

Non-Braid Graphs of Ring \mathbb{Z}_n

Era Setya Cahyati¹, Rizka 'Abid Fadhiilah², Ananditya Dwi Candra Bp³, Indah Emilia Wijayanti⁴

^{1,2,3,4}Departement of Mathematics, Gadjah Mada University, Indonesia

era.setya.cahyati@mail.ugm.ac.id¹, rizkaabid@mail.ugm.ac.id², anandityadwi97@mail.ugm.ac.id³,
ind.wijayanti@ugm.ac.id⁴

ABSTRACT

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The research in graph theory has been widened by combining it with ring. In this paper, we introduce the definition of a non-braid graph of a ring. The non-braid graph of a ring R , denoted by Υ_R , is a simple graph with a vertex set $R \setminus B(R)$, where $B(R)$ is the set of x where $x \in R$ such that $xyx = yxy$ for all $y \in R$. Two distinct vertices x and y are adjacent if and only if $xyx \neq yxy$. The method that we use to observe the non-braid graphs of \mathbb{Z}_n is by seeing the adjacency of the vertices and its braider. The main objective of this paper is to prove the completeness and connectedness of the non-braid graph of ring \mathbb{Z}_n . We prove that if n is a prime number, the non-braid graph of \mathbb{Z}_n is a complete graph. For all $n \geq 3$, the non-braid graph of \mathbb{Z}_n is a connected graph.



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A. INTRODUCTION

Some different areas in mathematics can be combined due to obtaining some interesting tools. A new interdisciplinary branch in mathematics such as algebraic graphs recently has powerful progress, in which involves graph theory and some algebraic structures. Associating a graph with an algebraic structure is a research subject that aims at exposing the relationship between algebra and graph theory and at advancing applications of one to the other (Maimani et al., 2011). There are plenty of results in associating a graph with an algebraic structure, some of them are in (Ghalandarzadeh & Rad, 2011), (Ma et al., 2014), and (Taloukolaei & Sahebi, 2018) which results in associating a graph with the module, group, and ring, respectively.

Among many results in associating a graph with the group, we mention for example Abdolahi et al. (2006), who discuss the non-commuting graph of a group. They explore how the graph-theoretical properties of the non-commuting graph of a group G can affect the group-theoretical properties of G . Some important results are the number of edges of the non-commuting graph shows the association of a group to the graph, the number of edges of the non-commuting graph and commutativity degree are in the opposite proportion (Abdollahi et al., 2006). Furthermore, Tolue et al. (2014) introduce a more general situation by defining the

g -non-commuting graph of finite groups and investigating its planarity and regularity, its clique number, and dominating number.

Some authors also pay attention to the observation of graphs and rings. For a ring R , Gupta (2013) defined a simple undirected graph $\Gamma_2(R)$ with all the non-zero elements of R as vertices, and two vertices a, b are adjacent if and only if either $ab = 0$ or $ba = 0$ or $a + b$ is a zero-divisor (including 0). He considers the connectedness and completeness of $\Gamma_2(\mathbb{Z}_n)$. Omid and Vatandoost (2011) introduce the commuting graph of the ring R . Let R be a non-commutative ring and let $Z(R)$ denote the center of R . The commuting graph of R denoted by $\Gamma(R)$, is a graph with vertex set $R \setminus Z(R)$ and two vertices a and b are adjacent if $ab = ba$ (Omid & Vatandoost, 2011). The complement of commuting graph of the ring R is considered by Erfanian et al. (2015). They introduce the non-commuting graph of ring R and study various graph-theoretical properties of this graph. Let R be the non-commutative ring and $C(R)$ is the centre of R . The non-commuting graph of ring R , denoted by Γ_R , is a simple graph whose vertex set is $R \setminus C(R)$ and two distinct vertices x and y are connected by one edge if and only if $xy \neq yx$ (Erfanian et al., 2015). Some further results regarding the non-commuting graph of a ring are conducted by Dutta and Basnet (2017a). They prove that Γ_R is not isomorphic to certain graphs of any finite non-commutative ring R . Moreover, some connections between Γ_R and the commuting probability of R are also obtained. It is shown that the non-commuting graphs of two \mathbb{Z} -isoclinic rings are isomorphic if the centres of the rings have the same order (Dutta & Basnet, 2017a). Recall that two rings R_1 and R_2 are said to be \mathbb{Z} -isoclinic if there exist additive group isomorphisms $\phi: \frac{R_1}{Z(R_1)} \rightarrow \frac{R_2}{Z(R_2)}$ and $\psi: [R_1, R_1] \rightarrow [R_2, R_2]$ such that $\psi([u, v]) = [u', v']$ whenever $\phi(u + Z(R_1)) = u' + Z(R_2)$ and $\phi(v + Z(R_1)) = v' + Z(R_2)$ (Dutta et al., 2015).

