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# Dynamical Analysis of the Symbiotic Model of Commensalism in Four Populations with Michaelis-Menten type Harvesting in the First Commensal Population

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**ABSTRACT**

This study discusses the dynamical analysis of the symbiosis commensalism and parasitism models in four populations with Michaelis-Menten type harvesting in the first commensal population. This model is formed from a construction of the symbiotic model of commensalism and parasitism by harvesting the commensal population. This construction is by adding a new population, namely the second commensal population. Furthermore, it will be investigated that the four populations can coexist. The first analysis is to identify the conditions of existence at all equilibrium points along with the conditions for their existence and local stability around the equilibrium point along with the stability requirements. From this model, it is obtained sixteen points of equilibrium, namely one point of extinction in the four populations, four points of extinction in all three populations, six points of extinction in both populations, four points of extinction in one population and one point where the four populations can coexist. Of the sixteen points, only four points can be asymptotically stable if they meet the stability conditions that have been determined. Finally, a numerical simulation is performed to describe the model behavior. In this study, the method used in numerical simulation is the RK-4 method. The numerical simulation results that have been obtained support the dynamical analysis results that have been carried out previously.

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## A. INTRODUCTION

Symbiosis, namely the relationship between living things. Symbiosis is divided into four, namely parasitism, mutualism, commensalism, amensalism and neutralism. Symbiosis of commensalism is an relationship between living things where one does not benefit or is harmed (the host) while the other benefits (commensal). In this case, for example, orchids and ferns with mango trees. The orchids and ferns benefit from living on the mango tree, while the mango tree does not get any influence. Furthermore, symbiosis of parasitism is an relationship between living things where one is harmed (the host) while the other is benefited (the parasite). In this case, for example, parasite plants with mango trees. The parasite plant gets food from the mango tree, while the mango tree feels disadvantaged (Yukalov et al., 2012).

The dynamics of the symbiotic model has become one of the important topics in mathematics. In 1838, Pierre Verhulst introduced the logistics model for the first time. This is due to the fact that the population is too large, so a process of limitation must be carried out (John et al., 2008). In (Puspitasari & Kusumawinahyu, 2021) research, a logistic model was used to describe the commensal, parasite, and host growth. (Puspitasari & Kusumawinahyu, 2021) also introduced harvesting with the Michaelis-Menten type. Harvesting of the Michaelis-Menten type is harvesting with a saturated model or with a saturation point (Gupta et al., 2012), (Gupta & Chandra, 2013), (Hu & Cao, 2017),(Saha et al., 2018), (W. Liu & Jiang, 2018),(Y. Liu et al., 2018), (Chen, 2019),(Fattahpour et al., 2019), (Satar & Naji, 2019),(Xue et al., 2019), (Lai et al., 2020),(Zuo et al., 2020). The study of the symbiotic model continues to develop by adding various assumptions to make the model more realistic and complex. These developments include using various response functions (John et al., 2008), (Sun et al., 2012), (Ahmed Buseri Ashine et al., 2017), (Ma et al., 2017), (PK & S, 2017), (Kenassa Edessa, 2018), (Pavan Kumar et al., 2018), (Sarkar et al., 2020), the Alle effect (Ongun & Ozdogan, 2017), (Chen, 2018), (Ye et al., 2019), (Wei et al., 2020) etc. This causes the solution not easy to determine analytically, so a numerical approach is needed. One of the numerical approaches used to find solutions from a continuous model is to uses the Runge Kutta method (Yang & Shen, 2015), (Paul et al., 2016), (Stephen Olaniyan et al., 2020).

Based on the description above, this study will construct the symbiotic commensalism model by harvesting the commensal population in the study (Puspitasari & Kusumawinahyu, 2021). The construction is by adding the second commensal population. In this article, we produce the equilibrium point and the conditions of existence and local stability at the equilibrium point and their conditions. Finally, a numerical simulation is used to verify the dynamic analysis results.

## B. METHODS

This study uses a research method that consists of the following stages.

### 1. Specifying the Model

The model in this study was obtained from the symbiosis commensalism model with harvesting in the commensal population carried out by (Puspitasari & Kusumawinahyu, 2021). The model is

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k_1}+a\frac{y}{k_1}\right)-\frac{qEx}{m_1E+m_2x}, \\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{k_2}-b\frac{z}{k_2}\right), \\ \frac{dz}{dt} &= r_3 z\left(1-\frac{z}{k_3}+c\frac{y}{k_3}\right), \end{aligned} \tag{1}$$

with $x(t), y(t)$ and $z(t)$ interprets the first commensal population, host population, and parasite population. All parameters used are not negative. $r_1, r_2$, and $r_3$ interpret the intrinsic growth of $x, y$, and $z$. $k_1, k_2$, and $k_3$ interpret the carrying capacities of $x, y$, and $z$. The parameter $a$ is the relationship between $x$ and $y$. The parameter $b\ and\ c$ are the relationship between $y$ and $z$. The parameter $E$ is a fishing business used for harvest, $q$ is the catching power coefficient and $m_1, m_2$ are the suitable constants. The model will be constructed. Constructed of the model is by adding the second commensal population. This second commensal population does not harm other populations.

