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Boundedness of Solution Operator Families for the Navier-Lamé Equations with Surface Tension in Whole Space

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ABSTRACT

In this paper, we consider the boundedness of the operator families in whole space for Navier-Lame model problem in bounded domain of $\mathbb{N}$ dimensional Euclidean space ($\mathbb{N} \geq 2$). To find the boundedness of the operator families, first of all we construct model problem in the form of the resolvent problem by using Laplace transform. Then, using Fourier transform, we get the solution formula of the model problem. In this paper, we use the qualitative methods to construct solution formula of velocity ($\mathbf{u}$). This step is fundamental stage to find the well-posedness of the model problem. As we known that fluid motion can be described in partial differential equation (PDE). Essential point in PDE are finding existence and uniqueness of the model problem. One methods of investigating the well-posedness is $\mathcal{R}$-boundedness of the solution operator families of the model problem. We can find the $\mathcal{R}$-boundedness of the solution operator families not only in whole-space, half-space, bent-half space and in general domain. In this paper we investigate the $\mathcal{R}$-boundedness of the solution operator families only in whole space. By using this $\mathcal{R}$-boundedness, we can find that the multipliers which form of the operator families are bounded with some positive constant.

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A. INTRODUCTION

A Fluid is any substance that can flow in the form of gas or liquid. Based on the movement, fluids are divided in two type, static fluids and dynamic fluids. Fluids also have the property not to resist deformation and the ability to flow. As generally, fluids have an ability to take shape of the container. Meanwhile, based on the shear stress, fluids also divided into two part, Newtonian and non-Newtonian fluids.

A Newtonian fluid is a fluid in which the viscous stresses arising from its flow, at every point, are linearly correlated to the local rate of change of its deformation over time which known as strain rate(Panton, 2013). Sometimes, this condition called that the forces are proportional to the rates of change of the fluid's velocity vector as one moves away from the point in various directions and follow Newton's law of viscosity. An example of a Newtonian fluids is water. Fluid also divide in two part that are compressible and incompressible. Related to this kinds of fluid, there are many researchers who conduct this problem. For example, in 2018, Watanabe investigated the compressible-incompressible two-phase flows phase transition: Model problem. He investigated two-phase flows separated by a sharp interface not only with a phase transition but also with a surface tension (Watanabe, 2018).

A non-Newtonian fluid is a fluid that does not follow Newton's law of viscosity, i.e., constant viscosity independent of stress. In non-Newtonian fluids, viscosity can change when under force to either more liquid or more solid. Tooth paste is an example of the non-Newtonian fluids. Since it become runnier when shaken.

In our daily lives, there are many condition related to the fluids phenomena. Therefore, studying fluid dynamics become an interesting point. The air we breathe, water flowing through the tap, blood stream flowing within our body, etc. Those are examples of fluid phenomenon. These examples plays an essential role in making life possible on out earth. Over years, scientists and researchers have contributed in this field of science to uncover the interesting phenomena and behaviour of fluids under varied condition. Moreover, the nature phenomena made it possible to be understood.

Fluid dynamics helps human to imagine the movement of fluids or gases. The fluid movement is described by conservation of mass, conservation of momentum and a transport energy. There are two critical elements of fluid dynamics that are viscosity and fluid flow regimes. One of the mechanical behaviour of the fluid is polymers. DNA is one of the biological synthesis polymers which is the important component of our body. Others example of the fluid motion are turbulent character such as river and ocean currents. Many technological applications use the phenomena of turbulence to achieve their purpose. Navier-Lame equation is one of the model fluid flow which is important. Since, the well-posedness of the model problem can be used as reference for other model such as Oldroyd-B model fluid flow.

Recently, many researchers investigated the behaviour of the fluid motion. Majority of them conducted in numerical analysis point of view, rarely of them focused on mathematical side. Therefore, this situation become motivation of author to conducted fluid dynamics in mathematical point of view. In 2012, Girault et.al using finite element methods to investigated the Navier-Stokes of the model fluid flow in theory and algorithms (Girault & Raviart, 2012).

