that all the vertices in G can be colored in the game of coloring vertices at G, called games chromatic numbers, which are denoted by $\boldsymbol{\chi_g(G)}$.

It has been proven before, that several graph classes have proved the game chromatic number. However, there are still other interesting graph classes where the chromatic number of the game has not been found. One of the interesting graphs to explore the chromatic number of the game is the resulting graph from the *shackle* operation. The shackle operation is an operation between two or more graphs which results in a new graph. The graph resulting from the shackling operation or called the shackle graph is denoted $Shack(G_1, G_2, \ldots, G_t)$ is a shackle graph that is formed from t, a copy of the graph G is denoted by $Shack(G, t)$ with $t \geq 2$ and t is natural number (Maryati, Salman, Baskoro, Ryan, & Miller, 2010a). The shackle operation in this research is vertex *shackle*. Vertex shackle operations is denoted by $Shack(G, v, t)$ it means that the graph is constructed from any graph G as many as t copies and point v as the *linkage vertex*. Shackle graphs have been widely used as the subject of discrete mathematics research, one of which is (Saifudin, 2020) research which examines the power domination number on shackle graph. However, research on game chromatic numbers on shackle graphs has not been found. Therefore, this study aims to explore game chromatic numbers in several classes of shackle graphs.

This study examines and proves the game chromatic number of the graph class resulting from the *shackle* operation. The studied graphs are $shack(K_n, v_i, t)$, graphs $shack(S_n, v_i, t)$, and graphs $shack\big(K_{n,n}, v_i, t\big)$. The graph $shack(K_n, v_i, t)$is a complete graph K_n which is copied as many as t with the connecting vertex v_i where $i = 1,2,3, \ldots, n$. While graph $shack(S_n, v_i, t)$ is circle graph with n vertex, which is copied as many as t with the connecting point v_i where $i = 1,2,3, \ldots, n$. And graph $shack\big(K_{n,n}, v_i, t\big)$ is *bipartite* complete graph which is copied as many as t with the connecting point v_i where $i = 1,2,3, \ldots, n$.

Suppose $k \geq 2$ is integers. Define a shackle as a graph construction by connected non-trivial graphs $G_1, G_2, G_3, \ldots, G_k$ therefore G_s and G_t don't have a common vertex for each $s, t \in [1, k]$ with $|s - t| \geq 2$ and for each $i \in [1, k - 1] G_i$ and G_{i+1} has exactly one common vertex called the linkage vertex, and $k - 1$ the linkage vertex is different. Shackle graph is donated by $Shack(G_1, G_2, \ldots, G_k)$ (Maryati, Salman, Baskoro, Ryan, & Miller, 2010b). Based on the above definition, here is the graph construction resulting from the shackle graph operation K_n, S_n, dan $K_{n,n}$.