## 2. Dynamic Analysis and Numerical Simulation

The definition and theorem used in the dynamical analysis of this research are as follows.

**Definition 1.** The point $\vec{p}^*$ can be said to be the equilibrium point of the equation $\frac{d\vec{p}}{dt} = \vec{g}(\vec{p}), \vec{p} \in \mathbb{R}^n$ if it meets the condition $\frac{d\vec{p}}{dt} = \vec{0}$ (Trahan et al., 1979).

**Theorem 1.** If the eigenvalue of the Jacobi matrix $Dg(p^*)$ are $(\lambda_1,\ \lambda_2,\ \lambda_3, \text{dan } \lambda_4)$, then there are several local stability criterion as follows:

a. If all the eigenvalue in the Jacobi matrix $Dg(\vec{p}^*)$ have a negative real part or $Re(\lambda_i) < 0, \forall i = 1, 2, 3,$ then the equilibrium point is said to be asymptotically stable.

b. If there is an eigenvalue in the Jacobi matrix $Dg(\vec{p}^*)$ has a positive real part or $Re(\lambda_i) > 0, \exists i = 1, 2, 3,$ then the equilibrium point is said to be unstable.

(Trahan et al., 1979)

The numerical simulation in this article uses the Runge Kutta order 4 (RK-4) method and uses the Matlab software (R2015b).

## C. RESULT AND DISCUSSION

### 1. Dynamical Analysis

The symbiotic mathematical model in this paper is a construction of model (1) by adding the second commensal population so that the model becomes as follows.

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x\left(1 - \frac{x}{k_1} + a\frac{y}{k_1}\right) - \frac{qEx}{m_1E + m_2x},\\
\frac{dy}{dt} &= r_2 y\left(1 - \frac{y}{k_2} - b\frac{z}{k_2}\right),\\
\frac{dz}{dt} &= r_3 z\left(1 - \frac{z}{k_3} + c\frac{y}{k_3}\right),\\
\frac{dp}{dt} &= r_4 p\left(1 - \frac{p}{k_4} + d\frac{y}{k_4}\right),
\end{aligned} \tag{2}$$

where $r_4$ is the intrinsic growth of $p$ and $k_4$ is the carrying capacities of $p$. $d$ show the relationship between $y$ and $p$.

By solving the following equation

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = \frac{dp}{dt} = 0,$$

so that system (2) becomes

$$r_1 x\left(1 - \frac{x}{k_1} + a\frac{y}{k_1}\right) - \frac{qEx}{m_1E + m_2x} = 0, \tag{3}$$

$$r_2 y\left(1 - \frac{y}{k_2} - b\frac{z}{k_2}\right) = 0, \tag{4}$$

$$r_3 z\left(1-\frac{z}{k_3}+c\frac{y}{k_3}\right)=0, \qquad (5)$$
$$r_4 p\left(1-\frac{p}{k_4}+d\frac{y}{k_4}\right)=0. \qquad (6)$$

The (3) equation has the following solution
$$x=0 \qquad (3.a)$$
or
$$r_1\left(1-\frac{x}{k_1}+a\frac{y}{k_1}\right)-\frac{qE}{m_1E+m_2x}=0, \qquad (3.b)$$
then
$$x^*=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$
The (4) equation has the following solution
$$y=0 \qquad (4.a)$$
or
$$1-\frac{y}{k_2}-b\frac{z}{k_2}=0, \qquad (4.b)$$
then
$$y^*=k_2-bz^*.$$
The (5) equation has the following solution
$$z=0 \qquad (5.a)$$
or
$$1-\frac{z}{k_3}+c\frac{y}{k_3}=0, \qquad (5.b)$$
then
$$z^*=k_3+cy^*.$$
The (6) equation has the following solution
$$p=0 \qquad (6.a)$$
or
$$1-\frac{p}{k_4}+d\frac{y}{k_4}=0, \qquad (6.b)$$
then
$$p^*=k_4+dy^*.$$

From the solution of equation (3)- (6) there are sixteen equilibrium point exist, if they satisfy Theorem 1 as follows.