In 20th century, Jacques Hadamard a French mathematician was defined that well-posed as a condition that the mathematical model of physical phenomena hold three properties that are the solution of the model problem exists, unique and the behaviour of the solution changes continuously with the initial conditions. For researchers who conducting in this mathematical model the main purpose of the research is investigating not only local well-posedness but also global well-posedness with many different methods.

In this article, we consider the $\mathcal{R}$-boundedness of the solution operator families for Navier-Lam$\acute{e}$ equation with surface tension in whole space. As we known, that the Navier-Lam$\acute{e}$ equation is the fundamental equation of motion in classical linear elastodynamics (Eringen & Suhubi, 1975). Navier-Lam$\acute{e}$ equation in cylinder coordinate has been investigated by Sakhr(Sakhr & Chronik, 2017). In 2020, (D. Liu & Li, 2020) have studied the blood flow problem in a blood vessel. This problem related to elastic Navier-Lame equations.

To Find the $\mathcal{R}$-sectoriality, first of all, we are applying Fourier transformation to the model problem. Then by using Weis's operator valued Fourier multiplier theorem, we estimate all multiplier. This $\mathcal{R}$-sectoriality introduced for first time by (Denk et al., 2005). The $\mathcal{R}$-bounded operator families arising from the study of Barotropic compressible flows with free surface has been studied by Zhang(Zhang, 2020). Some exact solution of Lam$\acute{e}$ equations with $n = 3$ by using Lie point transformations has been proved by Ozer (Özer, 2003). On the other hand, (Cao, 2009) investigated the solution of Navier equations and their representation structure. Flag partial differential equations and representation of Lie algebra studied by Xu (Xu, 2008).

Recently, there are many researchers who concern studying $\mathcal{R}$-boundedness. In 2014, Murata investigated the $\mathcal{R}$-boundedness of the Stokes operator families with slip boundary condition (Murata, 2014). On the other hand, (Maryani, 2016b) proved the maximal $L_p - L_q$ regularity for Oldroyd-B model fluid flow without surface tension in bounded and unbounded domain. In the same year, (Maryani, 2016a) investigated the global well-posedness in some bounded domain case.

As we known that multiphase flows are new phenomenon in fluid motion which particularly relevant in subsurface flow. For this case, it was assumed that the fluids are well separated which mean the fluids do not mix each other and also there are no additional particles resolved in the fluids. For two-phase problem, (Maryani & Saito, 2017) studied the $\mathcal{R}$-boundedness of the solution operator families for Stokes equation. Furthermore, (Inna et al., 2020) investigated other fluid model i.e Korteweg equation. In that paper, they prove the $\mathcal{R}$-boundedness of the solution operator families for Korteweg model problem. Other researchers who concerned in Korteweg problem is (Murata & Shibata, 2020). They studied the global-well-posedness for the compressible fluid model case. Other researchers who consider two phase problem of compressible and incompressible viscous fluids motion without surface tension studied by (Kubo & Shibata, 2021). In 2020, Saito and Zhang, investigated elliptic problems with two phase incompressible flows in unbounded domain (Saito & Zhang, 2019). In one year later, (Zhang, 2020) considered The R-bounded operator families arising from the study of the barotropic compressible flows with free surface. On the other hand, the validity of the NSE for nanoscale liquid flows was investigated by (C. Liu & Li, 2011).

In 2021 (Maryani et al., 2021) investigated the Stokes equations in half-space. In the article, she showed the formula of the Stokes equations in half-space without surface tension. One year before (Alif et al., 2021) studied Stokes equation's formula in three dimension Euclidean space by using Fourier transform. In 2020, (Oishi, 2021) investigated the solution formula and R-boundedness for the generalized Stokes resolvent problem in an infinite layer with Neumann Boundary condition. On the R-boundedness also studied by (Götz & Shibata, 2014) in 2014. They investigated compressible fluid flow with free surface. Before we state our main result in the next session, we introduce our notation used throughout the paper.

