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The Characteristics of the First Kind of Chebyshev Polynomials and its Relationship to the Ordinary Polynomials

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ABSTRACT

In this article, we discuss the Chebyshev Polynomial and its characteristics. The second order difference equation and the process obtaining the explicit solution of the Chebyshev polynomial have been given for each real number. The symmetry and orthogonality of the Chebyshev polynomial has also been demonstrated using the explicit solutions obtained. Furthermore, we have also given how to approx the polynomial function using the Chebyshev polynomials.

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A. INTRODUCTION

The Chebyshev polynomials is a special orthogonal polynom like Legendre polynomials and Hermite polynomials. This polynom firstly published by Chebishev in 1854. Indeed, Chebishev was a mathematician who first popularized the analysis of this polynom. In mathematical analysis, Chebishev polynomials can be applied to solve Fredholm integral equation (Liu, 2009) and the characterization of analytical functions, (Dziok et al., 2015). This polynom also quite good for solving encryption key in cryptography (Bergamo et al., 2005). The application of Chebyshev polynomials in Statistics such us to estimate a parameter using Maximum Likelihood Estimation (MLE) (Jajang, 2019). Others applications of Chebyshev polynomial can be seen in some articles, such us (Kafash et al., 2012), (Khader, 2012), (Sedaghat et al., 2012), (Montijano et al., 2013), and (Capozziello et al., 2018).

The Chebyshev can be defined from the solution of Chebyshev differential equation as follows

$$(1-x^2)\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + n^2y = 0, \tag{1}$$

when $n = 0,1,2,3...$(Mason & Handscomb, 2002). This equation is a second orde differential equation with variable coefficients. The solution of this equation can be obtained by transformating it.

Suppose $x = \cos(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{-1}{\sin(t)}\right). \tag{2}$$

By deriving again the equation (2) with respect to x, it can be

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \left(\frac{1}{\sin^2(t)}\right)\frac{d^2y}{dt^2} - \frac{\cot(t)}{\sin^2(t)}\frac{dy}{dt}. \tag{3}$$

Subtituting (2) and (3) to (1), then it can be obtained the second orde differential equation as follows

$$\frac{d^2y}{dt^2} + n^2y = 0. \tag{4}$$

The general solution of (4) is

$$y(t) = Acos(nt) + Bsin(nt),$$

or

$$y(x) = Acos\left(ncos^{-1}(x)\right) + Bsin\left(ncos^{-1}(x)\right),$$

for $|x| < 1$.

Let $T_n(x) = cos\left(ncos^{-1}(x)\right)$ and $U_n(x) = sin\left(ncos^{-1}(x)\right)$, so T_n and U_n respectively can be defined as the first kind of Chebyshev polynomials and the second kind of Chebyshev polynomials for $|x| < 1$.

Suppose that $x = \cos h(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{1}{\sinh(t)}\right). \tag{5}$$

By deriving again the equation (2) with respect to x, it can be

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \left(\frac{1}{\sinh^2(t)}\right)\frac{d^2y}{dt^2} + \frac{\cosh(t)}{\sinh^3(t)}\frac{dy}{dt}. \tag{6}$$

Subtituting (5) and (6) to (1), then it can be obtained the second orde differential equation as follows:

$$\frac{d^2y}{dt^2} - n^2y = 0. \tag{7}$$

The general solution of (7) is

$$y(t) = Acosh(nt) + Bsinh(nt),$$

or

$$y(x) = Acosh\left(ncosh^{-1}(x)\right) + Bsinh\left(ncosh^{-1}(x)\right),$$

for $|x| \geq 1$.

Let $T_n(x) = cosh\left(ncosh^{-1}(x)\right)$ and $U_n(x) = sinh\left(ncosh^{-1}(x)\right)$, then T_n and U_n repectively are defined as the first kind of Chebyshev polynomials and the second kind of Chebyshev polynomials for $|x| \geq 1$. Thus it can be concluded that the definition of the first kind of Chebyshev polynomial is

$$T_n(x) = \begin{cases} cos\left(ncos^{-1}(x)\right); & |x| < 1 \\ cosh\left(ncosh^{-1}(x)\right) & ; |x| \geq 1. \end{cases} \tag{8}$$

Beside the first and the second kind, this polynomial has the third kind and the fourth, (Eslahchi et al., 2012). The general form of Chebyshev polynomial and its modifications can be seen in (Bhrawy & Alofi, 2013), (Cesarano, 2014), (Sweilam et al., 2015), (Hassani et al., 2019), and (Salih & Shihab, 2020). In this article, the study discussed is limited to the first kind of Chebyshev polynomial.

