

On the Explicit Formula for Eigenvalues, Determinant, and Inverse of Circulant Matrices

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ABSTRACT

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Determining eigenvalues, determinants, and inverse for a general matrix is computationally hard work, especially when the size of the matrix is large enough. But, if the matrix has a special type of entry, then there is an opportunity to make it much easier by giving its explicit formulation. In this article, we derive explicit formulas for determining eigenvalues, determinants, and inverses of circulant matrices with entries in the first row of those matrices in any formation of a sequence of numbers. The main method of our study is exploiting the circulant property of the matrix and associating it with cyclic group theory to get the results of the formulation. In every discussion of those concepts, we also present some computation remarks.



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A. INTRODUCTION

Discussion about circulant matrices has become more interesting research topics over last decades because of those can be associated with many areas of mathematical problems: numerical analysis, linear differential equations, operator theory, lightweight cryptography, and many others; hence also connected to computer science and engineering (D. Bozkurt & T.-Y. Tam, 2015). All of those take advantage of the nice structure of the circulant matrix so that the calculation of eigenvalues, eigenvectors, determinants, and inverse of the matrices can be formulated explicitly. This is what we are concerned about, in this paper we propose a kind of overview to derive the formulations based on the exploitation of cyclic group properties.

To broaden our insight, we refer to some papers that recently have studied the above problem with various specializations. Without intending to exclude any other articles whose similar topics to this topic but missed from our consideration, we start in 2012, similar problems but with different kinds of circulant matrices and different types of the sequence of

numbers, these are with the k -Fibonacci and k -Lucas numbers. The investigation is also about their invertibility. Then, (D. Bozkurt & T.-Y. Tam, 2016) establish some useful formulas for the determinants and inverses of circulant matrices using the nice properties of the number sequences, (X. Jiang & K. Hong, 2015) concerned with explicit inverse matrices of Tribonacci skew circulant type matrices, (Jiteng Jia & Sumei Li, 2015) found the formulation of the inverse and determinant of general bordered tridiagonal matrices, and (Ercan Altinisik et al., 2015) formulated the determinants and inverses of circulant matrices with complex Fibonacci numbers.

In 2016 we refer three papers: (Türkmen & Gökbaş, 2016) investigated norm of r -circulant matrices with the Pell and Pell-Lucas numbers, (B. Radicic, 2016) concerned on k -circulant matrices with geometric sequence, and (Nazmiye Yilmaz et al., 2016) about the g -circulant matrix involving the generalized k -Horadam numbers. Radicic continued her works on k -circulant matrices with arithmetic numbers (Biljana Radicic, 2017), Lucas numbers (Biljana Radicic, 2018a), and biperiodic Fibonacci and Lucas numbers (Biljana Radicic, 2018b). Three papers in 2018: (Mustafa Bahsi & Soleyman Solak, 2018) talked about the g -circulant matrices, (Yun Fun & Hualu Liu, 2018) about double circulant matrices, and (Jinjiang Yao & Jixiu Sun, 2018) about explicit determinants and inverses of skew left circulant matrices with the Pell-Lucas Numbers.

In 2019 Radicic (B. Radicic, 2019), (Biljana Radicic, 2019) still continued her works, here about k -circulant matrices involving the Jacobsthal and geometric numbers. We also refer to (Zhongyun Liu et al., 2019) about the eigen-structures of real skew circulant matrices with some applications, and with similar problems we can see also in (Xiaoting Chen, 2019), (Emrullah Kirkclar & Fatih Yilmaz, 2019), and (Jiang et al., 2019). Most recently, the problems concerning determinants, inverses, norms, and spreads of circulant matrices and their variations is still in great interest among researchers, we see for example in (Yunlan Wei et al., 2020), (A. C. F. Bueno, 2020), (Ma et al., 2021).

Inspired by all the above beautiful references, in this current paper we derive explicit formulas for determining eigenvalues, determinants, and inverses of circulant matrices with entry in general formation of numbers sequence, instead of a specific numbers sequence or defined by recurrence relation as we can see in the above references. The basic methods of the formulations are mainly by exploiting cyclic group properties which induced from the definition of the circulant matrix.

