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Inclusion Properties of Henstock-Orlicz Spaces

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ABSTRACT

Henstock-Orlicz spaces were generally introduced by Hazarika and Kalita in 2021. In general, a function is Lebesgue integral if only if that function and its modulus are Henstock-Kurzweil integrable functions. Moreover, suppose a function is a finite measurable function with compact supports. In that case, the function is a Henstock-Kurzweil integrable function if only if the function is a Lebesgue integrable function. Due to these properties, Henstock-Orlicz spaces were constructed by utilizing Young functions. This definition is almost similar to the definition of Orlicz spaces, but by embedding the Henstock-Kurzweil integral, and the norm used is the Luxembourg norm. Therefore, an analysis of properties in these spaces is needed carried out more deeply. This research was using a literature study on inclusion properties from scientific journals, especially those related to the Orlicz Spaces. And based on the definition of Henstock-Orlicz spaces and its norm, we formulate a hypothesis regarding the inlcusion properties. By deductive proof, we proof the hypothesis and state it as theorem. In this study, we obtain sufficient and necessary conditions for the inclusion properties in Henstock-Orlicz spaces.

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A. INTRODUCTION

The theory of Henstock-Kurzweil integral was developed by R. Henstock and J. Kurzweil independently during 1955-1957(Lee, 2011). Henstock-Kurzweil integral can be viewed as a generalized of Riemann integral (Hazarika & Kalita, 2021). All of Henstock-Kurzweil integrable functions contained on Henstock-Kurzweil Spaces. This spaces also contain Lebesgue integrable functions. The Lebesgue spaces of p-th integrable functions on $\mathbb{R}^n$ also called the Orlicz spaces with respect to the Young function so that it can be viewed as a generalization of the Lebesgue Spaces (Masta et al., 2016).

Next, in general, f and $|f|$ are Henstock-Kurzweil integrable functions if only if f is Lebesgue integrable function. Suppose f is a bounded measurable function with compact support, then f is Henstock-Kurzweil integrable function if only if f is a Lebesgue integrable function (Hazarika & Kalita, 2021). This Hazarika's Motivation to construct the Henstock-Orlicz spaces (briefly $\mathcal{H}$-Orlicz Spaces) by embedding the Henstock-Kurzweil integral and using the Young function. In 2021, Hazarika and Kalita introduced the $\mathcal{H}$-Orlicz spaces, which have some different properties of the Orlicz spaces such as $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}$-Orlicz spaces but not generally dense in Orlicz spaces.

Many researchers have studied Henstock-Kurzweil integral, Orlicz spaces, inclusion properties, and $\mathcal{H}$-Orlicz spaces. Several studies on Henstock-Kurzweil integrals include discussing some properties of Henstock-Kurzweil integrable function in n-dimensional spaces (Herlinawati, 2021), the different definition of Henstock-Kurzweil integral on a closed bounded interval by using primitive (Leng & Yee, 2018), Henstock-Kurzweil integral on Riezs Spaces (Boccuto et al., 2012), generalization of this integral (Malý & Kuncová, 2019), the distributional Henstock-Kurzweil integral and also applications in integral and differential equation (Liu, 2016), the Henstock-Kurzweil transform(Sánchez-Perales et al., 2012)(Sánchez-Perales et al., 2012)(Sánchez-Perales et al., 2012)(Sánchez-Perales et al., 2012)(Sánchez-Perales et al., 2012)(Sánchez-Perales et al., 2012) (Mendoza-Torres et al., 2015; Mendoza Torres et al., 2002; Sánchez-Perales et al., 2012; Talvila, 2002), and inclusion relations for several spaces such as Lebesgue spaces, Henstock-Kurzweil spaces, and bounded variation spaces (Mendoza Torres et al., 2009).