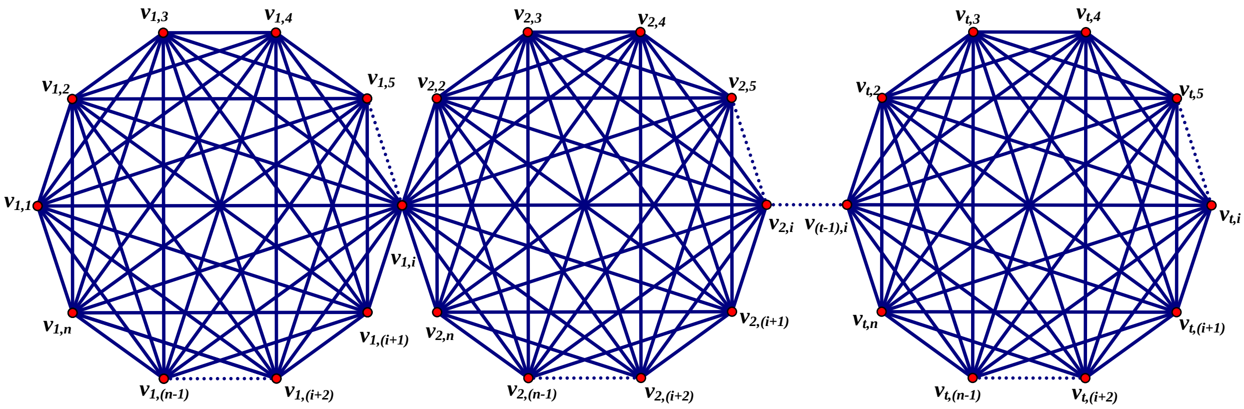

Figure 1. Graph $\boldsymbol{shack(K_n, v_{(j,i)}, t)}$

Based on Figure 1, Without loss of generality, for example, we select point $\boldsymbol{v_{(1,i)}}$, then Graph $\boldsymbol{shack(K_n, v_{(j,i)}, t)}$ is a graph that has a set of vertices $\boldsymbol{V = \{v_{(j,k)}: 1 \le j \le t,\ 1 \le k \le n, k \ne i\} \cup \{v_{(j,k)}: 1 \le j \le t, k = i\}}$ and set of edges $\boldsymbol{E = \{e_{(1,k,l)} = (v_{(1,k)}v_{(1,l\ne k)}): 1 \le k \le n, 1 \le l \le n\ \} \cup \{e_{(j,k,l)} = (v_{(j,k)}v_{(j,l\ne k)}): 2 \le j \le t, 1 \le k, l \le n\} \cup \{e_{(j+1,i,l)} = (v_{(j,i)}v_{(j+1,l\ne i)}): 1 \le j \le t-1,\ 2 \le l \le n\}}$. Where $\boldsymbol{|V| = (n-1)t - 1}$ and $\boldsymbol{|E| = tC_2^{n+1}}$. In addition, we get $\Delta\left(\boldsymbol{shack(K_n, v_{(j,i)}, t)}\right) = \boldsymbol{2(n-1)}$ and $\boldsymbol{\delta\left(shack(K_n, v_{(j,i)}, t)\right) = n - 1}$ (Firmansyah & Mujib, 2021).

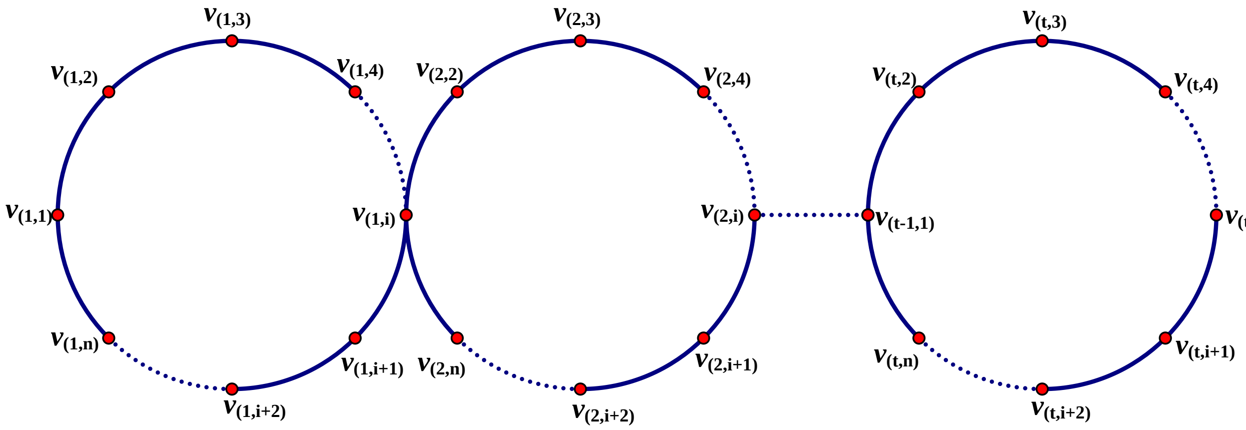

Figure 2. Graph $\boldsymbol{shack(S_n, v_{(j,i)}, t)}$

Figure 2 shows the construction of shack graph $shack(S_n, v_{(j,i)}, t)$. Graph $shack(S_n, v_{(j,i)}, t)$ has the set of vertices $V = \{\{v_{(j,k)}: 1 \le j \le t,\ 1 \le k \le n, k \ne i\} \cup \{v_{(j,k)}: 1 \le j \le t, k = i\}\}$ and set of edges $E = \{e_{(j,k)} = (v_{(j,k)}v_{(j,k+1)}): 1 \le j \le t, 1 \le k \le n\ \} \cup \{e_{(j+1,k)} = (v_{(j,i)}v_{(j+1,k)}): 1 \le j \le t, k = 2, n\ \}$. Obtained $|V| = (n-1)t - 1$, $|E| = tn$, $\Delta\left(shack(S_n, v_{(j,i)}, t)\right) = 4$, and $\delta\left(shack(S_n, v_{(j,i)}, t)\right) = 2$ (Firmansyah & Mujib, 2021).