**Theorem 2**. Conditions for the existence of an equilibrium point
If the solution to the system of equation (2) is $T_i, i=0,1,2,\dots,15$, then the equilibrium point of the system of equation (2) which has the following terms of existence.

a. $T_i, i=0,1,\dots,5$ are the equilibrium point in the system of equation (2).
b. $T_6$ and $T_7$ are the equilibrium point in the system of equation (2) if $k_2>bk_3$.

c. $T_8, T_9, T_{10}$, and $T_{12}$ are the equilibrium point in the system of equation (2) if $D_1 = {B_1}^2 - 4A_1C_1 \geq 0$, then will get ${x_a}^* = {x_8}^* = {x_9}^* = {x_{10}}^* = {x_{12}}^*$. Where, $A_1 = r_1m_2$, $B_1 = (m_1E - m_2k_1)r_1$, and $C_1 = (q - r_1m_1)k_1E$, so there are several possible values for ${x_a}^*$ as follows.
   i. $D_1 = 0$ and $B_1 < 0$, or
   ii. $D_1 > 0$ and $C_1 < 0$, or
   iii. $D_1 > 0, C_1 = 0$, and $B_1 < 0$, or
   iv. $D_1 > 0, C_1 > 0$, and $B_1 < 0$.

d. $T_{11}$ and $T_{13}$ are the equilibrium point of the system of equation (2) if $D_2 = {B_2}^2 - 4A_2C_2 \geq 0$, then will get ${x_b}^* = {x_{11}}^* = {x_{13}}^*$. Where, $A_2 = r_1m_2, B_2 = (m_1E - m_2k_1 - am_2k_2)r_1$ and $C_2 = \big((q - r_1m_1)k_1 - ar_1m_1k_2\big)E$, so there are several possible values for ${x_b}^*$ as follows.
   i. $D_2 = 0$ and $B_2 < 0$, or
   ii. $D_2 > 0$ and $C_2 < 0$, or
   iii. $D_2 > 0, C_2 = 0$, and $B_2 < 0$, or
   iv. $D_2 > 0, C_2 > 0$, and $B_2 < 0$.

e. $T_{14}$ and $T_{15}$ are the equilibrium point of the system of equation (2) if $D_3 = {B_3}^2 - 4A_3C_3 \geq 0$, then will get ${x_c}^* = {x_{14}}^* = {x_{15}}^*$. Where, $A_3 = (1 + bc)r_1m_2 > 0$, $B_3 = \big((1 + bc)(m_1E - m_2k_1) + (bk_3 - k_2)am_2\big)r_1$ and $C_3 = \big((1 + bc)(q - r_1m_1)k_1 + (bk_3 - k_2)ar_1m_1\big)E$, so there are several possible values for ${x_c}^*$ as follows.
   i. $D_3 = 0$ and $B_3 < 0$, or
   ii. $D_3 > 0$ and $C_3 < 0$, or
   iii. $D_3 > 0, C_3 = 0$, and $B_3 < 0$, or
   iv. $D_3 > 0, C_3 > 0$, and $B_3 < 0$.

**Proof:** Based on the existing reality, the population number will always be not negative, so the solution of the system of equation (2) must be non-negative.

a. $T_0 = (0,0,0,0), T_1 = (0,0,0,k_4), T_2 = (0,0,k_3,0), T_3 = (0,k_2,0,0), T_4 = (0,0,k_3,k_4)$, and $T_5 = (0,k_2,0,k_4 + dk_2)$ is always not negative, so that $T_i, i = 0,1,\ldots,5$ is the equilibrium point for the system of equation (2).

b. $T_6 = \left(0, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, 0\right)$ and $T_7 = \left(0, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$ is not negative, so that $T_6$ and $T_7$ is the equilibrium point for the system of equation (2).

c. If $D_1 \geq 0$, then have
   i. the twin solution is not negative that is $T_8 = \left(\frac{-B_1}{2A_1}, 0,0,0\right), T_9 = \left(\frac{-B_1}{2A_1}, 0,0,k_4\right), T_{10} = \left(\frac{-B_1}{2A_1}, 0, k_3, 0\right)$, and $T_{12} = \left(\frac{-B_1}{2A_1}, 0, k_3, k_4\right)$, or
   ii. the one solution is not negative that is $T_8 = \left(\frac{-B_1 + \sqrt{D_1}}{2A_1}, 0,0,0\right), T_9 = \left(\frac{-B_1 + \sqrt{D_1}}{2A_1}, 0,0,k_4\right), T_{10} = \left(\frac{-B_1 + \sqrt{D_1}}{2A_1}, 0, k_3, 0\right)$, and $T_{12} = \left(\frac{-B_1 + \sqrt{D_1}}{2A_1}, 0, k_3, k_4\right)$, or
   iii. the one solutions are not negative namely $T_8 = \left(\frac{-B_1}{A_1}, 0,0,0\right), T_9 = \left(\frac{-B_1}{A_1}, 0,0,k_4\right), T_{10} = \left(\frac{-B_1}{A_1}, 0, k_3, 0\right)$, and $T_{12} = \left(\frac{-B_1}{A_1}, 0, k_3, k_4\right)$, or

iv. the two solutions are not negative namely $T_8 = \left(\frac{-B_1 \pm \sqrt{D_1}}{2A_1}, 0,0,0\right), T_9 = \left(\frac{-B_1 \pm \sqrt{D_1}}{2A_1}, 0,0,k_4\right), T_{10} = \left(\frac{-B_1 \pm \sqrt{D_1}}{2A_1}, 0, k_3, 0\right)$, and $T_{12} = \left(\frac{-B_1 \pm \sqrt{D_1}}{2A_1}, 0, k_3, k_4\right)$.