Moreover, research on the Orlicz space and inclusion properties that applicable in several spaces has been carried out, such as the generalization of Orlicz spaces (Ebadian & Jabbari, 2021), some characterization of generalized Orlicz spaces (Ferreira et al., 2020), some properties of Discrete Orlicz space (Prayoga et al., 2020), some necessary and sufficient conditions for inclusion properties on Orlicz spaces and another type of this spaces (Masta et al., 2016), the inclusion properties of two version of Orlicz-Morrey spaces (Masta et al., 2017b), the inclusion property on Orlicz-Morrey Spaces (Masta et al., 2017a), and the inclusion properties of discrete Morrey spaces (Prayoga et al., 2020).

Furthermore, research on Henstock-Orlicz spaces was carried out by several mathematicians such as about the definition of $\mathcal{H}$-Orlicz and its dense space (Hazarika & Kalita, 2021), the theory of $\mathcal{H}$-Orlicz with respect to vector measure (Kalita et al., 2020), and the relatively weakly compactness of some $\mathcal{H}$-Orlicz spaces of Henstock-Gel'fand integrable function (Kalita & Hazarika, 2021).

Based on the aforementioned studies, the inclusion properties of several functional spaces such as Orlicz space, weak-Orlicz space, Morrey space, Orlicz-Morrey space, and Discrete Morrey space, and others, have been proven. Moreover, several mathematicians have made observations in the $\mathcal{H}$-Orlicz spaces. Therefore, here we are interested to study about inclusions relation in the $\mathcal{H}$-Orlicz spaces. The aims of this research is to obtain some sufficient and necessary conditions for the inclusion properties on $\mathcal{H}$-Orlicz spaces.

B. METHODS

The research method in this article begins with a literature study on inclusion properties from books and scientific journals, especially those related to the $\mathcal{H}$-Orlicz Spaces and inclusion properties on some functional spaces. The reference also have been published at least the last 15 years. Reference sources were obtained from digital search engines such as Google Scholar, and from journals subscribed to by the institution, as well as scientific books from the institutional library.

Next, in particular, it is preceded by studying the Henstock-Kurzweil spaces, the Orlicz spaces, and the $\mathcal{H}$-Orlicz spaces and then examining the properties of inclusions. Moreover, several definitions and lemmas are needed in the discussion related to the results of this article.

Definition 1.1. (Royden & Fitzpatrick, 2010) Let X be a set, then a collection $\mathcal{A}$ of subsets of X is is called a σ-algebra if it satisfies the following properties:

(i) $\emptyset \in \mathcal{A}$.

(ii) If $C \in \mathcal{A}$, then $X \setminus C \in \mathcal{A}$.

(iii) If $A_1,\ A_2,\ A_3,\ \ldots \in \mathcal{A}$, then $A = A_1 \cup A_2 \cup A_3 \cup \ldots \in \mathcal{A}$.

Furthermore, elements of the σ-algebra are called measurable sets. In this case, X is a measurable set. Then an ordered pair $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a σ-algebra over X, is called a measurable space. Next, let μ be a measure on X, then $(X, \mathcal{A}, \mu)$ is called a measure space. In some literature, notation of a measure space notation is simplified to (X, μ). Moreover, a function between two measurable spaces is called a measurable function if the pre-image of every measurable set is measurable.

Definition 1.2. (Lee, 2011) Let $[a, b]$ be an interval on $\mathbb{R}$, then a function $\delta: [a, b] \to \mathbb{R}$ is a gauge on $[a, b]$, and $[a, b] = \bigcup_{k=1}^{m} [u_k, v_k]$ where $u_k, v_k \in \mathbb{R}$ for $k = 1,2, \ldots, m$. Moreover, a partition $(t_1, [u_1, v_1]), \ldots, (t_m, [u_m, v_m])$ of $[a, b]$, for some point $t_k \in \mathbb{R}$, is said to be δ-fine if $[u_k, v_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$.

Definition 1.3. (Lee, 2011) Let $\mu([u_k, v_k]) = v_k - u_k$ be a measure of interval $[u_k, v_k]$ for $k = 1,2, \ldots, m$ and $t_k \in [u_k, v_k]$. A function $f: [a, b] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable on $[a, b]$ if there exist $A \in \mathbb{R}$ with the following property: given $\varepsilon > 0$, there exist a gauge δ on $[a, b]$ such that

$$|S(f, P) - A| < \varepsilon$$

with $S(f, P) = \sum_{k=1}^{m} f(t_k)\mu([u_k, v_k])$, for each δ-fine partition P of $[a, b]$.