Dutta and Basnet (2017b) introduce a generalization of the non-commuting graph for the ring, namely the relative non-commuting graph of a finite ring. Let S be a subring of a finite ring R and $C_R(S) = \{r \in R : rs = sr \forall s \in S\}$. The relative non-commuting graph of the subring S in R , denoted by $\Gamma_{S,R}$, is a simple undirected graph whose vertex set is $R \setminus C_R(S)$ and two distinct vertices a, b are adjacent if and only if a or $b \in S$ and $ab \neq ba$ (Dutta & Basnet, 2017b). They discuss some properties of $\Gamma_{S,R}$, determine the diameter, girth, some dominating sets, and chromatic index for $\Gamma_{S,R}$. Another generalization of non-commuting graph for ring is introduced by Nath et al. (2021), namely the r -non-commuting graph of a finite ring R , for some r in R . This graph, which is denoted by Γ_R^r , is a simple undirected graph whose vertex set is R and two vertices x and y are adjacent if and only if $[x, y] \neq r$ and $[x, y] \neq -r$ (Nath et al., 2021). Nath et al. prove that Γ_R^r is neither a regular graph nor a lollipop graph if R is non-commutative. Moreover, they characterize the finite non-commutative ring such that Γ_R^r is a tree or a star graph.

For the detailed explanation of some notions in graph theory, we refer to (Chartrand et al., 2015) and (Wilson, 2010). Allcock (2002) introduced the notion of the braid. Two group elements x and y braid, if they satisfy $xyx = yxy$ (Allcock, 2002). In other words, two group elements x and y non-braid, if they satisfy $xyx \neq yxy$. So far, there is no graph representing the non-braid property of two elements in ring R . Hence, in this paper, we will define a graph that represents that property.

By adopting the concept of defining the non-commuting graph of ring R , we define a non-braid graph of the ring R based on the non-braid property of two elements in ring R under multiplication. We introduce the definition of a non-braid graph of the ring as follows. Let R be a ring and $B(R) = \{x \in R \mid \forall y \in R, xyx = yxy\}$. We called $B(R)$ as the *braider* of R . The non-braid graph of R , denoted by Y_R , is a simple graph whose vertex set is $R \setminus B(R)$. Two distinct vertices x and y are connected by an edge if and only if $xyx \neq yxy$. The vertices x and y that satisfy these condition is denoted by $x \sim y$.

Since the non-commuting graph of ring R cannot be constructed for the commutative ring, we are interested to investigate the existence of a non-braid graph of a commutative ring, Example of a commutative ring is ring \mathbb{Z}_n (Malik et al., 1997). Some authors already associated a graph with ring \mathbb{Z}_n , as in (Chelvam & Asir, 2011), (Patra & Kalita, 2014), (Pirzada et al., 2020), and (Aditya & Muchtadi-Alamsyah, 2021). In this paper, we discuss the non-braid graph of ring \mathbb{Z}_n . It is important to know the kind of non-braid graph based on n of \mathbb{Z}_n . Hence as the purpose of the study we observe the relationships of n and the completeness and connectedness of the graph. We show sufficient condition of n such that the non-braid graph of \mathbb{Z}_n is complete and prove that all $Y_{\mathbb{Z}_n}$ is connected.

B. RESULT AND DISCUSSION

Gupta (2013) defined a simple undirected graph $\Gamma_2(R)$ and determined the connectedness of $\Gamma_2(\mathbb{Z}_n)$. In this paper, we conduct a research in connectedness and completeness of non-braid graph of ring \mathbb{Z}_n . The notion of non-braid graph is motivated by commuting graph from Omid and Vatandoost (2011) and the non-commuting graph from Erfanian et al. (2015).

1. Definition of Non-braid Graph

In this section, we define a non-braid graph of ring R .

Definition 2.1. Let R be a ring and $B(R) = \{x \in R \mid \forall y \in R, xyx = yxy\}$. A non-braid graph of ring R , denoted by Y_R , is a simple graph whose vertex set is $R \setminus B(R)$. Two distinct vertices x and y are connected by an edge if and only if $xyx \neq yxy$. Furthermore, x and y that satisfy this condition is denoted by $x \sim y$.

For a special case, we discuss the non-braid graph of ring \mathbb{Z}_n which is denoted by $Y_{\mathbb{Z}_n}$. We start by investigating the existence of a non-braid graph of ring \mathbb{Z}_n . We take a note for $n = 1$, that is $\mathbb{Z}_1 = \{\bar{0}\}$. It is clear that $B(\mathbb{Z}_1) = \{\bar{0}\}$, so we get $V(Y_{\mathbb{Z}_1}) = \mathbb{Z}_1 \setminus B(\mathbb{Z}_1) = \emptyset$. However, $V(Y_{\mathbb{Z}_1})$ cannot be empty, meaning that a non-braid graph of \mathbb{Z}_1 cannot be formed. We then take a note for $n = 2$, that is $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Since $\bar{0}\bar{0}\bar{0} = \bar{0}\bar{0}\bar{0}$, $\bar{0}\bar{1}\bar{0} = \bar{1}\bar{0}\bar{1}$, and $\bar{1}\bar{1}\bar{1} = \bar{1}\bar{1}\bar{1}$, so $B(\mathbb{Z}_2) = \{\bar{0}, \bar{1}\}$, which results in $V(Y_{\mathbb{Z}_2}) = \mathbb{Z}_2 \setminus B(\mathbb{Z}_2) = \emptyset$. However, $V(Y_{\mathbb{Z}_2})$ cannot be empty, which means that a non-braid graph of \mathbb{Z}_2 cannot be constructed.