B. METHODS

In Section C.1, firstly we review the notion of n th root of unity in the system of complex numbers. Then, we derive a group cyclic notion that comes from the set of all n th roots of unity in the system of complex numbers. This group cyclic notion will become the basic theory of the subsequent sections which concern the formulations of eigenvalues, determinants, and inverses for circulant matrices of general type of entry.

In Section C.2, firstly we give an overview of how to get an explicit formula of eigenvalues for a general circulant matrix as presented in Theorem 1 whose proof is mainly based on the basic theory of the cyclic group explained in Section C.1. Then, the formulation of the determinant is easily derived from the above eigenvalues formulation using the spectral theory of the circulant matrix. As a corollary of the theorem, we also explain its relationship

with the determinant of the left circulant matrix based on the theory of elementary row operations on a matrix.

In Section C.3, we derive an explicit formula of the inverse of circulant matrices presented in Theorem 2. To prove this theorem, we need two lemmas which we derive by exploiting cyclic group properties explained in Section C.1 and the proof based on the theory of elementary row operations on a matrix. We also present computation remarks in some specific topics of the discussion in connection with fast Fourier transform algorithm. At the end of this paper, we give a calculation illustration of all the results and close the paper by concluding remark.

C. RESULT AND DISCUSSION

1. Review n th Roots of Unity in Complex Numbers

We denote \mathbb{C} be the field of complex numbers. For a positive integer n , n th roots of unity over \mathbb{C} we mean as the solution of the equation $x^n - 1 = 0$. A set of those roots is $S = \{z \in \mathbb{C} | z^n = 1\}$ which is in fact a subgroup of the multiplication group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. To formulate the elements in S based on arithmetic of \mathbb{C} , firstly we will use the Euler's formula which states that for any $x \in \mathbb{R}$ we have $e^{ix} = \cos x + i \sin x$. Afterwards, we apply the theorem of De Moivre which states that $(\cos x + i \sin x)^n = \cos nx + i \sin nx$. Thus, by taking $x = \frac{2\pi}{n}$, we have

$$e^{\frac{2\pi}{n}} = \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \right)^n = \cos 2\pi + i \sin 2\pi = 1 \tag{1}$$

Since for $k = 1, 2, \dots, n - 1$,

$$s_k = \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \right)^k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} \neq 1, \tag{2}$$

we may conclude that s_k is a solution of $x^n - 1 = 0$, and hence we can rewrite $S = \{s_0, s_1, \dots, s_{n-1}\}$, then finally we may also conclude that S is a *cyclic group* of order n . Therefore, again we can rewrite S as

$$S = \langle \alpha \rangle = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\} \tag{3}$$

where $\alpha = \cos \frac{2\pi l}{n} + i \sin \frac{2\pi l}{n}$ for some $\text{gcd}(l, n) = 1$, and we call α as a primitive (generator) of S .

2. An Overview on Explicit Formula for the Eigenvalues and Determinant

Given any sequence $a_0, a_1, \dots, a_{n-2}, a_{n-1}$ of complex numbers, we use the usual notation from the references to define the $n \times n$ circulant matrix as

$$\text{Circ}(a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & a_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & a_2 \\ a_2 & a_3 & \dots & a_{n-1} & a_0 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_0 \end{pmatrix} \tag{4}$$

Below is the well-known theorem about the eigenvalues of the above circulant matrix. Here, we give a detail proof for the sake of subsequently discussions.

Theorem 1. Suppose we have $A = \text{Circ}(a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1}) \in \mathbb{C}^{n \times n}$. For every $k = 0, 1, 2, \dots, n - 1$, let λ_k be the eigenvalue of A corresponding to u_k be an eigenvector of A , then

$$\lambda_k = \sum_{j=0}^{n-1} a_j \alpha^{jk} \text{ and } u_k = \begin{pmatrix} 1 \\ \alpha^k \\ \alpha^{2k} \\ \vdots \\ \alpha^{(n-2)k} \\ \alpha^{(n-1)k} \end{pmatrix} \tag{5}$$

for some α primitive of S .