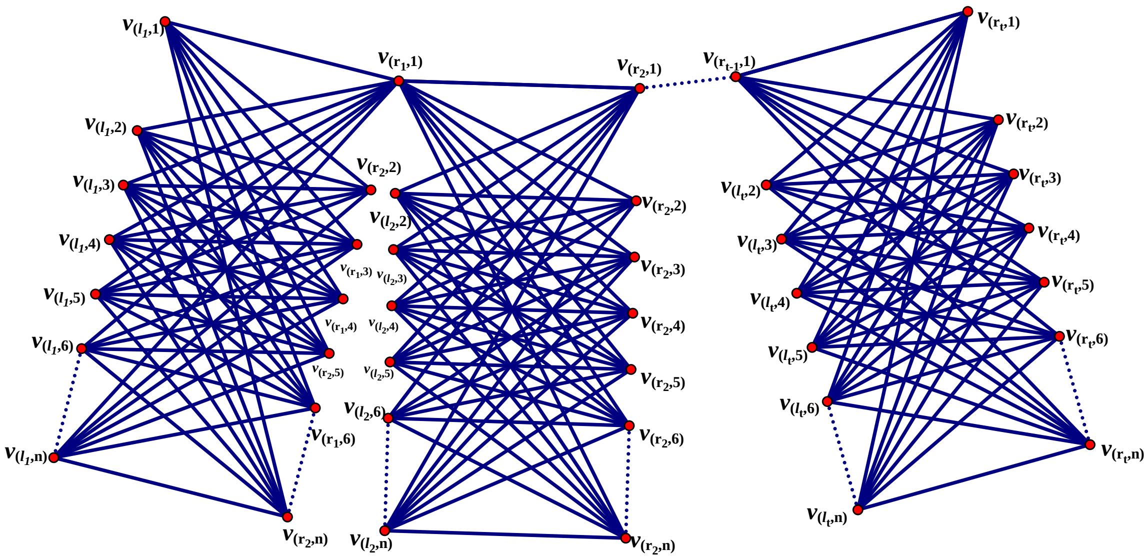

Figure 3. Graph $\boldsymbol{shack(K_{(n,n)}, v_{(j,i)}, t)}$

The shack graph construction $shack(K_{(n,n)}, v_{(j,i)}, t)$ is shown in Figure 3. The *shackle* graph $shack(K_{(n,n)}, v_{(j,i)}, t)$ has the set of vertices $V = \left\{\left\{v_{(l_j,k)}: 1 \le j \le t,\ 1 \le k \le n, k \ne i\right\} \cup \left\{v_{(j,k)}: 1 \le j \le t, k = i\right\}\right\}$ and the set of edges $E = \left\{e_{(j,k)} = \left(v_{(l_j,k)} v_{(r_j,i)}\right): 1 \le j \le t, 1 \le k, i \le n\ \right\} \cup \left\{e_{(j,k)} = \left(v_{(l_j,k)} v_{(r_{j+1},i)}\right): 1 \le j \le t-1, 1 \le k, i \le n\ \right\}$ .Obtained $|V| = 2nt$, $|E| = tn^2$ $\left(shack\ (K_{n,n}, v_{(j,i)}, t)\right) = 2n$ and $\delta\left(shack(K_{n,n}, v_{(j,i)}, t)\right) = n$ (Firmansyah & Mujib, 2021).

B. METHODS

This research is qualitative research with the aim of proofing game chromatic number theorems. The class of graphs that is studied is the resulting graph of shackle operation on the complete K_n, graph, the S_n, cycle graph, and the Complete Bipartite $K_{n,n}$ graph. In terms of notation, research determines, $\chi_g\left(shack(K_n, v_{(j,i)}, t)\right)$, $\chi_g\left(shack(S_n, v_{(j,i)}, t)\right)$, and $\chi_g\left(shack(K_{n,n}, v_{(j,i)}, t)\right)$. The research stages are:

1. Exploring the concept of shackle operations on graphs.
2. Constructing the generality from the graph $shack(K_n, v_{(j,i)}, t)$, $shack(S_n, v_{(j,i)}, t)$ and $shack(K_{n,n}, v_{(j,i)}, t)$.
3. Exploring and simulating coloring game strategies on $shack(K_n, v_{(j,i)}, t)$, $shack(S_n, v_{(j,i)}, t)$, and $shack(K_{n,n}, v_{(j,i)}, t)$
4. Making Conjecture of game chromatic numbers from graphs of $shack(K_n, v_{(j,i)}, t)$, $shack(S_n, v_{(j,i)}, t)$, and $shack(K_{n,n}, v_{(j,i)}, t)$.
5. Proving the game chromatic number on the graph $shack(K_n, v_{(j,i)}, t)$, $shack(S_n, v_{(j,i)}, t)$, and $shack(K_{n,n}, v_{(j,i)}, t)$.