In Proposition 2.1, we explain the non-braid graph of \mathbb{Z}_n where $n \geq 3$.

Proposition 2.1. If $n \geq 3$, then $V(Y_{\mathbb{Z}_n}) \neq \emptyset$. Furthermore, $Y_{\mathbb{Z}_n}$ is not a null graph. *Proof.* Suppose that $n \geq 3$. We will show that $V(Y_{\mathbb{Z}_n}) \neq \emptyset$. Let $\bar{1}, \overline{n-1} \in \mathbb{Z}_n$. Since $n \geq 3$, then $\bar{1} \neq \overline{n-1}$. We know that $\bar{1}$ and $\overline{n-1}$ are self-inverse elements, then $\overline{\bar{1}n-1\bar{1}} = \overline{n-1}$ and $\overline{n-1\bar{1}n-1} = \bar{1}$, which implies $\overline{\bar{1}n-1\bar{1}} \neq \overline{n-1\bar{1}n-1}$. Hence, $\bar{1}, \overline{n-1} \in B(\mathbb{Z}_n)$. Since

$V(Y_{\mathbb{Z}_n}) = \mathbb{Z}_n \setminus B(\mathbb{Z}_n)$ and $\bar{1}, \overline{n-1} \in B(\mathbb{Z}_n)$, then $\bar{1}, \overline{n-1} \in V(Y_{\mathbb{Z}_n})$. In other words, $V(Y_{\mathbb{Z}_n}) \neq \emptyset$. Furthermore, since $\overline{1n-11} \neq \overline{n-11n-1}$, then on $Y_{\mathbb{Z}_n}$, vertex $\bar{1}$ is adjacent to vertex $\overline{n-1}$. Hence, $Y_{\mathbb{Z}_n}$ is not a null graph. ■

Since non-braid graph $Y_{\mathbb{Z}_n}$ can only be constructed for $n \geq 3$, throughout we assume \mathbb{Z}_n where $n \geq 3$.

2. Completeness of The Non-braid Graph $Y_{\mathbb{Z}_n}$

We give some results regarding the completeness of the non-braid graph $Y_{\mathbb{Z}_n}$.

Example 2.1. For every $\bar{x} \in \mathbb{Z}_6$, we get $\bar{x}\bar{0}\bar{x} = \bar{0}\bar{x}\bar{0}$, so $\bar{0} \in B(\mathbb{Z}_6)$. Then, the result of $\bar{x}\bar{y}\bar{x}$ and $\bar{y}\bar{x}\bar{y}$ for every $\bar{x}, \bar{y} \in \mathbb{Z}_6 \setminus \{\bar{0}\}$ where $\bar{x} \neq \bar{y}$ is shown on Table 1.

Table 1. The result of $\bar{x}\bar{y}\bar{x}$ and $\bar{y}\bar{x}\bar{y}$ for every $\bar{x}, \bar{y} \in \mathbb{Z}_6 \setminus \{\bar{0}\}$ where $\bar{x} \neq \bar{y}$

\bar{x}	\bar{y}	$\bar{x}\bar{y}\bar{x}$	$\bar{y}\bar{x}\bar{y}$	Description
$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1} \sim \bar{2}$
$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1} \not\sim \bar{3}$
$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{1} \not\sim \bar{4}$
$\bar{1}$	$\bar{5}$	$\bar{5}$	$\bar{1}$	$\bar{1} \sim \bar{5}$
$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{0}$	$\bar{2} \not\sim \bar{3}$
$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{2}$	$\bar{2} \sim \bar{4}$
$\bar{2}$	$\bar{5}$	$\bar{2}$	$\bar{2}$	$\bar{2} \not\sim \bar{5}$
$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{0}$	$\bar{3} \not\sim \bar{4}$
$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{3}$	$\bar{3} \not\sim \bar{5}$
$\bar{4}$	$\bar{5}$	$\bar{2}$	$\bar{4}$	$\bar{4} \sim \bar{5}$

Based on Table 1, we have $\bar{3} \in B(\mathbb{Z}_6)$, so the vertex set of non-braid graph $Y_{\mathbb{Z}_6}$ is $V(Y_{\mathbb{Z}_6}) = \mathbb{Z}_6 \setminus B(\mathbb{Z}_6) = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}$. Furthermore, the edge set of non-braid graph $Y_{\mathbb{Z}_6}$ is $E(Y_{\mathbb{Z}_6}) = \{(\bar{1}, \bar{2}), (\bar{1}, \bar{5}), (\bar{2}, \bar{4}), (\bar{4}, \bar{5})\}$, so we get a non-braid graph $Y_{\mathbb{Z}_6}$ which is illustrated by Figure 1.