Proof. For all $k = 0, 1, 2, \dots, n - 1$ it is clear $\mu = \alpha^k \in S$. In this proof, we will show that vector $u = (1, \mu, \mu^2, \dots, \mu^{n-1}) \in \mathbb{C}^n$ is an eigenvector of A : For this purpose, since S is a cyclic group, consider that

$$Au = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & a_1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & a_2 \\ a_2 & a_3 & \dots & a_{n-1} & a_0 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_0 \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \vdots \\ \mu^{n-2} \\ \mu^{n-1} \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{pmatrix} = v \tag{6}$$

if and only if

$$\begin{aligned} v_0 &= a_0 + a_1\mu + a_2\mu^2 + \dots + a_{n-1}\mu^{n-1} \\ v_1 &= a_{n-1} + a_0\mu + a_1\mu^2 + \dots + a_{n-3}\mu^{n-2} + a_{n-2}\mu^{n-1} \\ &= a_{n-1}\mu^n + a_0\mu + a_1\mu^2 + \dots + a_{n-3}\mu^{n-2} + a_{n-2}\mu^{n-1} \\ &= (a_0 + a_1\mu + \dots + a_{n-3}\mu^{n-3} + a_{n-2}\mu^{n-2} + a_{n-1}\mu^{n-1})\mu = v_0\mu \\ v_2 &= a_{n-2} + a_{n-1}\mu + a_0\mu^2 + a_1\mu^3 + \dots + a_{n-3}\mu^{n-1} \\ &= a_{n-2}\mu^n + a_{n-1}\mu^{n+1} + a_0\mu^2 + a_1\mu^3 + \dots + a_{n-3}\mu^{n-1} \\ &= (a_0 + a_1\mu + a_2\mu^2 + \dots + a_{n-1}\mu^{n-1})\mu^2 = v_0\mu^2 \end{aligned} \tag{7}$$

and so on until we have $v_{n-1} = v_0\mu^{n-1}$. Based on this fact, if we denote $\lambda = v_0$, then it is clear that $Au = \lambda u$. Furthermore, since $\mu = \alpha^k$ for all $k = 1, 2, \dots, n - 1$, then λu can be stated as $\lambda_k u_k$ of which the formulation of λ_k and u_k given in the theorem. \square

Computation aspect of Theorem 1 is given in the following remark.

Remark 1. (A computation note for eigenvalues) Formulation of eigenvalues in Equation 5 can be calculated using matrix multiplication $Pa = \lambda$ written as

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{(n-1)2} & \dots & \alpha^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-2} \\ \lambda_{n-1} \end{pmatrix} \tag{8}$$

which is in fact a kind of discrete Fourier transform, so those eigenvalues can be computed efficiently using fast Fourier transform algorithm. Also, we note that the column vectors of P

are the eigenvectors which can be computed efficiently by exploiting the recursive properties of the cyclic group S . We present the explicit formula for the determinant of those circulant matrices as the first corollary of Theorem 1.

Corollary 1. Given the matrix A in Theorem 1 and suppose that the eigenvalues λ_k of A has been computed efficiently, then the determinant of A is formulated as

$$\det(A) = \prod_{k=0}^{n-1} \lambda_k \tag{9}$$

Proof. The matrix P , defined in Remark 1, is in fact a Vandermonde matrix, so it is easy to verify that $\det(P) = \prod_{i < j} (\alpha^j - \alpha^i) \neq 0$ which means that all n column vectors of P (i.e. all eigenvectors of A) are linearly independent, then we conclude that A is a simple matrix, and hence we can write $A = PDP^{-1}$. Now, we have

$$\det(A) = \det(PDP^{-1}) = \det(P) \cdot \det(D) \cdot \det(P^{-1}) = \det(D) = \prod_{k=0}^{n-1} \lambda_k \tag{10}$$

Remark 2. (A computation note for determinant) From Corollary 1, It is clear that the computation efficiency for $\det(A)$ depends on the efficiency of computing the eigenvalues of A , see Remark 1.

The second corollary of Theorem 1 is about the invertibility of A asserted as follows which the proof is very clear.

Corollary 2. The matrix A in Theorem 1 is invertible if only if $\lambda_k \neq 0$ for every $k = 0, 1, 2, \dots, n - 1$.