Since the value of $x_a{}^* > 0, y^* = 0, z^* \geq 0,$ and $p^* \geq 0$, then $T_8, T_9, T_{10}$, and $T_{12}$ is the equilibrium point for the system of equation (2).

d. If $D_2 \geq 0$, then have

i. the twin solution is not negative that is $T_{11} = \left(\frac{-B_2}{2A_2}, k_2, 0,0\right)$ and $T_{13} = \left(\frac{-B_2}{2A_2}, k_2, 0, k_4 + dk_2\right)$, or

ii. the one solution is not negative that is $T_{11} = \left(\frac{-B_2 + \sqrt{D_2}}{2A_2}, k_2, 0,0\right)$ and $T_{13} = \left(\frac{-B_2 + \sqrt{D_2}}{2A_2}, k_2, 0, k_4 + dk_2\right)$, or

iii. the one solutions are not negative namely $T_{11} = \left(\frac{-B_2}{A_2}, k_2, 0,0\right)$ and $T_{13} = \left(\frac{-B_2}{A_2}, k_2, 0, k_4 + dk_2\right)$, or

iv. the two solutions are not negative namely $T_{11} = \left(\frac{-B_2 \pm \sqrt{D_2}}{2A_2}, k_2, 0,0\right)$ and $T_{13} =$

Since the value of $x_b{}^* > 0, y^* \geq 0, z^* = 0,$ and $p^* \geq 0$, then $T_{11}$ and $T_{13}$ is the equilibrium point for the system of equation (2).

e. If $D_3 \geq 0$, then have

i. the twin solution is not negative that is $T_{14} = \left(\frac{-B_3}{2A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, 0\right)$ and $T_{15} = \left(\frac{-B_3}{2A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$, or

ii. the one solution is not negative that is $T_{14} = \left(\frac{-B_3 + \sqrt{D_3}}{2A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, 0\right)$ and $T_{15} = \left(\frac{-B_3 + \sqrt{D_3}}{2A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$, or

iii. the one solutions are not negative namely $T_{14} = \left(\frac{-B_3}{A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, 0\right)$ and $T_{15} = \left(\frac{-B_3}{A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$, or

iv. the two solutions are not negative namely $T_{14} = \left(\frac{-B_3 \pm \sqrt{D_3}}{2A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, 0\right)$ and $T_{15} = \left(\frac{-B_3 \pm \sqrt{D_3}}{2A_3}, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$.

Since the value of $x_c{}^* > 0, y^* > 0, z^* > 0,$ and $p^* \geq 0$, then $T_{14}$ and $T_{15}$ is the equilibrium point for the system of equation (2).

In studying the dynamics of the model in the system of equation (2) around the equilibrium point $E_i, i = 0,1, \ldots, 15$, a linear model is used in the system of equation (2). Furthermore, from the linearity, the Jacobian matrix is obtained from the system of equation (2) around the equilibrium point as follows.

$$J(x^*,y^*,z^*,p^*) = \begin{bmatrix} r_1-\frac{2r_1x^*}{k_1}+\frac{ar_1y^*}{k_1}-\frac{qm_1E^2}{(m_1E+m_2x^*)^2} & \frac{ar_1x^*}{k_1} & 0 & 0 \\ 0 & r_2-\frac{2r_2y^*}{k_2}-\frac{r_2bz^*}{k_2} & \frac{-br_2y^*}{k_2} & 0 \\ 0 & \frac{cr_3z^*}{k_3} & r_3-\frac{2r_3z^*}{k_3}+\frac{r_3cy^*}{k_3} & 0 \\ 0 & \frac{dr_4p^*}{k_4} & 0 & r_4-\frac{2r_4p^*}{k_4}+\frac{r_4dy^*}{k_4} \end{bmatrix},$$

The eigenvalues of the matrix $J(x^*,y^*,z^*,p^*)$ are obtained from $|J(x^*,y^*,z^*,p^*)-\lambda I|=0$ is

$$\lambda_1 = r_1-\frac{2r_1x^*}{k_1}+\frac{ar_1y^*}{k_1}-\frac{qm_1E^2}{(m_1E+m_2x^*)^2},$$

$$\lambda_4 = r_4-\frac{2r_4p^*}{k_4}+\frac{r_4dy^*}{k_4},$$

$\lambda_2$ and $\lambda_3$ are obtained by solving following characteristic equation

$$\left(r_2-\frac{2r_2y^*}{k_2}-\frac{br_2z^*}{k_2}-\lambda\right)\left(r_3-\frac{2r_3z^*}{k_3}+\frac{cr_3y^*}{k_3}-\lambda\right)-\left(\frac{cr_3z^*}{k_3}\right)\left(\frac{-br_2y^*}{k_2}\right)=0,$$

$$[\lambda^2+B\lambda+C],$$

where

$$B=-r_2-r_3+\frac{2r_2y^*}{k_2}+\frac{br_2z^*}{k_2}+\frac{2r_3z^*}{k_3}-\frac{cr_3y^*}{k_3}$$

and

$$C=\frac{r_2r_3}{k_2k_3}(k_2k_3+(-2k_3+ck_2-2cy^*)y^*+(-2k_2+bk_3+2bz^*)z^*+4y^*z^*).$$

Based on theorem 1, in determining stability $E_i, i=0,1,\ldots,15$ the system of equation (2) is expressed in the following theorem form.