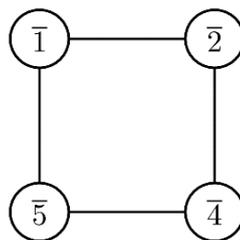


Figure 1. Non-braid Graph $Y_{\mathbb{Z}_6}$

Example 2.2. For every $\bar{x} \in \mathbb{Z}_4$, we get $\bar{x}\bar{0}\bar{x} = \bar{0}\bar{x}\bar{0}$, so $\bar{0} \in B(\mathbb{Z}_4)$. Then, the result of $\bar{x}\bar{y}\bar{x}$ and $\bar{y}\bar{x}\bar{y}$ for every $\bar{x}, \bar{y} \in \mathbb{Z}_4 \setminus \{\bar{0}\}$ where $\bar{x} \neq \bar{y}$ is shown on Table 2.

Table 2. The result of $\bar{x}\bar{y}\bar{x}$ and $\bar{y}\bar{x}\bar{y}$ for every $\bar{x}, \bar{y} \in \mathbb{Z}_4 \setminus \{\bar{0}\}$ where $\bar{x} \neq \bar{y}$

\bar{x}	\bar{y}	$\bar{x}\bar{y}\bar{x}$	$\bar{y}\bar{x}\bar{y}$	Description
$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1} \sim \bar{2}$
$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1} \sim \bar{3}$
$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{2}$	$\bar{2} \sim \bar{3}$

Based on the Table 2, we get that all vertices are adjacent to others. So, we get a non-braid graph $Y_{\mathbb{Z}_4}$ illustrated by Figure 2.

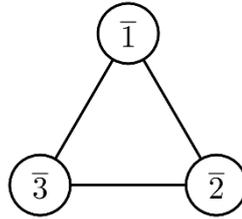


Figure 2. Non-braid graph $Y_{\mathbb{Z}_4}$

Examples 2.1 and 2.2 show us that for any number n , we cannot obtain a certain non-braid graph, it could be complete or not. Hence in general, a non-braid graph $Y_{\mathbb{Z}_n}$ is not always a complete graph. However, we prove that it is complete when n is a prime number, as we give in the following theorem.

Theorem 2.2. If p is prime, then $Y_{\mathbb{Z}_p}$ is a complete graph.

Proof. Let $\bar{x}, \bar{y} \in \mathbb{Z}_p \setminus B(\mathbb{Z}_p)$ where $\bar{x} \neq \bar{y}$. We will show that $\bar{x} \sim \bar{y}$. Assume that $\bar{x} \not\sim \bar{y}$. It means $\overline{xyx} = \overline{yxy}$. Since \mathbb{Z}_p is a field, the cancellation properties hold and we get $\bar{x} = \bar{y}$. This contradicts the hypothesis. Hence, $\bar{x} \sim \bar{y}$. ■

The following is an example of a non-braid graph of ring \mathbb{Z}_7 which is a complete graph.

Example 2.3. For every $\bar{x} \in \mathbb{Z}_7$, we get $\bar{x}\bar{0}\bar{x} = \bar{0}\bar{x}\bar{0}$, so $\bar{0} \in B(\mathbb{Z}_7)$. Then, the result of $\bar{x}\bar{y}\bar{x}$ and $\bar{y}\bar{x}\bar{y}$ for every $\bar{x}, \bar{y} \in \mathbb{Z}_7 \setminus \{\bar{0}\}$ where $\bar{x} \neq \bar{y}$ is shown in Table 3.

Table 3. The result of $\bar{x}\bar{y}\bar{x}$ and $\bar{y}\bar{x}\bar{y}$ for every $\bar{x}, \bar{y} \in \mathbb{Z}_7 \setminus \{\bar{0}\}$ where $\bar{x} \neq \bar{y}$