Notation $\mathbb{N}$denotes the sets of natural numbers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{C}$and$\mathbb{R}$ denote the sets of complex numbers and real numbers, respectively. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$and $\partial_x^\kappa = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$ with $x = (x_1, \dots, x_N)$. For scalar function f and N-vector of function $\mathbf{g}$, we get

$$\nabla f = (\partial_1 f, \dots, \partial_N f), \nabla\mathbf{g} = \{\partial_i g_j \mid i,j = 1, \dots, N\},$$
$$\nabla^2 f = \{\partial_i \partial_j f \mid i,j = 1, \dots, N\},\ \nabla^2\mathbf{g} = \{\partial_i \partial_j g_k \mid i,j,k = 1, \dots, N\}.$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by

$$\mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx,\ \ \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\, d\xi. \tag{1a}$$

We also write $\hat{f}(\xi) = \mathcal{F}_x[f](\xi)$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform, respectively, which are defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\, dt,\ \ \mathcal{L}^{-1}[g](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\tau)\, d\tau,$$

with$\lambda = \gamma + i\tau \in \mathbb{C}$.

For $\mathbf{x} = (x_1, \ldots, x_N)$and $\mathbf{y} = (y_1, \ldots, y_N)$, we set $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{N} x_j y_j$. For scalar functions f, g and N-vectors of function , $\mathbf{g}$, we get$(\mathbf{k}, \mathbf{g})_D = \int_D \mathbf{k} \cdot \mathbf{g}\, dx$, $(k, g)_\Gamma = \int_\Gamma kg\, d\sigma$, $(\mathbf{k}, \mathbf{g})_\Gamma = \int_\Gamma \mathbf{k} \cdot \mathbf{g}\, d\sigma$, where σ is the surface element of Γ. For $N \times N$ matrices of function $\mathbf{F} = (F_{ij})$and $\mathbf{G} = (G_{ij})$, we get $(\mathbf{F}, \mathbf{G})_D = \int_D \mathbf{F} : \mathbf{G}\, dx$, $(\mathbf{F}, \mathbf{G})_\Gamma = \int_\Gamma \mathbf{F} : \mathbf{G}\, d\sigma$, where $\mathbf{F} : \mathbf{G} \equiv \sum_{i,j=1}^{N} F_{ij} G_{ij}$. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a, b, \ldots$. The values of constants $C$and $C_{a,b,\ldots}$ denote a positive constant which maybe different even in a single chain of inequalities. We use small boldface letter, e.g. **u**to denote vector-valued functions and capital boldface letters, e.g. **H** to denote matrix-valued functions, respectively. But, we also use the Greek letters, e.g $\sigma, \rho, \theta, \tau, \omega$ such as mass densities.

In this paper we construct the solution formula of velocity (**u**) for the model problem for Navier-Lam$\acute{e}$ equation in whole space. For this purpose, we apply Laplace transform for the resolvent problem then by using Fourier transform and inverse Fourier transform we get the operator families of the solution. In this step, we also use Weis's operator valued Fourier multiplier theorem.

B. METHODS

The research methodology which used in this paper is literature review of the related articles. In this article, we defined the solution of the velocity of the Navier-Lam$\acute{e}$ equations in whole space case. The procedures are in the following, first of all, transforming model problem by using Laplace transform to be resolvent problem. Then applying Fourier transform and inverse Fourier transform in whole space case, we have the solution formula of velocity **u.** The last step, we estimate the multiplier of the solution formula **u,** by using Weis's multipliers theorem. In this paper, we study $\mathcal{R}$-boundedness of the solution operator families for Navier-Lam$\acute{e}$ equation with surface tension in whole space. This $\mathcal{R}$-boundedness is the tools to prove further research which related to maximal $L_p - L_q$ regularity class. We prove the $\mathcal{R}$-bounded solution operators of the generalized resolvent problem of the Navier-Lam$\acute{e}$ equations by using Weis's operator valued Fourier multiplier.