**Theorem 3.** Conditions for the stability of an equilibrium point
If the solution to the system of equation (2) is $T_i, i=0,1,2,\ldots,15$, then the stability of the equilibrium point of the system of equation (2) which has the following conditions.

a. The point $T_0=(0,0,0,0)$ has unstable properties.
b. The point $T_1=(0,0,0,k_4)$ has unstable properties.
c. The point $T_2=(0,0,k_3,0)$ has unstable properties.
d. The point $T_3=(0,k_2,0,0)$ has unstable properties.
e. The point $T_4=(0,0,k_3,k_4)$ has asymptotically stable properties.
f. The point $T_5=(0,k_2,0,k_4+dk_2)$ has unstable properties.
g. The point $T_6=\left(0,\frac{k_2-bk_3}{1+bc},\frac{k_3+ck_2}{1+bc},0\right)$ has unstable properties.
h. The point $T_7=\left(0,\frac{k_2-bk_3}{1+bc},\frac{k_3+ck_2}{1+bc},k_4+\frac{d(k_2-bk_3)}{1+bc}\right)$ has asymptotically stable properties.
i. The point $T_8=({x_a}^*,0,0,0)$ has unstable properties.
j. The point $T_9=({x_a}^*,0,0,k_4)$ has unstable properties.
k. The point $T_{10}=({x_a}^*,0,k_3,0)$ has unstable properties.

l. The point $T_{11} = (x_b{}^*, k_2, 0,0)$ has unstable properties.
m. The point $T_{12} = (x_a{}^*, 0, k_3, k_4)$ has asymptotically stable properties.
n. The point $T_{13} = (x_b{}^*, k_2, 0, k_4 + dk_2)$ has unstable properties.
o. The point $T_{14} = \left(x_c{}^*, \frac{k_2 - bk_3}{1+bc}, \frac{k_3+ck_2}{1+bc}, 0\right)$ has unstable properties.
p. The point $T_{15} = \left(x_c{}^*, \frac{k_2 - bk_3}{1+bc}, \frac{k_3+ck_2}{1+bc}, k_4 + \frac{d(k_2-bk_3)}{1+bc}\right)$ has asymptotically stable properties.

**Proof:**

a. The eigenvalues of the $J(T_0)$ are $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = r_2 > 0$, $\lambda_3 = r_3 > 0$, and $\lambda_4 = r_4 > 0$. It is proved, that theorem 3 satisfies theorem 1(b).
b. The eigenvalues of the $J(T_1)$ are $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = r_2 > 0$, $\lambda_3 = r_3 > 0$, and $\lambda_4 = -r_4$. It is proved, that theorem 3 satisfies theorem 1(b).
c. The eigenvalues of the $J(T_2)$ are $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = r_2 - \frac{br_2k_3}{k_2}$, $\lambda_3 = -r_3$, and $\lambda_4 = r_4 > 0$. It is proved, that theorem 3 satisfies theorem 1(b).
d. The eigenvalues of the Jacobian $J(T_3)$ are $\lambda_1 = r_1 + \frac{ar_1k_2}{k_1} - \frac{q}{m_1}$, $\lambda_2 = -r_2$, $\lambda_3 = r_3 + \frac{cr_3\boldsymbol{k_2}}{k_3} > 0$, and $\lambda_4 = r_4 + \frac{dr_4\boldsymbol{k_2}}{k_4} > 0$. It is proved, that theorem 3 satisfies theorem 1(b).
e. The eigenvalues of the $J(T_4)$ are $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = r_2 - \frac{br_2k_3}{k_2}$, $\lambda_3 = -r_3 < 0$, and $\lambda_4 = -r_4 < 0$. Further, if $r_1 - \frac{q}{m_1} < 0$ and $r_2 - \frac{br_2k_3}{k_2} < 0$, then $\lambda_{1,2} < 0$. It is proved, that theorem 3 satisfies theorem 1(a).
f. The eigenvalues of the $J(T_5)$ are $\lambda_1 = r_1 + \frac{ar_1k_2}{k_1} - \frac{q}{m_1}$, $\lambda_4 = -r_2$, $\lambda_3 = r_3 + \frac{cr_3\boldsymbol{k_2}}{k_3} > 0$, and $\lambda_4 = r_4 - \frac{2r_4(k_4+dk_2)}{k_4} + \frac{r_4dk_2}{k_4}$. It is proved, that theorem 3 satisfies theorem 1(b).
g. The eigenvalues of the $J(T_6)$ are $\lambda_1 = r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{q}{m_1}$, $\lambda_4 = r_4 + \frac{dr_4}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right) > 0$ and $\lambda_{2,3} = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$, where $A = 1, B = -r_2 - r_3 + \frac{2r_2y^*}{k_2} + \frac{br_2z^*}{k_2} + \frac{2r_3z^*}{k_3} - \frac{cr_3y^*}{k_3}$, and $C = \frac{r_2r_3}{k_2k_3}(k_2k_3 + (-2k_3 + ck_2 - 2cy^*)y^* + (-2k_2 + bk_3 + 2bz^*)z^* + 4y^*z^*)$. It is proved, that theorem 3 satisfies theorem 1(b).
h. The eigenvalues of the $J(T_7)$ are $\lambda_1 = r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{q}{m_1}$, $\lambda_4 = r_4 - \frac{2r_4}{k_4}\left(k_4 + \frac{d(k_2-bk_3)}{1+bc}\right) + \frac{r_4d}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right)$ and $\lambda_{2,3} = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$, where $A = 1, B = -r_2 - r_3 + \frac{2r_2y^*}{k_2} + \frac{br_2z^*}{k_2} + \frac{2r_3z^*}{k_3} - \frac{cr_3y^*}{k_3}$, and $C = \frac{r_2r_3}{k_2k_3}(k_2k_3 + (-2k_3 + ck_2 - 2cy^*)y^* + (-2k_2 + bk_3 + 2bz^*)z^* + 4y^*z^*)$. Further, if $r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{q}{m_1} < 0, r_4 - \frac{2r_4}{k_4}\left(k_4 + \frac{d(k_2-bk_3)}{1+bc}\right) + \frac{r_4d}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right) < 0$ and $B, C > 0$, then $\lambda_{1,2,3,4} < 0$. It is proved, that theorem 3 satisfies theorem 1(a).
i. The eigenvalues of the $J(T_8)$ are $\lambda_1 = r_1 - \frac{2r_1x_a{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_a{}^*)^2}$, $\lambda_2 = r_2 > 0$, $\lambda_3 = r_3 > 0$, and $\lambda_4 = r_4 > 0$. It is proved, that theorem 3 satisfies theorem 1(b).