\bar{x}	\bar{y}	$\bar{x}\bar{y}\bar{x}$	$\bar{y}\bar{x}\bar{y}$	Description
$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1} \sim \bar{2}$
$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1} \sim \bar{3}$
$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{2}$	$\bar{1} \sim \bar{4}$
$\bar{1}$	$\bar{5}$	$\bar{5}$	$\bar{4}$	$\bar{1} \sim \bar{5}$
$\bar{1}$	$\bar{6}$	$\bar{6}$	$\bar{1}$	$\bar{1} \sim \bar{6}$
$\bar{2}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2} \sim \bar{3}$
$\bar{2}$	$\bar{4}$	$\bar{2}$	$\bar{4}$	$\bar{2} \sim \bar{4}$
$\bar{2}$	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2} \sim \bar{5}$
$\bar{2}$	$\bar{6}$	$\bar{3}$	$\bar{2}$	$\bar{2} \sim \bar{6}$
$\bar{3}$	$\bar{4}$	$\bar{1}$	$\bar{6}$	$\bar{3} \sim \bar{4}$
$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{5}$	$\bar{3} \sim \bar{5}$
$\bar{3}$	$\bar{6}$	$\bar{5}$	$\bar{3}$	$\bar{3} \sim \bar{6}$
$\bar{4}$	$\bar{5}$	$\bar{3}$	$\bar{2}$	$\bar{4} \sim \bar{5}$
$\bar{4}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{4} \sim \bar{6}$
$\bar{5}$	$\bar{6}$	$\bar{3}$	$\bar{5}$	$\bar{5} \sim \bar{6}$

Based on Table 3, we get that all vertices are adjacent to others. So, we get a non-braid graph $Y_{\mathbb{Z}_7}$ illustrated by Figure 3.

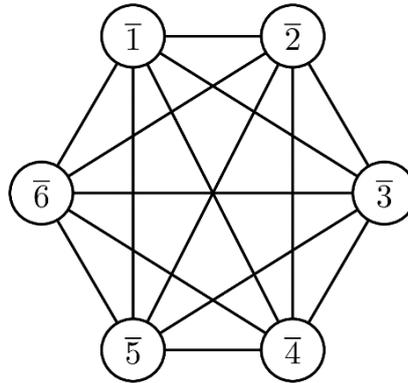


Figure 3. Non-braid graph $Y_{\mathbb{Z}_7}$

3. Braider of \mathbb{Z}_n

Regarding of connection of non-braid graph $Y_{\mathbb{Z}_n}$ we start to find out the braider of \mathbb{Z}_n . To begin with, we study a useful property of ring \mathbb{Z}_n for finding out the braider of \mathbb{Z}_n .

Lemma 2.3. Let \mathbb{Z}_{2m} be a ring where $m \in \mathbb{N}$ and $m > 1$. The number m is odd if and only if \overline{m} is an idempotent element.

Proof. (\Rightarrow) We show that \overline{m} is an idempotent element by showing that $\overline{m^2} = \overline{m}$. Since m is odd and $m > 1$, then $m = 2k + 1$ for some $k \in \mathbb{N}$. Note that

$$m^2 - m = m(m - 1) = m(2k + 1 - 1) = m(2k) = (2m)k$$

thus $\overline{m^2 - m} = \overline{0}$ and $\overline{m^2} = \overline{m}$.

(\Leftarrow) Suppose that \overline{m} is an idempotent element. Assume that m is an even number. Then, there is $k \in \mathbb{N}$ such that $m = 2k$. Note that

$$\overline{m} = \overline{m^2} = \overline{(2k)^2} = \overline{k(2(2k))} = \overline{k(2m)} = \overline{k2m} = \overline{0}$$

It means that there is $l \in \mathbb{Z}$ such that $m = l(2m)$. Then, we obtain

$$m = l(2m) \Leftrightarrow m(2l - 1) = 0 \Leftrightarrow m = 0 \text{ or } 2l - 1 = 0.$$

Since there is no $l \in \mathbb{Z}$ such that $2l = 1$, we have $m = 0$. Contradicts to the hypothesis that $m \neq 0$. Hence, m must be an odd number. ■

The results of the braider of \mathbb{Z}_n for $n = 1, \dots, 10$ are given in the following table:

Table 4. The braider of \mathbb{Z}_n for $n = 1, \dots, 10$

n	$B(\mathbb{Z}_n)$
1	$\{\overline{0}\}$
2	$\{\overline{0}, \overline{1}\}$
3	$\{\overline{0}\}$
4	$\{\overline{0}\}$
5	$\{\overline{0}\}$
6	$\{\overline{0}, \overline{3}\}$
7	$\{\overline{0}\}$
8	$\{\overline{0}\}$
9	$\{\overline{0}\}$

$$\overline{10 \quad \{0, 5\}}$$

By using Lemma 2.3, we have the following result of the braider of \mathbb{Z}_n for every $n \in \mathbb{N}$.

Lemma 2.4. If $n \in \mathbb{N}$, then

$$B(\mathbb{Z}_n) = \begin{cases} \{\overline{0}, \overline{m}\}, & n = 2m \text{ where } m \text{ is odd} \\ \{\overline{0}\}, & \text{otherwise.} \end{cases}$$

Proof. Suppose $n \in \mathbb{N}$, then we have the following result:

- a. For $n \neq 2m$ where m is odd, we will show that $B(\mathbb{Z}_n) = \{\overline{0}\}$. Consider that $\overline{0} \in B(\mathbb{Z}_n)$, because for every $\overline{x} \in \mathbb{Z}_n$, we get $\overline{x} \overline{0} \overline{x} = \overline{x0x} = \overline{0} = \overline{0x0} = \overline{0} \overline{x} \overline{0}$. Thus, we have $\{\overline{0}\} \subseteq B(\mathbb{Z}_n)$. Next, we will show that $B(\mathbb{Z}_n) \subseteq \{\overline{0}\}$. Assume that $B(\mathbb{Z}_n) \not\subseteq \{\overline{0}\}$. It means that there is $\overline{x} \neq \overline{0}$ such that $\overline{x} \in B(\mathbb{Z}_n)$.