Remark 3. (A computation note for invertibility) A computation method to test the invertibility of A can be applied using fast Fourier transform algorithm, see Remark 1.

Given any sequence $b_0, b_1, b_2, \dots, b_{n-2}, b_{n-1}$ of complex numbers, we define the $n \times n$ left circulant matrix as

$$\text{LCirc}(b_0, b_1, b_2, \dots, b_{n-2}, b_{n-1}) = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{n-2} & b_{n-1} \\ b_1 & b_2 & b_3 & \dots & b_{n-1} & b_0 \\ b_2 & b_3 & b_4 & \dots & b_0 & b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-2} & b_{n-1} & b_0 & \dots & b_{n-4} & b_{n-3} \\ b_{n-1} & b_0 & b_1 & \dots & b_{n-3} & b_{n-2} \end{pmatrix} \tag{11}$$

From this definition, now we arrive at the last corollary of Theorem 1 given as follows.

Corollary 3. For integer $n \geq 3$, suppose we have $B = \text{LCirc}(b_0, b_1, b_2, \dots, b_{n-2}, b_{n-1}) \in \mathbb{C}^{n \times n}$ then

$$\det(B) = (-1)^m \cdot \det(A) \text{ with } m = \lfloor n - 1/2 \rfloor, \tag{12}$$

where $A = \text{Circ}(b_0, b_1, b_2, \dots, b_{n-2}, b_{n-1})$.

Proof. Elementary row operations on matrix B by interchanging rows i and $(n - i + 2)$ of B for every $i = 2, 3, \dots, \lfloor n + 1/2 \rfloor$ and $n \geq 3$ will produce matrix $A = \text{Circ}(b_0, b_1, b_2, \dots, b_{n-2}, b_{n-1})$. In this process, the number of exchanges are $m = \lfloor n - 1/2 \rfloor$ which is the number of row permutations. \square

3. An Explicit Formula for the Inverse

The following lemma will be used to prove the subsequent lemma.

Lemma 1. Recall the cyclic group $S = \langle \alpha \rangle = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$. If d is a positive integer and $d|n$, then we have

$$\sum_{l=0}^{t-1} \alpha^{ld} = 1 + \alpha^d + \alpha^{2d} + \dots + \alpha^{(t-1)d} = 0 \tag{13}$$

where t is the positive integer such that $n = td$.

Proof. Since S is a cyclic group and $d|n$, it clear that $\alpha^d \neq 1$, then we have

$$\begin{aligned} 1 + \alpha^d + \alpha^{2d} + \dots + \alpha^{(t-1)d} &= 1 + (\alpha^d) + (\alpha^d)^2 + \dots + (\alpha^d)^{t-1} \\ &= \frac{(\alpha^d)^t - 1}{\alpha^d - 1} = \frac{\alpha^n - 1}{\alpha^d - 1} = \frac{1 - 1}{\alpha^d - 1} = 0 \end{aligned} \tag{14}$$

□

To formulate the inverse of the circulant matrix A in Theorem 1, we need the following lemma.

Lemma 2. For integer $n \geq 3$ and from the matrix

$$P = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{(n-1)2} & \dots & \alpha^{(n-1)(n-1)} \end{pmatrix} \tag{15}$$

defined in Remark 1, we have

$$P^{-1} = \frac{1}{n} PT \tag{16}$$

where T be the permutation matrix resulting from elementary column operations on the identity matrix I_n by interchanging column j and $(n - j + 2)$ of I_n for every $j = 2, 3, \dots, \lfloor \frac{n+1}{2} \rfloor$.