Definition 1.4. (Masta et al., 2016) Let $\Phi: [0, \infty) \to [0, \infty)$ be a Young function, then Φ is convex, left-continuous, $\Phi(0) = 0$, and $\lim\limits_{t \to \infty} \Phi(t) = \infty$.

Lemma 1.1. (Masta et al., 2016) Suppose that Φ is a Young function and $\Phi^{-1}(s) = \inf\{t \geq 0 : \Phi(t) > s\}$. Then

(1) $\Phi^{-1}(0) = 0$.

(2) $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ for $s_1 \leq s_2$.

(3) $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ for $0 \leq s < \infty$.

(4) Let $K > 0$. Then $\Phi_1(t) \leq \Phi_2(Kt)$ if only if $K\Phi_1^{-1}(t) \geq \Phi_2^{-1}(t)$ for every $t \geq 0$.

(5) Let $K > 0$. Then $\Phi_1(t) \leq \Phi_2 K(t)$ if only if $\Phi_1^{-1}(Kt) \geq \Phi_2^{-1}(t)$ for every $t \geq 0$

Lemma 1.2. (Masta et al., 2016) Let Φ be a Young function, then $\Phi(\mathrm{Ct}) \leq \mathrm{C}\,\Phi(\mathrm{t})$ for $\mathrm{t} > 0$ and $0 \leq \mathrm{C} \leq 1$.

From the definitions and lemmas above, first, we studied the similarities between the definitions of the Orlicz spaces and the $\mathcal{H}$-Orlicz spaces which both utilize the Young's function. Then we studied the definition and properties of Young's function in each spaces. The second, we studied the inclusion properties in various functional spaces. For the last step, based on the definition of $\mathcal{H}$-Orlicz spaces and its norm (given in section C), we formulated a hypothesis regarding the inlcusion properties. By deductive proof, we proven the hypothesis and stated it as theorem. Finally, sufficient and necessary conditions will be found for inclusion in the $\mathcal{H}$-Orlicz spaces.

C. RESULT AND DISCUSSION

In this section, the definition and norm of $\mathcal{H}$-Orlicz spaces are given as follows:

Definition 1.5. Let $(X, \mathcal{B}, \mu)$ be a measure space and $\Phi: [0,\infty) \to [0,\infty)$ be a Young function. The $\mathcal{H}$ -Orlicz space $\mathcal{H}_\Phi(X)$ is the set of measurable functions $f: X \to \mathbb{R}$ such that $(HK)\int_X \Phi(af)d\mu < \infty$ for some $a > 0$.

Definition 1.6. Consider $f \in \mathcal{H}_\Phi(X)$ and $b \in \mathbb{R}$. Then the norm is the Luxemburg norm as follow:

$$\|f\|_{\mathcal{H}_\Phi} = \inf\left\{b > 0 : (HK)\int_X \Phi\left(\frac{f}{b}\right)d\mu < 1\right\}.$$

Some properties of $\mathcal{H}$-Orlicz spaces have been discovered by Hazarika and Kalita. Here we studied inclusion property of these spaces. In (Masta et al., 2016), Welland proved the inclusion property as follow: Let X be finite measure, and Φ, Ψ be two Young functions. If there is $C > 0$ such that $\Phi(t) \le \Psi(Ct)$ for every $t > 0$ then $L_\Phi(X) \subseteq L_\Psi(X)$, wheares Masta et al. proved following statements are equivalent: (1) $\Phi(t) \le \psi(Ct)$ for every $t > 0$; (2) $L_\psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$; (3) for every $f \in L_\psi(\mathbb{R}^n)$, we have $\|f\|_{L_{\Phi(\mathbb{R}^n)}} \le C\,\|f\|_{L_{\psi(\mathbb{R}^n)}}$; (4) $wL_\psi(\mathbb{R}^n) \subseteq wL_\Phi(\mathbb{R}^n)$; (5)) for every $f \in wL_\psi(\mathbb{R}^n)$, we have $\|f\|_{wL_{\Phi(\mathbb{R}^n)}} \le C\,\|f\|_{wL_{\psi(\mathbb{R}^n)}}$, with L_Φ, L_ψ and $wL_{\Phi,} wL_\psi$ are orlicz space and Orlicz space are weak associated with the each Young functions, repectively. On the other hand, in (Masta et al., 2016, 2017b) also proved inclusion properties in different functional spaces.