Case 1. If \overline{x} is not an idempotent element, then there is $\overline{1} \in \mathbb{Z}_n$ such that $\overline{1} \overline{x} \overline{1} = \overline{x}$ and $\overline{x} \overline{1} \overline{x} = \overline{x}^2$. Since \overline{x} is not an idempotent element, $\overline{x}^2 \neq \overline{x}$, which results in $\overline{1} \overline{x} \overline{1} \neq \overline{x} \overline{1} \overline{x}$.

Case 2. If \overline{x} is an idempotent element, then we claim that there is $\overline{n-1} \in \mathbb{Z}_n$ such that $\overline{n-1} \overline{x} \overline{n-1} \neq \overline{x} \overline{n-1} \overline{x}$. Assume that $\overline{n-1} \overline{x} \overline{n-1} = \overline{x} \overline{n-1} \overline{x}$. Since \overline{x} is an idempotent element, we have

$$\begin{aligned} \overline{n-1} \overline{x} \overline{n-1} &= \overline{x} \overline{n-1} \overline{x} \\ \overline{n-1}^2 \overline{x} &= \overline{x}^2 \overline{n-1} \\ \overline{1} \overline{x} &= \overline{x} \overline{n-1} \\ \overline{x} &= \overline{x(n-1)} \\ \overline{x} &= \overline{xn-x} \\ \overline{x} &= \overline{xn-x} \\ \overline{x} &= -\overline{x} \\ \overline{x} + \overline{x} &= \overline{x} - \overline{x} \\ 2\overline{x} &= \overline{0} \end{aligned} \quad \dots (1)$$

Subsequently, since $n \neq 2m$ where m is odd, there are two possibilities for the value of n , namely, n is odd or $n = 2k$ where k is even and $k \neq 0$. It is obvious that for odd n , there is no \overline{x} that satisfies $2\overline{x} = \overline{0}$, which is a contradiction. Next, for $n = 2k$ where k is even and $k \neq 0$. It is clear that \overline{x} which satisfies $2\overline{x} = \overline{0}$ is only $\overline{x} = \overline{k}$. Since $\overline{x} = \overline{k}$ is an idempotent element, by Lemma 2.3, we get that k is odd, which is a contradiction. Hence, it is proven that $\overline{n-1} \overline{x} \overline{n-1} \neq \overline{x} \overline{n-1} \overline{x}$.

By cases 1 and 2, we obtain that every $\overline{x} \neq \overline{0}$ satisfies $\overline{x} \notin B(\mathbb{Z}_n)$ which is a contradiction. Thus, we have $B(\mathbb{Z}_n) \subseteq \{\overline{0}\}$. Hence, it is proven that $B(\mathbb{Z}_n) = \{\overline{0}\}$.

- b. For $n = 2m$ where m odd, we will show that $B(\mathbb{Z}_{2m}) = \{\overline{0}, \overline{m}\}$. It is clear that $\overline{0} \in B(\mathbb{Z}_{2m})$. We will show that $\overline{m} \in B(\mathbb{Z}_{2m})$, by showing that $\overline{m} \overline{x} \overline{m} = \overline{x} \overline{m} \overline{x}$ for every $\overline{x} \in \mathbb{Z}_{2m}$.

Case 1. For $\overline{x} = \overline{2a}$ where $a \in \mathbb{Z}$, we get $\overline{x} \overline{m} = \overline{0}$, so $\overline{m} \overline{x} \overline{m} = \overline{x} \overline{m} \overline{x}$.

Case 2. For $\overline{x} = \overline{2a+1}$ where $a \in \mathbb{Z}$, we will show that $\overline{m} \overline{x} \overline{m} = \overline{x} \overline{m} \overline{x}$.

Note that, by Lemma 2.3, we have

$$\overline{m \bar{x} m} = \overline{\bar{x} m \bar{x}} \Leftrightarrow \overline{m^2 x} = \overline{x^2 m} \Leftrightarrow \overline{m x} = \overline{x^2 m} \Leftrightarrow \overline{m(2a+1)} = \overline{(2a+1)^2 m}.$$

Thus, it is equivalent to show that $\overline{m(2a+1)} = \overline{(2a+1)^2 m}$. Subsequently,

$$\begin{aligned} (2a+1)^2 m - m(2a+1) &= m(2a+1)(2a+1-1) \\ &= m(2a+1)(2a) \\ &= (2m)(2a^2+a) \end{aligned}$$

so $\overline{(2a+1)^2 m - m(2a+1)} = \overline{0}$, which result in $\overline{m(2a+1)} = \overline{(2a+1)^2 m}$.