Proof. From the structure of P , we have the first fact that P is a symmetric matrix. The second fact, for every $i = 1, 2, \dots, n$, let P_i be the i -th row of P , then it is easy to see that the set of all entries of P_i is a cyclic subgroup of order some integer positive $d|n$ of the cyclic group $S = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$. From those facts, and by applying Lemma 1, then we have $P_i P_i^T = \sum_{l=1}^n 1 = n$, for every $i = 2, 3, \dots, \lfloor \frac{n+2}{2} \rfloor$ and when $k = (n - i + 2)$:

$$P_i P_j^T = \sum_{l=0}^{n-1} (\alpha^{[(i-1)+(n-i+2-1)] \bmod n})^l = \sum_{l=0}^{n-1} (\alpha^0)^l = \sum_{i=1}^n 1 = n \tag{17}$$

otherwise

$$P_i P_j^T = \sum_{l=0}^{n-1} (\alpha^{[(i-1)+(j-1)] \bmod n})^l = \sum_{l=0}^{n-1} (\alpha^s)^l = d \sum_{i=1}^n \alpha^{ld} = d(0) = 0 \tag{18}$$

where $S = [(i - 1) + (j - 1)] \bmod n$, $d = \gcd(s; n)$, and $t = \frac{n}{d}$. Thus, let Q be the resulting matrix from elementary column operations on P by interchanging column j and $(n - j + 2)$ of P for every $j = 2, 3, \dots, \lfloor \frac{n+1}{2} \rfloor$, then we have

$$PQ = \text{diag}[n, n, \dots, n] \Leftrightarrow P\left(\frac{1}{n}Q\right) = I_n \Leftrightarrow P^{-1} = \frac{1}{n}Q = \frac{1}{n}PT \tag{19}$$

where T be the permutation matrix resulting from elementary column operations on the identity matrix I_n by interchanging column j and $(n - j + 2)$ of I_n for every $j = 2, 3, \dots, \lfloor n + 1/2 \rfloor$. \square

By those lemmas and all the previous discussion results, here we are at the last theorem of this paper.

Theorem 2. For integer $n \geq 3$, given the matrix A in Theorem 1 and suppose that all the eigenvalues λ_k of A has been computed efficiently, then the inverse of A is formulated as

$$A^{-1} = \frac{1}{n} \text{Circ}(e_0, e_1, e_2, \dots, e_{n-1}) \text{ with } \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{pmatrix} = (PT) \begin{pmatrix} \frac{1}{\lambda_0} \\ \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \\ \vdots \\ \frac{1}{\lambda_{n-2}} \\ \frac{1}{\lambda_{n-1}} \end{pmatrix} \tag{20}$$

where P and T are the matrices that have been asserted in Lemma 2.

Proof. Recall from the proof of Corollary 1, we already have that $A = PDP^{-1}$, so by Lemma 2, we have

$$A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1} = PD^{-1}\left(\frac{1}{n}PT\right) = \frac{1}{n}(PD^{-1})(PT) \tag{21}$$

if only if

$$nA^{-1} = (PD^{-1})(PT) = \begin{pmatrix} \frac{1}{\lambda_0} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \frac{1}{\lambda_0} & \alpha & \alpha^2 & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \frac{1}{\lambda_0} & \alpha^2 & \alpha^4 & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \frac{1}{\lambda_0} & \alpha^3 & \alpha^6 & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\lambda_0} & \alpha^{n-2} & \alpha^{n-4} & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \frac{1}{\lambda_0} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \frac{1}{\lambda_0} & \alpha^{n-1} & \alpha^{n-2} & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \\ \frac{1}{\lambda_0} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \dots & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \alpha^{n-1} & \alpha^{n-2} & \dots & \alpha^2 & \alpha \\ 1 & \alpha^{n-2} & \alpha^{n-4} & \dots & \alpha^4 & \alpha^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{n-4} & \alpha^{n-2} \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{n-2} & \alpha^{n-1} \end{pmatrix} \tag{22}$$

From this fact, let vector $R_1 = (e_0, e_1, e_2, \dots, e_{n-1})$ be the first row of nA^{-1} , then

$$R_1 = \begin{pmatrix} \frac{1}{\lambda_0} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & & & \\ & \frac{1}{\lambda_{n-2}} & \frac{1}{\lambda_{n-1}} & & & \\ & & & & & \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \alpha^{n-1} & \alpha^{n-2} & \dots & \alpha^2 & \alpha \\ 1 & \alpha^{n-2} & \alpha^{n-4} & \dots & \alpha^4 & \alpha^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{n-4} & \alpha^{n-2} \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{n-2} & \alpha^{n-1} \end{pmatrix} \quad (23)$$

and then

$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{pmatrix} = (PT) \begin{pmatrix} \frac{1}{\lambda_0} \\ \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \\ \vdots \\ \frac{1}{\lambda_{n-2}} \\ \frac{1}{\lambda_{n-1}} \end{pmatrix} \quad (24)$$

Furthermore, it is easy to verify that the second, third, ..., n -th rows of nA^{-1} is rotating R_1 one step to the right recursively. Thus, we may conclude that nA^{-1} is circulant, and so is A^{-1} .