Furthermore, the inclusion property of $\mathcal{H}$ -Orlicz space is proven by connecting the inclusion relation of two Young functions with the inclusion of two $\mathcal{H}$-Orlicz spaces with respect to each Young function, then we have the following theorem.

Theorem 1.1. Let Φ and ψ be Young Functions. If there exists $C > 0$ such that $\Phi(t) \le \psi(Ct)$ for every $t > 0$, then $\mathcal{H}_\psi(\mathbb{R}^n) \subseteq \mathcal{H}_\Phi(\mathbb{R}^n)$ with $\|f\|_{\mathcal{H}_\Phi} \le C\,\|f\|_{\mathcal{H}_\psi}$ for every $f \in \mathcal{H}_\psi(\mathbb{R}^n)$.

Proof. First, to prove $\mathcal{H}_\psi(\mathbb{R}^n) \subseteq \mathcal{H}_\Phi(\mathbb{R}^n)$, it is necessary to show that for every $f \in \mathcal{H}_\psi(\mathbb{R}^n)$ then $f \in \mathcal{H}_\Phi(\mathbb{R}^n)$. Suppose $f \in \mathcal{H}_\psi(\mathbb{R}^n)$. Because $\Phi(t) \le \psi(Ct)$ for every $t > 0$, then

$$(HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{C\,\|f\|_{\mathcal{H}_\psi}}\right)d\mu \le (HK)\int_{\mathbb{R}^n}\psi\left(C\left(\frac{f}{C\,\|f\|_{\mathcal{H}_\psi}}\right)\right)d\mu\,(HK) = \int_{\mathbb{R}^n}\psi\left(\frac{Cf}{C\,\|f\|_{\mathcal{H}_\psi}}\right)d\mu$$
$$= (HK)\int_{\mathbb{R}^n}\psi\left(\frac{f}{\|f\|_{\mathcal{H}_\psi}}\right)d\mu$$

Because $f \in \mathcal{H}_\psi(\mathbb{R}^n)$, then by definition of the norm, we have $(HK)\int_{\mathbb{R}^n}\psi\left(\frac{f}{\|f\|_{\mathcal{H}_\psi}}\right)d\mu \le 1$. Hence $(HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{C\,\|f\|_{\mathcal{H}_\psi}}\right)d\mu \le 1$. From Definition 1.5, by taking $a = C\,\|f\|_{\mathcal{H}_\psi}$, we obtain $f \in \mathcal{H}_\Phi(\mathbb{R}^n)$.

Next, we will show that $\|f\|_{\mathcal{H}_\Phi} \le C\,\|f\|_{\mathcal{H}_\psi}$ for every $f \in \mathcal{H}_\psi(\mathbb{R}^n)$. Because $(HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{C\,\|f\|_{\mathcal{H}_\psi}}\right)d\mu \le 1$ for every $f \in \mathcal{H}_\psi(\mathbb{R}^n)$, by Definition 1.6, we have $\|f\|_{\mathcal{H}_\Phi} = inf\left\{b > 0 : (HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{b}\right)d\mu \le 1\right\} = inf\left\{C\,\|f\|_{\mathcal{H}_\psi}\right\} \le C\,\|f\|_{\mathcal{H}_\psi}$. So, $\|f\|_{\mathcal{H}_\Phi} \le C\,\|f\|_{\mathcal{H}_\psi}$. ■

Theorem 1.2. Let $f \in \mathcal{H}_\Phi(\mathbb{R}^n)$. Then $\|f\|_{\mathcal{H}_\Phi} \le 1$ if only if $(HK)\int_{\mathbb{R}^n}\Phi(f)d\mu \le 1$.