C. RESULT AND DISCUSSION

The Previous research that is related to the game chromatic number is important to be revealed here as a theoretical basis for determining game chromatic numbers from resulting graphs of *shackle* operations. Bartnicki, et al. (2008) have proved that the game of chromatic number for any graph is in the range between the chromatic numbers with $\boldsymbol{\Delta(G)+1}$, in more detail is presented in the following theorem.

Theorem 1 (Bartnicki et al., 2008) If G is the graph with the largest degree $\Delta(G)$, so $\chi(G) \leq \chi_g(G) \leq \Delta(G)+1$

Proof:

Based on the definition of $\chi(G)$, if the provided color is less than $\chi(G)$, then the graph G cannot be colored with k points. Thus, the game chromatic number of graph G cannot be less than $\chi(G)$ so that $\chi_g(G) \geq \chi(G)$ is obtained.

Then, if the first player can color in such a way that for each point $v \in V(G)$ and each neighbor of v has a different color, then the first player will always win. This can be done if the color $\Delta(G)+1$ is available. So we get $\chi_g(G) \leq \Delta(G)+1$.

Thus, $\chi(G) \leq \chi_g(G) \leq \Delta(G)+1$ ■

This theorem gives the interval limitation of the game chromatic number of the graph G, where the lower limit of the game chromatic number is the chromatic number of the graph G with the upper limit being the largest degree plus one. For that, to determine the chromatic number of the game from graph G, first determine the chromatic number of the graph G. Furthermore, this theorem becomes the basis for researchers to determine the chromatic number from the results of the *shackle* operation which is proven in the following theorem.

1. Game Chromatics Number of $\boldsymbol{shack(K_n, v_{(j,i)}, t)}$

Theorem 2. $\chi_g\left(shack(K_n, v_{(j,i)}, t)\right) = n$

Proof:

Based on Figure 1, the game chromatic number will be determined from the graph $shack(K_n, v_{(j,i)}, t)$. It is known that $\chi\left(shack(K_n, v_{(j,i)}, t)\right) = n$ and $\Delta\left(shack(K_n, v_{(j,i)}, t)\right) = 2(n-1)$, based on Theorem 1, then $n \leq \chi_g\left(shack(K_n, v_{(j,i)}, t)\right) \leq 2(n-1)$. Suppose given the color set $C = \{1,2,3,\dots,n\}$. Since $d\left(v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i\right) = n-1$, then with n available colors vertex $v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i$ will always be colorable. For this reason, the main strategy is to determine the vertex $v_{(j,k)}: 1 \leq j \leq t, k = i$ is colored first. Since $d\left(v_{(j,k)}: 1 \leq j \leq t, k = i\right) = 2(n-1)$. Therefore, it will be shown that the first player (A) always has a chance to color the vertex $v_{(j,k)}: 1 \leq j \leq t, k = i$ with the color set C.

Without loss of generality, suppose A colors vertex $v_{(1,i)}$ with color 1. Then vertex $v_{(2,i)}$ cannot be colored with color 1. Because $v_{(2,i)}$ is adjacent to $v_{(1,i)}$. However, $v_{(1,i)}$ is independent of $v_{(j,k)}: 3 \leq j \leq t, k = i$. For this reason, A always has a chance to color the vertex $v_{(j,k)}: 3 \leq j \leq t, k = i$ with the same color as $v_{(1,i)}$. There are two possible first moves of the second player (B):

Case 1. If B colors vertex $v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i$
Without loss of generality, suppose vertex $v_{1,1}$ with color 2, then A will color vertex $v_{(j,i)}, 3 \leq j \leq t$. Player A always has chance to color vertex $v_{(j,i)}, 3 \leq j \leq t$ before the vertex $v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i$. As a result, all vertices will always be able to be colored with n color. Thus $\chi_g\left(shack\left(K_n, v_{(j,i)}, t\right)\right) \leq n$.