Below is the computation remark of the above theorem.

Remark 4. Rewrite a part of the formulation of A^{-1} in Equation 20 as

$$e = Pu \text{ where } e = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{pmatrix}, u = T\lambda^{-1} \text{ and } \lambda^{-1} = \begin{pmatrix} \frac{1}{\lambda_0} \\ \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \\ \vdots \\ \frac{1}{\lambda_{n-2}} \\ \frac{1}{\lambda_{n-1}} \end{pmatrix} \quad (25)$$

Assuming that we have already computed λ by considering Remark 1, in the next step is to compute λ^{-1} which is just a fast computation way in arithmetic of complex numbers, then followed by computing u which is definitely very fast because of just permuting λ^{-1} by T : Finally, to compute e can be done efficiently by applying fast Fourier transform algorithm, again see Remark 1.

The inverses relationship between circulant and left circulant matrices we present as a corollary of Theorem 2 as follows.

Corollary 4 Given $B = \text{LCirc}(b_0, b_1, \dots, b_{n-2}, b_{n-1}) \in \mathbb{C}^{n \times n}$, then

$$B^{-1} = A^{-1}T \quad (26)$$

where $A = \text{Circ}(b_0, b_1, \dots, b_{n-2}, b_{n-1})$ and T be the permutation matrix resulting from elementary row operations on the identity matrix I_n by interchanging rows i and $(n - i + 2)$ for every $i = 2, 3, \dots, \lfloor n + 1/2 \rfloor$. Furthermore, if we denote $A^{-1} = \frac{1}{n} \text{Circ}(e_0, e_1, \dots, e_{n-2}, e_{n-1})$, then

$$B^{-1} = \frac{1}{n} \text{LCirc}(f_0, f_1, \dots, f_{n-2}, f_{n-1}) \tag{27}$$

where

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix} = T \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{pmatrix} \tag{28}$$

Proof. Elementary row operations on matrix B by interchanging rows i and $(n - i + 2)$ of B for every $i = 2, 3, \dots, \lfloor n + 1/2 \rfloor$ and $n \geq 3$ will produce matrix $A = \text{Circ}(b_0, b_1, \dots, b_{n-2}, b_{n-1})$. In this process, we have

$$TB = A \Leftrightarrow B = T^{-1}A \Leftrightarrow B^{-1} = A^{-1}T \tag{29}$$

□

Before we close the paper by concluding remark, below we give an illustration connected to all formulations that have been discussed previously.

A Numerical Example:

For given

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$$

then

$$\alpha = \cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6} = \frac{1}{2}i\sqrt{3} + \frac{1}{2},$$

$$\alpha^2 = \frac{1}{2}i\sqrt{3} - \frac{1}{2}, \text{ and } \alpha^3 = \alpha^2 \left(\frac{1}{2}i\sqrt{3} + \frac{1}{2} \right) = -1,$$

Thus, we have $P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & 3 & 4 & 5 \\ 1 & \alpha^2 & -\alpha & 1 & \alpha^2 & -\alpha \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 \\ 1 & -\alpha^2 & -\alpha & -1 & \alpha^2 & \alpha \end{pmatrix}$ and

$$\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = P \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 21 \\ -3i\sqrt{3}-3 \\ -i\sqrt{3}-3 \\ -3 \\ i\sqrt{3}-3 \\ 3i\sqrt{3}-3 \end{pmatrix}$$

$$\det(A) = (21)(-3i\sqrt{3}-3)(-i\sqrt{3}-3)(-3)(i\sqrt{3}-3)(3i\sqrt{3}-3)$$

$$B = TA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\det(B) = (-1)^{\lfloor \frac{6-1}{2} \rfloor} \cdot \det(A) = (1)(-27216) = -27216$$