Proof. Suppose $\|f\|_{\mathcal{H}_\Phi} \le 1$, we will show that $(HK)\int_{\mathbb{R}^n}\Phi(f)d\mu \le 1$. From Definition 1.6, we have $(HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{b}\right)d\mu \le 1$ for some $0 < b \le 1$. Then

$$(HK)\int_{\mathbb{R}^n}\Phi(f)d\mu \le (HK)\int_{\mathbb{R}^n}\frac{1}{b}\Phi(f)d\mu = (HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{b}\right)d\mu \le 1.$$

The last equation is obtained by using Lemma 1.2. Therefore, $(HK)\int_{\mathbb{R}^n}\Phi(f)d\mu \le 1$.

On the other hand, suppose $(HK)\int_{\mathbb{R}^n}\Phi(f)d\mu < 1$, we will show that $\|f\|_{\mathcal{H}_\Phi} \le 1$. By Definition 1.6, $\|f\|_{\mathcal{H}_\Phi} = inf\left\{b > 0 : (HK)\int_{\mathbb{R}^n}\Phi\left(\frac{f}{b}\right)d\mu \le 1\right\}$, then we have $0 < b \le 1$, so $\inf b \le 1$. Therefore, $\|f\|_{\mathcal{H}_\Phi}$. ■

From Theorem 1.1, we obtain that if $\Phi \le \psi$ then $\mathcal{H}_\psi(\mathbb{R}^n) \subseteq \mathcal{H}_\Phi(\mathbb{R}^n)$ with $\|f\|_{\mathcal{H}_\Phi} \le C\,\|f\|_{\mathcal{H}_\psi}$ for every $f \in \mathcal{H}_\psi(\mathbb{R}^n)$. For this result, the converse also holds. Moreover, to prove this converse, we need to use the characteristic function of the ball in $\mathbb{R}^n$. In addition, the following lemma's are needed.

Lemma 1.3. (Masta et al., 2016) Let Φ be a Young function, $a \in \mathbb{R}^n, r > 0$ and $B(a,r)$ is a ball in $\mathbb{R}^n$. Then

$$\left\|\chi_{B(a,r)}\right\|_{\mathcal{H}_\Phi} = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu\left(B(a,r)\right)}\right)}$$

where $\mu\left(B(a,r)\right)$ is the measure of the ball $B(a,r)$ with centered at a and radius r.

Next, we come to the main result of this article. The sufficient and necessary conditions for inclusion properties of $\mathcal{H}$-Orlicz spaces are proved in the following theorem.

Theorem 1.3. Let Φ and ψ be Young Functions and $C > 0$. Then the following statements are equivalent.

(1) $\Phi(t) \leq \psi(Ct)$ for every $t > 0$.

(2) $\mathcal{H}_{\psi}(\mathbb{R}^n) \subseteq \mathcal{H}_{\Phi}(\mathbb{R}^n)$.

(3) $\|f\|_{\mathcal{H}_{\Phi}} \leq C\, \|f\|_{\mathcal{H}_{\psi}}$ for every $f \in \mathcal{H}_{\psi}(\mathbb{R}^n)$.

Proof. First, we will show that (1) implies (2), then (2) implies (3). The proof is similar as the proof of Theorem 1.1. In breafly, suppose that $f \in \mathcal{H}_{\psi}(\mathbb{R}^n)$, we have $(HK)\int_{\mathbb{R}^n} \psi\left(\frac{f}{\|f\|_{\mathcal{H}_{\psi}}}\right) d\mu \leq 1$. By (1), we obtain

$$(HK)\int_{\mathbb{R}^n} \Phi\left(\frac{f}{C\,\|f\|_{\mathcal{H}_{\psi}}}\right) d\mu \leq (HK)\int_{\mathbb{R}^n} \psi\left(\frac{Cf}{C\,\|f\|_{\mathcal{H}_{\psi}}}\right) d\mu = (HK)\int_{\mathbb{R}^n} \psi\left(\frac{f}{\|f\|_{\mathcal{H}_{\psi}}}\right) d\mu \leq 1.$$