j. The eigenvalues of the $J(T_9)$ are $\lambda_1 = r_1 - \frac{2r_1 x_a{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_a{}^*)^2}$, $\lambda_2 = r_2 > 0, \lambda_3 = r_3 > 0$, and $\lambda_4 = -r_4$. It is proved, that theorem 3 satisfies theorem 1(b).

k. The eigenvalues of the $J(T_{10})$ are $\lambda_1 = r_1 - \frac{2r_1 x_a{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_a{}^*)^2}$, $\lambda_2 = r_2 - \frac{br_2k_3}{k_2}$, $\lambda_3 = -r_3$, and $\lambda_4 = r_4 > 0$. It is proved, that theorem 3 satisfies theorem 1(b).

l. The eigenvalues of the $J(T_{11})$ are $\lambda_1 = r_1 + \frac{ar_1k_2}{k_1} - \frac{2r_1x_b{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_b{}^*)^2}$, $\lambda_2 = -r_2$, $\lambda_3 = r_3 + \frac{cr_3\boldsymbol{k_2}}{k_3} > 0$, and $\lambda_4 = r_4 + \frac{dr_4\boldsymbol{k_2}}{k_4} > 0$. It is proved, that theorem 3 satisfies theorem 1(b).

m. The eigenvalues of the $J(T_{12})$ are $\lambda_1 = r_1 - \frac{2r_1 x_a{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_a{}^*)^2}$, $\lambda_2 = r_2 - \frac{br_2k_3}{k_2}$, $\lambda_3 = -r_3 < 0$, and $\lambda_4 = -r_4 < 0$. Further, if $r_1 - \frac{2r_1 x_a{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_a{}^*)^2} < 0$ and $r_2 - \frac{br_2k_3}{k_2} < 0$, then $\lambda_{1,2} < 0$. It is proved, that theorem 3 satisfies theorem 1(a).

n. The eigenvalues of the $J(T_{13})$ are $\lambda_1 = r_1 + \frac{ar_1k_2}{k_1} - \frac{2r_1x_b{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_b{}^*)^2}$, $\lambda_4 = -r_2$, $\lambda_3 = r_3 + \frac{cr_3\boldsymbol{k_2}}{k_3} > 0$, and $\lambda_4 = r_4 - \frac{2r_4(k_4+dk_2)}{k_4} + \frac{r_4dk_2}{k_4}$. It is proved, that theorem 3 satisfies theorem 1(b).