As we know that the mathematical model of fluid motion formed by conservation of mass and conservation of momentum. These conservations guarantee the boundary condition of the model problem. To prove the existence of $\mathcal{R}$-bounded solution operators, first of all, we transform model problem to resolvent problem by using Laplace transform. Then, applying Fourier transform to model for getting multiplier of the operator families. This methods have been introduced by(Enomoto & Shibata, 2013). As we explained above that the research methods in this paper, we apply Fourier transform and Laplace transform to the model problem to get the solution operator families. Then by using Weis's multiplier theorem, we prove the existence of the model problem in whole space. The procedures of the research can be described in the following diagram (Figure 1)

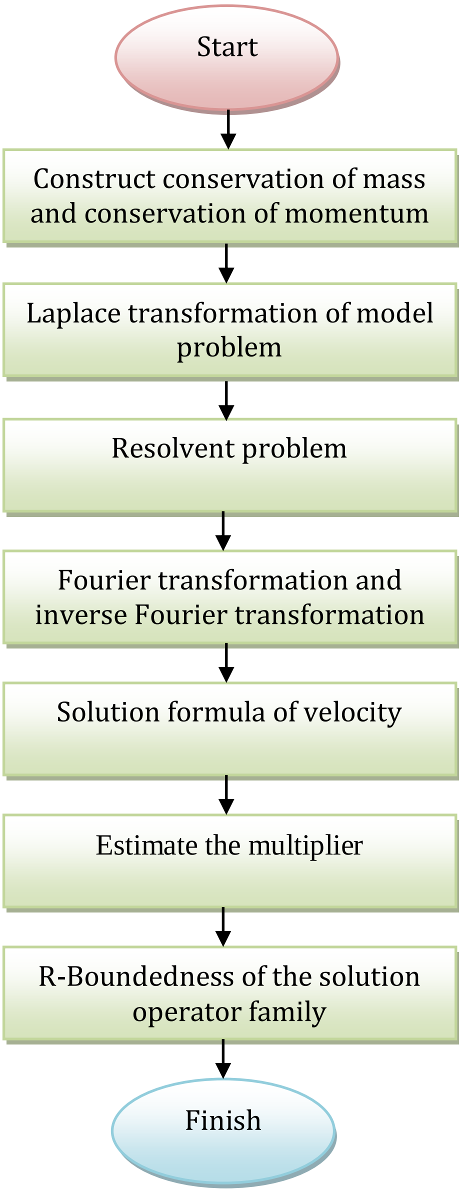

Figure 1. Flowchart of the research

The model problem of Navier-Lam$\acute{e}$ equation with surface tension will be explained in the following section.

C. RESULT AND DISCUSSION

1. Construction

Let $\mathbf{u}$ and Ω be a velocity field and a bounded domain in N-dimensional Euclidean space $\mathbb{R}^N$ $(N \geq 2)$, respectively. The formula of the generalized resolvent problem of Navier-Lam$\acute{e}$ equations in bounded domain with surface tension is described in the following

$$\begin{aligned} \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} &= \mathbf{f} && \text{in } \Omega \\ (\boldsymbol{\alpha}\mathbf{D}(\mathbf{u}) - (\beta-\alpha)\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n} - \sigma(\Delta'_{\Gamma}\eta)\mathbf{n} &= \mathbf{g} && \text{in } \Gamma \\ \lambda\eta + \mathbf{a}'\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n} &= \mathrm{d} && \text{in } \Gamma \end{aligned} \tag{1}$$

where $\mathbf{a}' = (a_1, \ldots, a_N) \in \mathbb{R}^{\boldsymbol{N-1}}$ and $\mathbf{a}'\cdot\nabla'\eta = \sum_{j=1}^{N-1} a_j\partial_j\eta$. We assume that

$$|\mathbf{a}'| \leq a_0$$

for some constant $a_0 > 0$. The $\mathbf{n} = (0, \dots, 0, -1)$ be the unit outer normal to $\Gamma$and $\mathbf{D}(\mathbf{u}), \mathbf{u} = (u_1, \dots, u_N)$, the doubled deformation tensor whose (i,j) components are $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i \quad (\partial_i = \partial/\partial x_i)$, $\mathbf{I}$ the $N \times N$ the identity matrix, α, β are positive constants (α and β are the first and second viscosity coefficients, respectively) such that $\beta - \alpha > 0$. Meanwhile, Δ'_Γ is the Laplace-Beltrami operator on Γ.