Case 2. If B colors vertex $v_{(j,i)}, 2 \leq j \leq t$.
Without loss of generality, suppose vertex $v_{(2,i)}$ with color 2, it will ease A to ensure to color vertex $v_{(j,i)}, 2 \leq j \leq t$ before the other vertices. So that, the next step A will keep coloring vertex $v_{(j,i)}, 3 \leq j \leq t$. Player A always has chance to color vertex $v_{(j,i)}, 3 \leq j \leq t$ before the vertices $v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i$ are colored all. As a result, all vertices will always be able to be colored with n color. Thus $\chi_g\left(shack\left(K_n, v_{(j,i)}, t\right)\right) \leq n$.
Based on the both case and theorem 1, then $n \leq \chi_g\left(shack\left(K_n, v_{(j,i)}, t\right)\right) \leq n$.
Thus, $\chi_g\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n$ ∎

2. Game Chromatic Number of $\boldsymbol{shack\left(S_n, v_{(j,i)}, t\right)}$

Theorem 3. $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 3$
Proof:
Based on Figure 2, the game chromatic numbers will be determined from graph $shack\left(S_n, v_{(j,i)}, t\right)$. It is known that $\chi\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 3$ and $\Delta\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 4$, based on theorem 1, thus $3 \leq \chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) \leq 4$.
Will be shown that $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) \geq 3$. Suppose color set is given $\{1,2\}$. Without loss of generality, suppose A colors vertex $v_{(j,i)}$ with color 1, then B will color vertex $v_{(j,i+2)}$ with color 2. As a result, vertex $v_{(j,i+1)}$ can be colored. Therefore $3 \leq \chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) \leq 4$.
Will be shown $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 3$.
Suppose the set of colors is given $C = \{1,2,3\}$. Since $d\left(v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i\right) = 2$, thus with 3 available colors vertices $v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i$ can be always be colored. So that, the main strategy is making sure the vertices $v_{(j,k)}: 1 \leq j \leq t, k = i$ are colored first.

Since $d\left(v_{(j,k)}: 1 \leq j \leq t, k = i\right) = 4$. Therefore, will be shown that first player (A) always has chance to color vertex $v_{(j,k)}: 1 \leq j \leq t, k = i$ with the color set C.
Without loss of generality, suppose A colors vertex $v_{(1,i)}$ with color 1. Since $v_{(1,i)}$ is separated from each other $v_{(j,k)}: 2 \leq j \leq t, k = i$. For that, A always has chance to color vertex $v_{(j,k)}: 2 \leq j \leq t, k = i$ with the same color as $v_{(1,i)}$. There are two possible steps of the second player (B):

Case 1. If B colors vertex $v_{(j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n, k \neq i$
Since $v_{(1,i)}$ is colored with color 1, the vertex that is possibly colored by B is vertex $v_{(j,i-1)}, v_{(j,i+1)}, v_{(j+1,2)}\ atau\ v_{(J+1,n)}, 2 \leq j \leq t$. Without loss of generality, suppose vertex $v_{(2,i-1)}$ with color 2, then A will color vertex $v_{(2,i)}$ with color 1. Thus, player A always has chance to color vertex $v_{(j,i)}, 1 \leq j \leq t$ before its neighbor's vertex is colored. Thus $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) \leq 3$.

Case 2. If B colors vertex $v_{(j,i)}, 2 \leq j \leq t$.
Without loss of generality, suppose vertex $v_{(2,i)}$ with color 2, it will ease A to ensure coloring vertex $v_{(j,i)}, 2 \leq j \leq t$ before the other vertices. So that, the next step A will keep coloring vertex $v_{(j,i)}, 3 \leq j \leq t$. Regardless of the next move of player B. Player A always has chance to color vertex $v_{(j,i)}, 3 \leq j \leq t$ before the neighboring vertex is colored. As a result, all the vertices can always be colored with 3 colors. Thus, $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) \leq 3$.
Based on the both cases, thus$3 \leq \chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) \leq 3$.
Thus, $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 3$ ■

3. Game Chromatics Number of $\boldsymbol{shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)}$

Theorem 4. $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 3$
Proof:
The game of chromatics numbers will be determined by the graph $shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$
Noted that $\chi\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 2$ and $\Delta\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 2n$, based on theorem 1, thus $2 \leq \chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) \leq 2n$.
It will be shown that $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) \geq 3$. Suppose the set of colors is given $\{1,2\}$. Without loss of generality, suppose A colors vertex $v_{(r_j,i)}$ with color 1, then B will color vertex $v_{(l_j,i+2)}$ with color 2. As the result, vertex $v_{(r_j,i+1)}$ cannot be colored. Therefore, $3 \leq \chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) \leq 2n$. Will be shown $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 3$.