o. The eigenvalues of the $J(T_{14})$ are $\lambda_1 = r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{2r_1x_c{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_c{}^*)^2}$, $\lambda_4 = r_4 + \frac{dr_4}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right) > 0$ and $\lambda_{2,3} = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$, where $A = 1, B = -r_2 - r_3 + \frac{2r_2y^*}{k_2} + \frac{br_2z^*}{k_2} + \frac{2r_3z^*}{k_3} - \frac{cr_3y^*}{k_3}$, and $C = \frac{r_2r_3}{k_2k_3}(k_2k_3 + (-2k_3 + ck_2 - 2cy^*)y^* + (-2k_2 + bk_3 + 2bz^*)z^* + 4y^*z^*)$. It is proved, that theorem 3 satisfies theorem 1(b).

p. The eigenvalues of the $J(T_{15})$ are $\lambda_1 = r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{2r_1x_c{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_c{}^*)^2}$, $\lambda_4 = r_4 - \frac{2r_4}{k_4}\left(k_4 + \frac{d(k_2-bk_3)}{1+bc}\right) + \frac{r_4d}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right)$ and $\lambda_{2,3} = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$, where $A = 1, B = -r_2 - r_3 + \frac{2r_2y^*}{k_2} + \frac{br_2z^*}{k_2} + \frac{2r_3z^*}{k_3} - \frac{cr_3y^*}{k_3}$, and $C = \frac{r_2r_3}{k_2k_3}(k_2k_3 + (-2k_3 + ck_2 - 2cy^*)y^* + (-2k_2 + bk_3 + 2bz^*)z^* + 4y^*z^*)$. Further, if $r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{q}{m_1} < 0, r_4 - \frac{2r_4}{k_4}\left(k_4 + \frac{d(k_2-bk_3)}{1+bc}\right) + \frac{r_4d}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right) < 0$ and $B, C > 0$, then $\lambda_{1,2,3,4} < 0$. It is proved, that theorem 3 satisfies theorem 1(a).

## 2. Numerical Simulation

Several numerical simulations that match the results of the analysis described earlier will be provided at this stage. The numerical simulation shows the local stability for $T_4, T_7, T_{12}$, and $T_{15}$. In selecting the parameters used, namely based on the conditions in the results of the previous analysis. This is because there is no real data that corresponds to this model. Therefore, the parameter values used in the first simulation are as follows.
$q = r_2 = r_3 = r_4 = a = c = d = k_1 = 1, r_1 = 0.4,\ b = 0.1,\ E = 0.001,\ k_2 = 0.8, k_3 = 9, k_4 = 3, m_1 = 2$ and $m_2 = 0.9$, while in the second simulation values and values were taken $q = 7, r_1 = r_2 = r_3 = r_4 = b = c = d = 0.01, a = 1,\ b = 0.1,\ E = 0.0001, k_1 = k_2 = 2, k_3 = 9, k_4 = 3, m_1 = 2,$ and $m_2 = 4$. By using the parameter values in the first simulation, the

equilibrium point is obtained $T_4 = (0,0,9,3)$ and $T_{12} = (0.99,0,9,3)$which is locally asymptotically stable as it satisfies $r_1 - \frac{q}{m_1} = -0.1 < 0$, $r_2 - \frac{br_2k_3}{k_2} = -0.125 < 0$, and $r_1 - \frac{2r_1x_a{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_a{}^*)^2} = -0.324829 < 0$, see Figure 1. This shows that at point $T_4$ the first commensal population and the host will become extinct, while the second commensal and parasite populations can survive. At point $T_{12}$, the first commensal population will become extinct, while the second host, parasite and commensal population can survive. Then, by using the parameter values in the second simulation, the equilibrium point is obtained $T_7 = (0,1.9,9.019,3.019)$ and $T_{15} = (3.9,1.9,9.019,3.019)$ with the conditions of existence are $k_2 = 2 > 0.09 = bk_3, A_3 = 0.04 > 0, B_3 = -0.1564, C_3 = 0.00139$ and $D_3 = 0.02424$. $T_7$ and $T_{15}$ which is locally asymptotically stable as it satisfies $r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{q}{m_1} = -0.136363 < 0, r_1 + \frac{ar_1}{k_1}\left(\frac{k_2-bk_3}{1+bc}\right) - \frac{2r_1x_c{}^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x_c{}^*)^2} = -0.019459 < 0, r_4 - \frac{2r_4}{k_4}\left(k_4 + \frac{d(k_2-bk_3)}{1+bc}\right) + \frac{r_4d}{k_4}\left(\frac{k_2-bk_3}{1+bc}\right) = -0.0100637 < 0, B = 0.01957 > 0$, and $C = 0.00010472 > 0$, see Figure 2. This shows that at point $T_4$ the first commensal population and the host will become extinct, while the second commensal and parasite populations can survive. At point $T_{12}$, the first commensal population will become extinct, while the second host, parasite and commensal population can survive.