$$\begin{pmatrix} \lambda_0^{-1} \\ \lambda_1^{-1} \\ \lambda_2^{-1} \\ \lambda_3^{-1} \\ \lambda_4^{-1} \\ \lambda_5^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{21} \\ \frac{1}{-3i\sqrt{3}-3} \\ \frac{1}{-i\sqrt{3}-3} \\ \frac{1}{-3} \\ \frac{1}{i\sqrt{3}-3} \\ \frac{1}{3i\sqrt{3}-3} \end{pmatrix} = \begin{pmatrix} \frac{1}{21} \\ -\frac{1}{21}i\sqrt{3}-\frac{1}{12} \\ -\frac{1}{21}i\sqrt{3}-\frac{1}{4} \\ \frac{1}{-3} \\ \frac{1}{21}i\sqrt{3}-\frac{1}{4} \\ \frac{1}{21}i\sqrt{3}-\frac{1}{12} \end{pmatrix}$$

$$E = PU = P \begin{pmatrix} \frac{1}{21} \\ -\frac{1}{21}i\sqrt{3}-\frac{1}{12} \\ -\frac{1}{21}i\sqrt{3}-\frac{1}{4} \\ \frac{1}{-3} \\ \frac{1}{21}i\sqrt{3}-\frac{1}{4} \\ \frac{1}{21}i\sqrt{3}-\frac{1}{12} \end{pmatrix} = \begin{pmatrix} -\frac{20}{21} \\ \frac{22}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \end{pmatrix}$$

$$A^{-1} = \frac{1}{6} \text{Circ} \begin{pmatrix} -\frac{20}{21} & \frac{22}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} \end{pmatrix}$$

$$F = TE = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{20}{21} \\ \frac{22}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \end{pmatrix} = \begin{pmatrix} -\frac{20}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \\ \frac{1}{21} \end{pmatrix}$$

$$B^{-1} = \frac{1}{6} \text{LCirc} \left(-\frac{20}{21} \quad \frac{1}{21} \quad \frac{1}{21} \quad \frac{1}{21} \quad \frac{1}{21} \quad \frac{22}{21} \right)$$

To determine the eigenvalues, determinants, and inverses for general matrices can be done using simply methods which can be found in any standard books of linear algebra. But, that is a computationally hard work, especially when the size of the matrix is large enough. It is because the determinant calculation based on recursive method, and the calculation of inverse and eigenvalues depends on its determinant. When the matrix has a special structure, such as circulant, and more specific also having special type of entry, such as Fibonacci sequence, then the calculation can be made much easier by giving their explicit formulations.

For the case of circulant matrices with general type of entry, one of the explicit formulations we have derived and discussed in the above theorems, their proofs, and their corollaries in this article as the results of our research. If we compare these results to the previous results in the references is that the previous results using more specific type of entry of the matrix and more focus on the problems of mostly determining the determinant and the inverse. It means that our results more general which the explicit formulation can be applied to circulant matrices with any type of sequence of numbers, and of course more complex calculation.

D. CONCLUSIONS AND SUGGESTIONS

A direct method to get the eigenvalues, determinants, and, inverses for general matrices can be done using simply methods which can be found in any standard books of linear algebra. However, for the circulant matrices, one can make it faster by applying numerical methods such as using fast Fourier transform algorithm. Now in this paper, we present a different approach to get that explicit formulation of the inverse is just by matrix multiplication. Then, the computation of this matrix multiplication can also be accelerated by applying fast Fourier transform algorithm. The most important method of that formulation is based exploitation of cyclic group properties which could be explored further to other cases such as for either skew circulant matrix or circulant matrices over finite fields. Thus, the results of this paper still need to be continued, there are at least three subjects that would become our ongoing and nearly future works. (1) To find a subgroup cyclic of the group \mathbb{C}^* that can be used to derive explicit formulas for eigenvalue, determinant, and inverse for general skew circulant matrices, (2) Exploring the methods of this paper for the case of circulant and skew circulant matrices

over finite fields, (3) Studying and exploring fast Fourier transform algorithm especially for the case of circulant and skew circulant matrices over finite fields.

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