B. METHODS

This research was conducted based on a literature study in the form of books and scientific journals, especially those related to the Chebyshev polynomial. The initial study conducted was to study the definition of Chebishev polynomial as a solution to a differential equation. Then discussed the form of the explicit solution of the Chebyshev polynomial and some of the characteristics of this polynomial. Furthermore, research was developed on the application of Chebyshev's polynomials to express the function of ordinary polynomials along with the theorems that strengthen this analysis.

C. RESULT AND DISCUSSION

1. Recursive Formula and Explicit Solution

Just like any other classical orthogonal polynomials, the Chebyshev polynomials in equation (8) explicitly can be obtained from the Rodrigue formula as follows

$$T_n(x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}(1-x)^{n-\frac{1}{2}}$$

(Koepf, 1999).

Some notes from this Rodrigue formula can be seen in (Qi et al., 2018). By using this formula, here is given some Chebyshev polynomials for several n values.

Table 1. Some Chebyshev Polynomials

n	$T_n(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$
5	$16x^5-20x^3+5x$
6	$32x^6-48x^4+18x^2-1$
7	$64x^7-112x^5+56x^3-7x$
8	$128x^8-256x^6+160x^4-32x^2+1$
9	$256x^9-576x^7+432x^5-120x^3+9x$
10	$512x^{10}-1280x^8+1120x^6-400x^4+50x^2-1$

However, the process to determine these solutions where using derivative to n will be difficult for big value of n. The explicit solution of Chebyshev polynomials can help to determine the Chebyshev polynomial of n. Other than used this derived form, the Chebyshev polynomial could be obtained by using a generating function than can be studied in (Cesarano, 2012). In (Cesarano, 2012) also has been given some characteristics of it. Other characteristics can be seen in (Kim et al., 2014). One of the characteristics of this polynomials is can be expressed in recursive form. The recursive formula is given in the following theorem.

Theorem 1 (Recursive Formula of the Chebyshev Polynomials). Suppose that $x = cos(t)$, then the recursive solution of the first kind Chebyshev polynomials $T_n(x)$ can be obtained from this difference equation

$$T_{n+2}(x) - 2x\,T_{n+1}(x) + T_n = 0. \tag{9}$$

Proof: For $|x| < 1,\ x = \cos(t)$, then

$$T_n(x) = T_n(t) = \cos(nt),$$
$$T_{n+1}(t) = \cos((n+1)t) = \cos(nt)\cos(t) - \sin(nt)\sin(t),$$
$$T_{n+2}(t) = \cos((n+2)t) = \cos(nt)\cos(2t) - \sin(nt)\sin(2t).$$

Next, by using equations T_n, T_{n+1}, and T_{n+2}, we have

$T_{n+2}(t) - 2\cos(t)\,T_{n+1} + T_n$
$= \cos(nt)\cos(2t) - \sin(nt)\sin(2t) - 2\cos(nt)\cos^2 t + 2\sin(nt)\sin(t)\cos(t) + \cos(nt).$
$= \cos(nt)\,[\cos(2t) - 2\cos^2(t)] - \sin(nt)\sin(2t) + 2\sin(nt)\sin(t)\cos(t) + \cos(nt).$

By using the fact that $\cos(2t) = 2\cos^2(t) - 1$ and $2\sin(t)\cos(t) = \sin(2t)$, so it can be obtained

$$T_{n+2}(t) - 2\cos(t)\,T_{n+1} + T_n = 0. \tag{10}$$

The equation (10) equivalent with the difference equation (9). Using the sama arguments, for case $|x| \geq 1|$, let $x = \cosh(t)$, then it can be proved this recursive formula also true. Therefore, Theorem 1 is proved. ■

From this recursive formula, by using the mathematical induction, it is easy to prove that the first kind of Chebyshev polynomials is a polynomial with orde n.