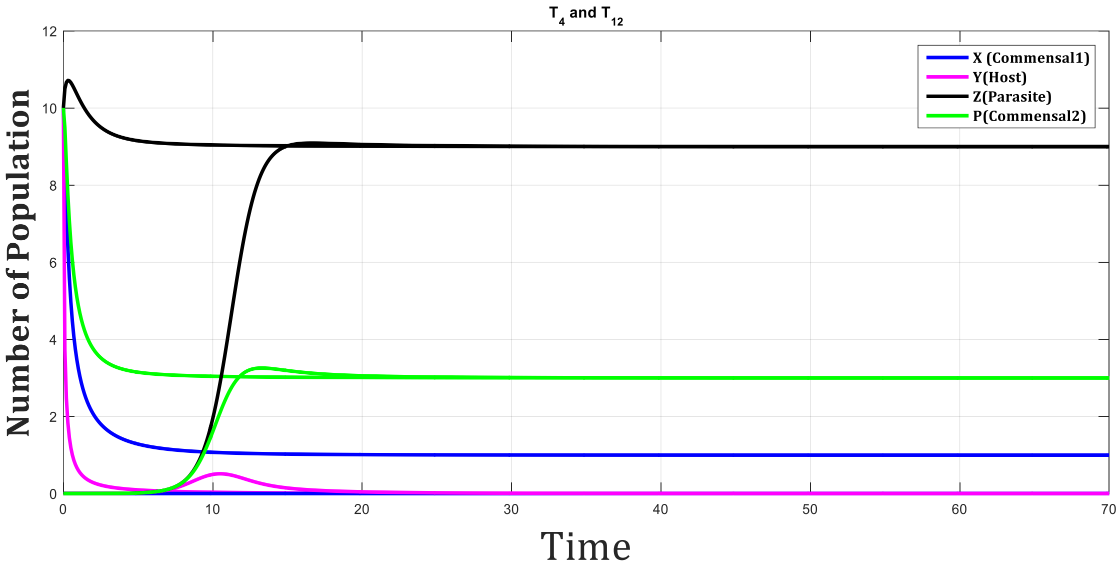


**Figure 1**. Numeric Simulations in $T_4$ and $T_{12}$

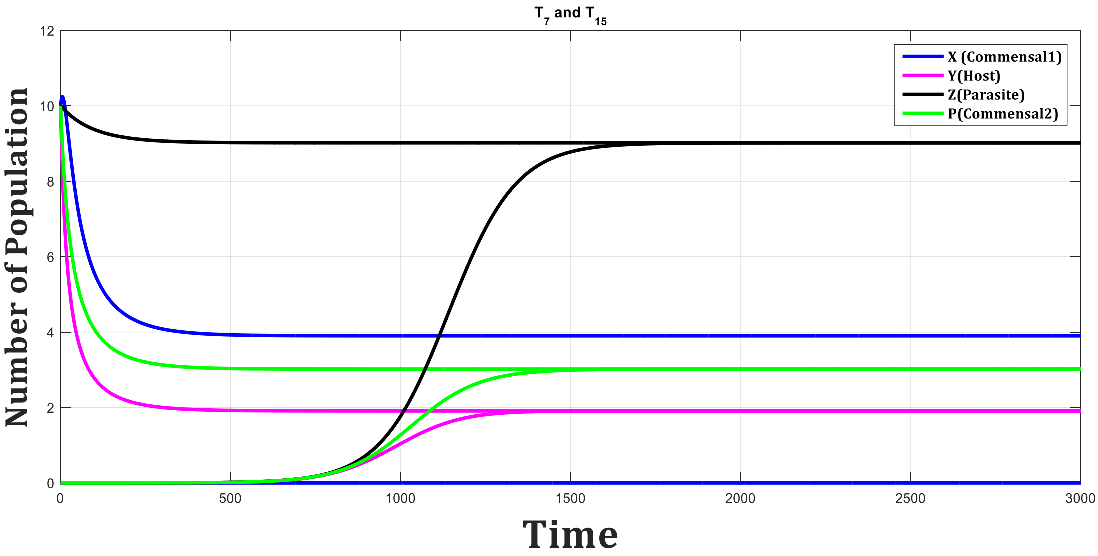


**Figure 2**. Numeric Simulations in $T_7$ and $T_{15}$

## D. CONCLUSION AND SUGGESTIONS

The model consist of four population, namely first commensal population, second commensal population, host population, and parasite population. The dynamical analysis in this study found sixteen equilibrium points with their existence and local asymptotic stability properties. The fourth, seventh, twelfth, and fifteenth points are locally asymptotically stable if they meet the specified stability conditions, while the other points are always unstable. From fourth point it can be interpreted that the parasite population, the second commensal will never become extinct, seventh point means that the host, parasite and second commensal population will never become extinct, twelfth point means that the first commensal population, parasites and the second commensal will never be extinct, while fifteenth point means that the four populations can live side by side. From the results of the numerical simulations that have been carried out, it shows that it is in accordance with the analysis being carried out. From the first simulation using the parameter values used, it can be seen that the graph converges towards fourth point and twelfth point, while the second simulation uses the parameter values used, it can be seen that the graph converges towards seventh point and fifteenth point.

Further research, it is advisable to add harvest to unharvested populations.

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