Suppose it is given the color set $C = \{1,2,3\}$. Since $d\left(v_{(l_j,k)}: 1 \leq j \leq t,\ 1 \leq k \leq n,\right) = n$ and $\left(v_{(r_j,k)}: 1 \leq j \leq t,\ 2 \leq k \leq n\right) = n$ are separated each other, thus, with the three colors available, those vertices are always able to be colored. For that, the main strategy is making sure that vertex $v_{(r_j,1)}: 1 \leq j \leq t$ is colored first. Since $d\left(v_{(r_j,1)}: 1 \leq j \leq t\right) = 2n$, thus, first player (A) always has chance to color point $v_{(r_j,1)}: 1 \leq j \leq t$ with the color set C.
Without loss of generality, suppose A colors vertex $v_{(r_1,1)}$ with color 1. Since $v_{(r_1,1)}$ is separated each other to $v_{(r_j,1)}: 3 \leq j \leq t$ thus, A always have chance to color vertex $v_{(r_j,1)}: 3 \leq j \leq t$ with the same color as $v_{(r_1,1)}$. There are two possible first steps of second player (B):

Case 1. If B color other than the vertex $v_{(r_j,1)}: 1 \leq j \leq t$
Since $v_{(r_1,1)}$ is colored with color 1, the vertex that is possibly colored by B is vertex $v_{(l_2,k)}, 2 \leq k \leq n$ or $v_{(r_3,k)}, 2 \leq k \leq n$. Without loss of generality, suppose point $v_{(l_2,k)}$ with color 2, thus A will color vertex $v_{(r_2,1)}$ with color 3. Therefore, Player A has chance to color vertex $v_{(r_j,1)}, 1 \leq j \leq t$ before all the neighbors are colored. Thus, $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) \leq 3$.

Case 2. If B color vertex $v_{(r_j,1)}: 1 \leq j \leq t$.
Without loss of generality, suppose vertex $v_{(r_2,1)}$ with color 2, it will ease A to ensure coloring vertex $v_{(r_j,1)}: 1 \leq j \leq t$ before the other vertices. For that, the next step of A will keep coloring vertex $v_{(r_j,1)}: 1 \leq j \leq t$. Regardless of the next step of player$B$. Player A always has chance to color vertex $v_{(r_j,1)}: 1 \leq j \leq t$ before the neighboring vertex is colored. As a result, all vertices can always be colored with 3 colors. Thus $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) \leq 3$.
Based on the both cases, thus $3 \leq \chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) \leq 3$.
Thus, $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 3$ ■
Based on the research results, it is known that $\chi_g\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n$, $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 3.$, and $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 3$. This shows that the game chromatic number of the shackle graph depends on the chromatic number of the constructing subgraph. In addition, the vertex linkage which has the maximum degree must be colored first. Several classes of graphs resulting from operations between graphs have been shown to have the game chromatic number. For example, a cartesian product graph (Bartnicki et al., 2008), a tensor product graph (Mujib & Assiyatun, 2011), an amalgamation graph (Maryati et al., 2010b), and even a hypergraph (Bohman et al., 2019). Therefore, this study contributes to the class of graphs that have proven the chromatic number of the game. For further research, it is expected to prove the generalization of game chromatic numbers from shackle graphs.

D. CONCLUSION AND SUGGESTIONS

Based on the research, it is obtained that game chromatics number of shackle graph $\chi_g\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n$. Game chromatics number of shackle graph $\chi_g\left(shack\left(S_n, v_{(j,i)}, t\right)\right) = 3$. Game chromatics number of shackle graph $\chi_g\left(shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)\right) = 3$. The strategy that determines the chromatics number of game changes from the constructing graph K_n, S_n, and $K_{n,n}$. The graph character is almost the same. Therefore, the strategy is generally the same. That is ensuring the point with the highest degree is colored. Suggestions for researchers, who are interested in studying game chromatic numbers, to generalize game chromatic numbers from shackle graphs.

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