Theorem 2 (The Explicit Solution of Chebyshev Polynomials). The Chebyshev plynomials *T_n can be determined by*

$$T_n(x) = \begin{cases} \frac{1}{2}\left[\left(x - i\sqrt{1-x^2}\right)^n + \left(x + i\sqrt{1-x^2}\right)^n\right] & ; |x| < 1 \\ \frac{1}{2}\left[\left(x - \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^n\right] & ; |x| \geq 1 \end{cases}$$

Proof:

The equation (9) is an homogeneous difference equation with coefficients which do not depend on n. Thus it can be obtained a characteristic equation as follows

$$p^2 - 2xp + 1 = 0.$$

The characteristic values of this equation are $p_2 = x + \sqrt{x^2 - 1}$ (Kelley & Peterson, 2001). For case 1, if $|x| < 1$ then p_1 and p_2 are complex numbers. The form of p_1 and p_2 can be written in following form:

$$p_1 = x - i\sqrt{1-x^2} \text{ and } p_2 = x + i\sqrt{1-x^2}.$$

Furthermore, the value of $r = \sqrt{x^2 + (\sqrt{1-x^2})^2} = 1$ and $\theta = cos^{-1}(x)$.

The general solution of $T_n(x)$ for case 1 is

$$T_n(x) = r^n A\cos(n\theta) + B\,\mathrm{r}^n\sin(n\theta) = A\cos(n\cos^{-1}x) + B\sin(n\cos^{-1}x).$$

By using the initial function $T_0(x) = 1$ and $T_1(x) = x$ it is obtained $A = 1$ and $B = 0$.
The particular solution of $T_n(x)$ for case 1 is

$$T_n(x) = \cos(n\cos^{-1}x).$$

Notice that

$$\cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n = (x + i\sqrt{1-x^2})^n, \tag{11}$$
$$\cos(n\theta) - i\sin(n\theta) = (\cos(\theta) - i\sin(\theta))^n = (x - i\sqrt{1-x^2})^n. \tag{12}$$

The elimination of equation (11) and equation (12) above yield

$$\cos\big((n\theta)\big) = \cos(n\cos^{-1}x)$$
$$= \frac{1}{2}\left[\left(x - i\sqrt{1-x^2}\right)^n + \left(x + i\sqrt{1-x^2}\right)^n\right] = T_n(x).$$

For case 2, suppose $|x| \geq 1, p_1$ and p_2 are real numbers. The general solution of $T_n(x)$ for case 2 is

$$T_n(x) = C(x - \sqrt{x^2 - 1})^n + D(x + \sqrt{x^2 - 1})^n.$$

By using the initial function $T_0(x) = 1$ and $T_1(x) = x$ it is obtained $C = D = \frac{1}{2}$. Therefore, the particular solution of $T_n(x)$ is equation (11). ■

2. Some properties of Chebyshev Polynomials

Theorem 3 (Symmetricity). The Chebyshev polynomial $T_n(x)$ is an even function for even n and odd function for odd n. Succinctly, it can be written

$$T_n(-x) = (-1)^n T_n(x).$$

Proof:

By applying Theorem 2, for case 1 $|x| < 1$ then

$$T_n(-x) = \frac{1}{2}\left[\left(-x - i\sqrt{1-(-x)^2}\right)^n + \left(-x + i\sqrt{1-(-x)^2}\right)^n\right]$$

$$= \frac{1}{2}\left[\left((-1)(x + i\sqrt{1-x^2})\right)^n + \left((-1)(x - i\sqrt{1-x^2})\right)^n\right]$$

$$= \frac{1}{2}(-1)^n\left[\left(x - i\sqrt{1-x^2}\right)^n + \left(x + i\sqrt{1-x^2}\right)^n\right]$$

$$= (-1)^n T_n(x).$$

The same idea can be used for case 2 $|x| \geq 1$. Therefore Theorem 3 is proved. ■