So, $f \in \mathcal{H}_{\Phi}(\mathbb{R}^n)$. As consequently, (2) also holds. Furthermore, because $(HK)\int_{\mathbb{R}^n} \Phi\left(\frac{f}{C\,\|f\|_{\mathcal{H}_{\psi}}}\right) d\mu \leq 1$, by Definition 1.6, $\|f\|_{\mathcal{H}_{\Phi}} \leq C\,\|f\|_{\mathcal{H}_{\psi}}$ for every $f \in \mathcal{H}_{\psi}(\mathbb{R}^n)$. We also obtain (2) implies (3).

Next, we have to show that (3) indicates (1) to complete the proof. Now, assume that (3) holds, by Lemma 1.3 and (3), we have

$$\frac{1}{\Phi^{-1}\left(\frac{1}{\mu(B(a,r))}\right)} = \left\|\chi_{B(a,r)}\right\|_{\mathcal{H}_{\Phi}} \leq \|f\|_{\mathcal{H}_{\Phi}} \leq C\,\|f\|_{\mathcal{H}_{\psi}} = \frac{C}{\psi^{-1}\left(\frac{1}{\mu(B(a,r))}\right)}$$

and we obtain $C\Phi^{-1}\left(\frac{1}{\mu(B(a,r))}\right) \geq \psi^{-1}\left(\frac{1}{\mu(B(a,r))}\right)$ for arbitrary $a \in \mathbb{R}^n$ and $r > 0$. By Lemma 1, we have

$$\Phi\left(\frac{1}{\mu(B(a,r))}\right) \leq C\psi\left(\frac{1}{\mu(B(a,r))}\right),$$

since $a \in \mathbb{R}^n$ and $r > 0$ are arbitrary, then for every $t > 0$ we obtained $\Phi(t) \leq \psi(Ct)$. ∎

Theorem 1.3 explains that inclusion property can be satisfied in the $\mathcal{H}$-Orlicz spaces. This is showed by being satisfied of sufficient and necessary conditions for inclusion properties in the $\mathcal{H}$-Orlicz space which includes the relationship between the inclusion of the two Young functions and the inclusion of the $\mathcal{H}$-Orlicz space with respect to each Young function, also about the inclusion of the norm of a function in the $\mathcal{H}$-Orlicz space respect to each Young function. With the proof of the theorem, it can enrich the development of theory about $\mathcal{H}$-Orlicz spaces, especially regarding the properties that satisfied in these spaces, and develop a theory that inclusion properties can also be applied to other functional spaces such as $\mathcal{H}$-Orlicz spaces.

D. CONCLUSION AND SUGGESTIONS

In this article, we have studied the inclusion relation between $\mathcal{H}$-Orlicz spaces. By using the norm of characteristic function of balls in $\mathbb{R}^n$, we obtained sufficient and necessary conditions for inclusion properties in the $\mathcal{H}$-Orlicz. In particular, if Φ and ψ be Young Functions and $C > 0$, then the following statements are equivalent:

(1) $\Phi(t) \leq \psi(Ct)$ for every $t > 0$

(2) $\mathcal{H}_\psi(\mathbb{R}^n) \subseteq \mathcal{H}_\Phi(\mathbb{R}^n)$

(3) $\|f\|_{\mathcal{H}_\Phi} \leq C\, \|f\|_{\mathcal{H}_\psi}$ for every $f \in \mathcal{H}_\psi(\mathbb{R}^n)$.

For further research, it can be study about the conditions that can cause the inclusion relation to be equal realtion and also inclusion relation between $\mathcal{H}$-Orlicz spaces with three or more Young functions. In addition, it is possible to study inclusion properties in other types of $\mathcal{H}$-Orlicz spaces.

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