Before state our main result, first of all we introduce the definition of $\mathcal{R}$-boundedness and the operator valued Fourier multiplier theorem due to Weis (Weis, 2001).

Definition 1. [$\mathcal{R}$-boundedness] A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^N \subset \mathcal{T}$, $\{f_j\}_{j=1}^N \subset X$ and sequences $\{r_j\}_{j=1}^N$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the inequality:

$$\left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j x_j \right\|_Y^p du \right\}^{1/p} \leq C \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\|_X^p du \right\}^{1/p}$$

the smallest such C is called $\mathcal{R}$-bounded of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Theorem 2. [Weis's Theorem] Let X and Ybe two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X,Y))$ such that

$$\mathcal{R}_{\mathcal{L}(X,Y)} \left(\{ \left(\tau \frac{d}{d\tau} \right)^\ell M(\tau) \;\Big|\; \tau \in \mathbb{R} \setminus \{0\} \} \right) \leq \kappa < \infty \quad (l = 0,1)$$

with some constant κ. Then, the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ defined by

$$T_M \phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)),$$

is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \leq C_\kappa ,$$

for some positive constant C depending on p, X and Y.

Following theorem and lemma are technical theorems which used to prove main theorem.

Theorem 3. Let $1 < q < \infty$ and $\sum_{\epsilon,\lambda_0}$ be a set in $\mathbb{C}$. Let $m(\lambda, \xi)$ be a function defined on $\sum_{\epsilon,\lambda_0} \times (\mathbb{R}^N \setminus \{0\})$ such that for any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a constant C_α depending solely on α and $\sum_{\epsilon,\lambda_0}$ such that

$$\left| \partial_\xi^\alpha m(\lambda, \xi) \right| \leq C_\alpha |\xi|^{-\alpha}$$

for any $(\lambda, \xi) \in \sum_{\epsilon,\lambda_0} \times (\mathbb{R}^N \setminus \{0\})$. Let K_λ be an operator defined by

$$K_\lambda f = \mathcal{F}_\xi^{-1}[m(\lambda, \xi) \hat{f}\ (\xi)].$$

Then the set $\{K_\lambda \mid \lambda \in \sum_{\epsilon,\lambda_0}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda \in \sum_{\epsilon,\lambda_0} \quad) \leq C_{q,N} \max_{|\alpha| \leq N+1} C_\alpha$$

with some constant $C_{q,N}$ depends solely on q and N.

Lemma 4. Let $0 < \epsilon < \frac{\pi}{2}$, $\sum_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}$. Then we have the following assertion,

1. For any $\lambda \in \sum_\epsilon$ and $\xi \in \mathbb{R}^N$ we have
$$|\alpha^{-1}\lambda + |\xi|^2| \geq \sin\frac{\epsilon}{2}(\alpha^{-1}|\lambda| + |\xi|^2).$$
2. Let $a, b \in \sum_\epsilon$ then
$$|a + b| \geq \sin\frac{\epsilon}{2}(|a| + |b|\ \ ).$$

The proof of Lemma 4 can be seen in (Shibata & Tanaka, 2004).

The following theorem is the main theorem of this article.