Here, some curves of Chebyshev polynomials

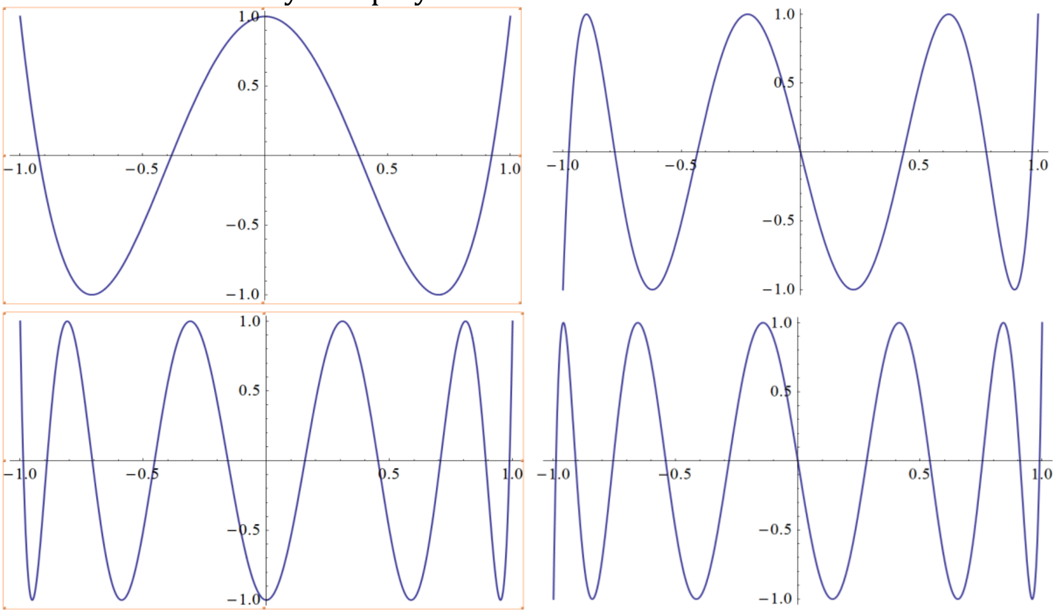

Figure 1. The curve of Chebyshev plynomials for $n = 4$ (left up), $n = 7$ (right up), $n = 10$ (left down), and $n = 11$ (right down).

Based on Figure 1, the curves for $n = 4$ and $n = 10$ are symmetric with respect to the Y Axis. This is due to even n, $T_n(x)$ is an even function. Meanwhile, for $n = 7$ and $n = 11$, the curves is symmetric with respect to the origin point. It is appropriate because $T_n(x)$ is an odd function for odd n.

Theorem 4. (Some Special Values). For each Chebyshev polynomial $T_n(x)$, these are valid:

a. $T_n(1) = 1$.

b. $T_n(1) = (-1)^n$.

c. $T_{2n}(0) = (-1)^n$.

d. $T_{2n+1}(0) = 0$.

Proof: Obviously, it can be proved by substituting the value of $x = 1$ and $x = 0$ to the explicit solution given by Theorem 2.

■

The Chebyshev polynomials are the orthogonal polynomials as well as Legendre polynomials and Hermit polynomials, (Boyd & Petschek, 2014). In (Atkinson, 1989), every orthogonal polynomial can be stated recursively as a second order difference equation. In this case, it is clear that the Chebyshev polynomial of this first type has been expressed in the second-order difference equation, namely in equation (10). The orthogonality of the Chebyshev polynomial is given in the following theorem.

Theorem 5. (The Orthogonality of Chebyshev Polynomials). Given the weight function $w(x) = (1 - x^2)^{-\frac{1}{2}}, for -1 < x < 1\ and\ m \neq n\ \in N\ apply$

$$\int_{-1}^{1} T_m(x)T_n(x)w(x)dx = 0.$$

Proof:

By using the definition $T_m(x) = cos(m\cos^{-1} x)$ and $T_n(x) = cos(n\cos^{-1} x)$, let $x = \cos(t)$ then

$$\int_{-1}^{1} T_m(x)T_n(x)w(x)dx = \int_{0}^{\pi} \cos(mt)\cos(nt)\frac{1}{\sqrt{1-\cos^2 t}}\sin(t)dt$$

$$= \int_{0}^{\pi} \cos(mt)\cos(nt)dt$$

$$= \int_{0}^{\pi} \frac{1}{2}[\cos(mt+nt) + \cos(mt-nt)]dt$$

$$= \frac{1}{2(m+n)}\sin\big((m+n)t\big) + \frac{1}{2(m-n)}\sin\big((m-n)t\big)\ |_0^{\pi} = 0.$$ ■

The same idea gives us $\int_{-1}^{1} T_m(x)T_n(x)w(x)dx = \frac{\pi}{2}$, for $m = n \neq 0$ and $\int_{-1}^{1} T_m(x)T_n(x)w(x)dx = \pi$, for $m = n = 0$.