By cases 1 and 2, it is proven that $\overline{m} \in B(\mathbb{Z}_{2m})$. Hence $\{\overline{0}, \overline{m}\} \subseteq B(\mathbb{Z}_{2m})$.

Next, we show that $B(\mathbb{Z}_{2m}) \subseteq \{\overline{0}, \overline{m}\}$. Assume that $B(\mathbb{Z}_{2m}) \not\subseteq \{\overline{0}, \overline{m}\}$. It means that there is $\bar{x} \in B(\mathbb{Z}_{2m})$, but $\bar{x} \notin \{\overline{0}, \overline{m}\}$. If \bar{x} is not an idempotent element, then there is $\bar{1} \in \mathbb{Z}_{2m}$ such that $\overline{\bar{1} \bar{x} \bar{1}} \neq \overline{\bar{x} \bar{1} \bar{x}}$. If \bar{x} is an idempotent element, then we claim that there is $n-1 \in \mathbb{Z}_{2m}$ such that $\overline{\bar{x} n-1 \bar{x}} \neq \overline{n-1 \bar{x} n-1}$. Assume $\overline{\bar{x} n-1 \bar{x}} = \overline{n-1 \bar{x} n-1}$. By equation (1), \bar{x} satisfies $2\bar{x} = \overline{0}$. We notice that \bar{x} which satisfies $2\bar{x} = \overline{0}$ is only $\bar{x} = \overline{m}$, but $\bar{x} \neq \overline{m}$. So, it is proven that $\overline{\bar{x} n-1 \bar{x}} \neq \overline{n-1 \bar{x} n-1}$. Therefore, $\bar{x} \notin B(\mathbb{Z}_{2m})$ is a contradiction. Thus, we have $B(\mathbb{Z}_{2m}) \subseteq \{\overline{0}, \overline{m}\}$.

Hence, it is proven that $B(\mathbb{Z}_{2m}) = \{\overline{0}, \overline{m}\}$. ■

4. The Connectedness of The Non-braid Graph $\mathbf{Y}_{\mathbb{Z}_n}$

Now we study the adjacency properties of two vertices that will be useful for investigating the connection of non-braid graph $\mathbf{Y}_{\mathbb{Z}_n}$. We will find out which vertices are always adjacent to each other or not. We find that the adjacency of a vertex depends on whether it is an idempotent element or not. In the next two following lemma we will give the adjacency of idempotent element. We start by showing the necessary and sufficient condition of adjacency for any vertex with vertex $\bar{1}$.

Lemma 2.5. Let $\bar{x} \in \mathbb{Z}_n \setminus B(\mathbb{Z}_n)$. Vertex \bar{x} is adjacent to vertex $\bar{1}$ if and only if \bar{x} is not an idempotent element.

Proof. (\Rightarrow) Suppose vertex \bar{x} is adjacent to vertex $\bar{1}$. It means that $\overline{\bar{x} \bar{1} \bar{x}} \neq \overline{\bar{1} \bar{x} \bar{1}}$. Thus, $\overline{\bar{x}^2} \neq \overline{\bar{x}}$. In other words, \bar{x} is not an idempotent element.

(\Leftarrow) Suppose \bar{x} is not an idempotent element. Note that $\overline{\bar{1} \bar{x} \bar{1}} = \overline{\bar{x}}$ and $\overline{\bar{x} \bar{1} \bar{x}} = \overline{\bar{x}^2}$. Since \bar{x} is not an idempotent element, then $\overline{\bar{x}^2} \neq \overline{\bar{x}}$, which implies $\overline{\bar{1} \bar{x} \bar{1}} \neq \overline{\bar{x} \bar{1} \bar{x}}$. Hence, vertex \bar{x} is adjacent to vertex $\bar{1}$. ■

By Lemma 2.5, we can conclude that the vertex \bar{x} which is not an idempotent element is adjacent to $\bar{1}$. Next, we give the result of the adjacency property for the vertex \bar{x} , which is an idempotent element.

Lemma 2.6. Let $\bar{x} \in \mathbb{Z}_n \setminus B(\mathbb{Z}_n)$. If \bar{x} is an idempotent element, then vertex \bar{x} is adjacent to vertex $\overline{n-1}$.