Theorem 5. Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and$\lambda_0 > 0$. Then, there exists an operator family $\mathcal{S}_0(\lambda) \in \mathrm{Hol}\,(\sum_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^N, W_q^2(\mathbb{R}^N)^N))$ such that for any $\lambda = \gamma + i\tau \in \sum_{\epsilon,\lambda_0}$, and $\mathbf{f} \in L_q(\mathbb{R}^N)^N$, $\mathbf{u} = \mathcal{S}_0(\lambda)\mathbf{f}$ is a unique solution of (1) and

$$\mathcal{R}_{\mathcal{L}\left(L_q(\mathbb{R}^N)^N, W_q^{2-j}(\mathbb{R}^N)^N\right)}\left(\{\left(\tau\frac{d}{d\tau}\right)^\ell (\lambda^{j/2}\mathcal{S}_0(\lambda)) \,\middle|\, \lambda \in \textstyle\sum_{\epsilon,\lambda_0},\}\right) \leq \kappa_1(\lambda_0)$$

for $\ell = 0{,}1$ and $j = 0, 1, 2$ where $\kappa_1(\lambda_0)$ is a constant depending on ϵ, λ_0, q and N.

2. Proof

In this subsection, we derive the proof of the Theorem 4. First of all, we consider the first equation of (1) i.e

$$\lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = \mathbf{f} \qquad \text{in } \Omega. \tag{2}$$

Applying div to equation (2), we have

$$(\lambda - (\alpha + \beta)\Delta)\mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{f}\,. \tag{3}$$

By substituting (3) to (2), we get

$$\boldsymbol{u} = (\lambda - \alpha\Delta)^{-1}\mathbf{f} + \beta(\lambda - \alpha\Delta)^{-1}\nabla(\lambda - (\alpha + \beta)\Delta)^{-1}\mathrm{div}\,\mathbf{f}. \tag{4}$$

Multiplying (4) by $\{(\lambda - \alpha\Delta) - (\lambda - (\alpha + \beta)\Delta)\}(\beta\Delta)^{-1}$ to the second term of the right-hand side of equation (4), we have the formula of $\boldsymbol{u} = (u_1, \dots, u_N)$ i.e

$$\boldsymbol{u} = (\lambda - \alpha\Delta)^{-1}\mathbf{f} + (\lambda - (\alpha + \beta)\Delta)^{-1}\Delta^{-1}\nabla\mathrm{div}\,\mathbf{f} - (\lambda - \alpha\Delta)^{-1}\Delta^{-1}\nabla\mathrm{div}\,\mathbf{f}. \tag{5}$$

Applying Fourier transform and then inverse Fourier transform to equation (5), we have Formula $\boldsymbol{u} = (u_1, \dots, u_N)$in the following

$$\boldsymbol{u} = \mathcal{F}_\xi^{-1}\left[\frac{\mathcal{F}[\mathbf{f}]}{\lambda + \alpha|\xi|^2}\right] + \mathcal{F}_\xi^{-1}\left[\frac{(\xi\xi\cdot\mathcal{F}[\mathbf{f}])|\xi|^{-2}}{\lambda + (\alpha+\beta)|\xi|^2}\right] - \mathcal{F}_\xi^{-1}\left[\frac{(\xi\xi\cdot\mathcal{F}[\mathbf{f}])|\xi|^{-2}}{\lambda + \alpha|\xi|^2}\right], \tag{6}$$

where $\mathcal{F}$ and $\mathcal{F}_\xi^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively which defined in (1a). We can write the equation (6) in the following

$$u_j(x) = \sum_{k=1}^{N} \frac{1}{\alpha}\, \mathcal{F}_\xi^{-1}\left[\frac{P(\xi)}{\alpha^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x) \quad + \sum_{k=1}^{N} \frac{1}{(\alpha+\beta)}\mathcal{F}_\xi^{-1}\left[\frac{Q(\xi)}{(\alpha+\beta)^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x), \tag{7}$$

where $P(\xi)$ and $Q(\xi)$ are $N \times N$ matrices whose (j,k) component $P_{jk}(\xi)$ and $Q_{jk}(\xi)$ are given by the formula$P_{jk}(\xi) = \delta_{jk} - \xi_j\xi_k|\xi|^{-2}$ and $Q_{jk}(\xi) = \xi_j\xi_k|\xi|^{-2}$, with δ_{jk} denote as Kronecker delta symbols defined by

$$\delta_{jk} = \begin{cases} 1, & \text{for } j = k \\ 0, & \text{for } j \neq k \end{cases}.$$