3. The Relationship Between Chebyshev Polynomials and Ordinary Polynomials

The ordinary polynomials we have known have the following form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

These polynomials can be written as a linear combination of the Chebyshev polynomials. The rule to determine the Chebyshev polynomials using the ordinary polynomials can be obtained as follows

$$T_0 = 1 \Leftrightarrow\ 1 = T_0.$$

$$T_1 = x \ \Leftrightarrow\ x = T_1.$$

$$T_2 = 2x^2 - 1 \ \Leftrightarrow\ \ x^2 = \frac{1}{2}(1 + T_2) = \frac{1}{2}(T_0 + T_2).$$

$$T_3 = 4x^3 - 3x \ \Leftrightarrow\ \ x^3 = \frac{1}{4}(T_3 + 3x) = \frac{1}{4}(3T_1 + T_3).$$

$$T_4 = 8x^4 - 8x^2 + 1 \ \Leftrightarrow\ \ x^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4),$$

and so on. By subtituting the polynomials $1, x, x^2, x^3$ ..., we can write the ordinary polynomials as a linear combination of the Chebsyshev polynomials. Here, given some of those relationship.

Table 2. The Linear Combination of the Chebyshev Polynomials

$p(x)$	The Linear Combination of The Chebyshev Polynomials
a_0	a_0T_0
$a_0 + a_1x$	$a_0T_0 + a_1T_1$
$a_0 + a_1x + a_2x^2$	$\left(a_0 + \frac{a_2}{2}\right)T_0 + a_1T_1 + \left(\frac{a_2}{2}\right)T_2$
$a_0 + a_1x + a_2x^2 + a_3x^3$	$\left(a_0 + \frac{a_2}{2}\right)T_0 + \left(a_1 + \frac{3a_3}{4}\right)T_1 + \left(\frac{a_2}{2}\right)T_2 + \left(\frac{a_3}{4}\right)T_3$
$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$	$\left(a_0 + \frac{a_2}{2} + \frac{a_3}{8}\right)T_0 + \left(a_1 + \frac{3a_3}{4}\right)T_1 + \left(\frac{a_2}{2} + \frac{a_4}{2}\right)T_2 + \left(\frac{a_3}{8}\right)T_3 + \left(\frac{a_4}{8}\right)T_4$

For example the polynomial $p(x) = 3x^2 + 2x - 1$ has values $a_0 = -1, a_1 = 2$, and $a_2 = 3$, so $p(x) = \left(-1 + \frac{3}{2}\right)T_0 + 2T_1 + \frac{3}{2}T_2$. In general, the relationship between the ordinary polynomials and the first kind Chebyshev polynomials is given in the following theorem.

Theorem 6. The polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$ *with* $a_n \neq 0$ *can be written as a linear combination of the Chebyshev polynomials*

$$p_n(x) = b_0T_0 + b_1T_1 + b_2T_2 + b_3T_3 + \cdots + b_nT_n.$$

Proof:

This theorem can be proved by using mathematical induction, for $n = 0$ it is true that

$$p(x) = a_0 = a_0T_0 .$$

Suppose that $n = k$ and it is true that

$$p_k(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_kx^k = b_0T_0 + b_1T_1 + b_2T_2 + b_3T_3 + \cdots + b_kT_k,$$

it will prove that for $n = k + 1$ Theorem 6 is also true.

Let $n = k + 1$, then

$$p_{k+1}(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{k+1}x^{k+1},$$

Because T_{k+1} is a polynomial with degree $k + 1$, so we have

$$a_{k+1}x^{k+1} = a'_{k+1}T_{k+1} + q_k(x),$$

with q_k is a polynomial with degree k. Next, it is obtained that

$$\begin{aligned} p_{k+1}(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a'_{k+1}T_{k+1} + q_k(x) \\ &= c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_kx^k + a'_{k+1}T_{k+1}, \end{aligned}$$

by using the hypothesis, we have

$p_{k+1}(x) = b_0T_0 + b_1T_1 + b_2T_2 + b_3T_3 + \cdots + b_kT_k + a'_{k+1}T_{k+1}$. ∎

D. CONCLUSION AND SUGGESTIONS

The explicit solution for the first kind Chebyshev polynomials and its properties have been studied. Some of the useful properties in determining a given Chebyshev polynomial are symmetry, special values, and orthogonal properties. The theorem relating the Chebyshev polynomial to the ordinary polynomial have also been given and its proof. The proof is given using mathematical induction. The technique of converting ordinary polynomials into Chebyshev polynomials is still carried out one by one by changing the basic polynomial form into Chebyshev polynomials.

The recommendation for the conducted research is to provide an algorithm to express the ordinary polynomials as the linear combination of the Chebyshev polynomials. The research can also be done by looking at the characteristics of the second kind of Chebyshev polynomial.

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