Proof. Suppose \bar{x} is an idempotent element. We will show that vertex \bar{x} is adjacent to vertex $\overline{n-1}$. For $n \neq 2m$ where m is odd, by the proof of Lemma 2.4, point 1, case 2, we have

$\overline{x} \overline{n-1} \overline{x} \neq \overline{n-1} \overline{x} \overline{n-1}$. Hence, vertex \overline{x} is adjacent to vertex $\overline{n-1}$. Next, for $n = 2m$ where m is odd. Assume that $\overline{x} \overline{n-1} \overline{x} = \overline{n-1} \overline{x} \overline{n-1}$. Since \overline{x} is an idempotent element, then \overline{x} satisfies $2\overline{x} = \overline{0}$ by equation 1. It is clear that \overline{x} which satisfies $2\overline{x} = \overline{0}$ is only $\overline{x} = \overline{m}$. By Lemma 2.4, point 2, we have $\overline{x} = \overline{m} \in B(\mathbb{Z}_n)$, which is a contradiction. Hence, for $n = 2m$ where m is odd, we have $\overline{x} \overline{n-1} \overline{x} \neq \overline{n-1} \overline{x} \overline{n-1}$ and consequently, vertex \overline{x} is adjacent to vertex $\overline{n-1}$. Therefore, it is proven that vertex \overline{x} is adjacent to vertex $\overline{n-1}$. ■

Adjacency of any vertices depends on not only an idempotent element, but also a unit element. In the following lemma, we show the adjacency properties of two vertices, which are unit elements.

Lemma 2.7. Let $\overline{u}, \overline{y} \in \mathbb{Z}_n \setminus B(\mathbb{Z}_n)$. If \overline{u} and \overline{y} are unit elements where $\overline{u} \neq \overline{y}$, then vertex \overline{u} is adjacent to vertex \overline{y} .

Proof. Suppose \overline{u} and \overline{y} are unit elements where $\overline{u} \neq \overline{y}$. Assume that vertex \overline{u} is not adjacent to vertex \overline{y} . It means that $\overline{u} \overline{y} \overline{u} = \overline{y} \overline{u} \overline{y}$. Note that

$$\begin{aligned} \overline{u} \overline{y} \overline{u} &= \overline{y} \overline{u} \overline{y} \\ \overline{u} \overline{u} \overline{y} &= \overline{u} \overline{y} \overline{y} \\ \overline{u}^{-1} \overline{u} \overline{u} \overline{y} \overline{y}^{-1} &= \overline{u}^{-1} \overline{u} \overline{y} \overline{y} \overline{y}^{-1} \\ \overline{u} &= \overline{y} \end{aligned}$$

This is a contradiction. Hence, vertex \overline{u} is adjacent to vertex \overline{y} . ■

By using Lemma 2.5, 2.6, and 2.7, we give the result of the connection of non-braid graph $Y_{\mathbb{Z}_n}$ for any n .

Theorem 2.8. If $n \geq 3$, then $Y_{\mathbb{Z}_n}$ is a connected graph.

Proof. Let $\overline{x}, \overline{y} \in \mathbb{Z}_n \setminus B(\mathbb{Z}_n)$ where $\overline{x} \neq \overline{y}$. We will show that there is a path from vertex \overline{x} to vertex \overline{y} . We consider two cases.

Case 1. If \overline{x} and \overline{y} are unit elements, then by Lemma 2.7, vertex \overline{x} is adjacent to vertex \overline{y} .

Case 2. For \overline{x} or \overline{y} which is a zero-divisor.

Without loss of generality, suppose \overline{x} is a zero-divisor. Since every element of \mathbb{Z}_n is either an idempotent element or not, then zero-divisor \overline{x} is either an idempotent element or not. By Lemma 2.5, 2.6, 2.7, we have the following result. If \overline{x} is not an idempotent element, then

- a. for a zero-divisor \overline{y} which is not an idempotent element or \overline{y} which is a unit element where $\overline{y} \neq \overline{1}$, we can construct a path

$$\overline{x} - \overline{1} - \overline{y}, \tag{2}$$

- b. for $\overline{y} = \overline{1}$, we have that vertex \overline{x} is adjacent to vertex \overline{y} ,

- c. for a zero-divisor \overline{y} which is an idempotent element, we can construct a path

$$\overline{x} - \overline{1} - \overline{n-1} - \overline{y}, \tag{3}$$

If \overline{x} is an idempotent element, then

- a. for a zero-divisor \overline{y} which is not an idempotent element, we can construct a path

$$\overline{x} - \overline{n-1} - \overline{1} - \overline{y}, \tag{4}$$

- b. for a zero-divisor \overline{y} which is an idempotent element or \overline{y} which is a unit element where $\overline{y} \neq \overline{n-1}$, we can construct a path

$$\bar{x} - \overline{n-1} - \bar{y}, \quad \dots(5)$$

c. for $\bar{y} = \overline{n-1}$, we have that vertex \bar{x} is adjacent to vertex \bar{y} .

By cases 1 and 2, it is proven that there is a path from vertex \bar{x} to vertex \bar{y} . Hence, $Y_{\mathbb{Z}_n}$ is a connected graph. ■

C. CONCLUSION AND SUGGESTIONS

If n is a prime number, the non-braid graph $Y_{\mathbb{Z}_n}$ is a complete graph. For all $n \geq 3$ the non-braid graph $Y_{\mathbb{Z}_n}$ is a connected graph.

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