Moreover, we define all operator family $\mathcal{S}_0(\lambda)$ respect to $\mathbf{f} \in L_q(\mathbb{R}^N)^N$in equation (6) by

$$\mathcal{S}_0(\lambda)\mathbf{f} = \mathcal{F}_\xi^{-1}\left[\frac{\mathcal{F}[\mathbf{f}]}{\lambda+\alpha|\xi|^2}\right] + \mathcal{F}_\xi^{-1}\left[\frac{(\xi\xi\cdot\mathcal{F}[\mathbf{f}])|\xi|^{-2}}{\lambda+(\alpha+\beta)|\xi|^2}\right] - \mathcal{F}_\xi^{-1}\left[\frac{(\xi\xi\cdot\mathcal{F}[\mathbf{f}])|\xi|^{-2}}{\lambda+\alpha|\xi|^2}\right]. \tag{8}$$

Furthermore, based on solution formula (7) in whole-space, we can write $u_j(x)$

$$\lambda u_j(x) = \sum_{k=1}^{N} \frac{\lambda}{\alpha}\, \mathcal{F}_\xi^{-1}\left[\frac{P(\xi)}{\alpha^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x) + \sum_{k=1}^{N} \frac{\lambda}{(\alpha+\beta)}\mathcal{F}_\xi^{-1}\left[\frac{Q(\xi)}{(\alpha+\beta)^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x),$$

$$|\lambda|^{1/2}\partial_\ell u_j(x) = \sum_{k=1}^{N} \frac{|\lambda|^{1/2}}{\alpha}\, \mathcal{F}_\xi^{-1}\left[\frac{i\xi_\ell P(\xi)}{\alpha^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x) + \sum_{k=1}^{N} \frac{|\lambda|^{1/2}}{(\alpha+\beta)}\mathcal{F}_\xi^{-1}\left[\frac{i\xi_\ell Q(\xi)}{(\alpha+\beta)^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x),$$

$$\partial_\ell\partial_m u_j(x) = \sum_{k=1}^{N} \frac{|\lambda|^{1/2}}{\alpha}\, \mathcal{F}_\xi^{-1}\left[\frac{i\xi_\ell P(\xi)}{\alpha^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x) \quad + \sum_{k=1}^{N} \frac{|\lambda|^{1/2}}{(\alpha+\beta)}\mathcal{F}_\xi^{-1}\left[\frac{i\xi_\ell Q(\xi)}{(\alpha+\beta)^{-1}\lambda+|\xi|^2}\hat{f}_k(\xi)\right](x). \tag{9}$$

By using Lemma 4 and Theorem 3, we can prove the main Theorem 5, which complete the proof of the main theorem. This result is different from (Enomoto & Shibata, 2013), in that article they studied Stokes equation model problem without surface tension.

D. CONCLUSION AND SUGGESTIONS

As we see in section C, that the operator families of equation (8), which carried out satisfy the Weis's Theorem. Then we called as $\mathcal{R}$-boundedness as Definition 1. Therefore, for Navier-Lame model problem in bounded domain of $\mathbb{N}$ dimensional Euclidean space ($\mathbb{N} \geq 2$), we can find $\mathcal{R}$-boundedness for operator families of model problem. This result of the $\mathcal{R}$-boundedness can be used for further research to investigate the boundedness in half-space. Then we can study the well-posedness of the model problem. This well-posedness is the main purpose for the partial differential equations problem.

For PDE's, it is important to prove well-posedness properties. One method to prove well-posedness is regularity, we know that $\mathcal{R}$-boundedness becomes one technique to find it. For the further research, one can investigate same problem in half-space case.

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