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Prime Graph over Cartesian Product over Rings and Its Complement

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ABSTRACT

The prime graph over commutative ring R ($PG(R)$)) is a graph construction with set of vertices $V(PG(R)) = R$ and vertices x and y are adjacent if $xRy = \{0\}$, for $x \neq y$. Girth is the shortest cycle length contains in $PG(R)$ or can be written $gr(PG(R))$. Order in $PG(R)$ denoted by $|V(PG(R))|$ and size in $PG(R)$ denoted by $|E(PG(R))|$. In this paper, we discussed prime graph over cartesian product over rings $\mathbb{Z}_m \times \mathbb{Z}_n$ and its complement. We focused only for $m = p_1$, $n = p_2$ and $m = p_1$, $n = {p_2}^2$, where p_1 and p_2 are prime numbers. Then, we obtained some properties related to order and size, degree, and girth. We also observe some examples. Moreover, we found that a correction in the statement of (Pawar & Joshi, 2019) about the complement graph over prime graph over a ring and gave a counter example for that.

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A. INTRODUCTION

A graph G consist of a finite non empty set whose elements are called vertices ($V(G)$) and edges ($E(G)$) such that the edges are identified by pairs of vertices (Axler & Ribet, 2008). A simple graph is a graph which containing no multiple edges and loop (Vasudev, 2006). A graph that containing no edge is trivial graph. A graph if exist at least one path is a connected graph, otherwise that graph is a disconnected graph. A vertex with zero degree called by isolated vertex (Vasudev, 2006). Girth of G ($gr(G)$) is the shortest cycle length in G (Chartrand, 2016). The number of edge connecting with x is degree of a vertex x (Vasudev, 2006). Size of G is number of edge and order is number of vertex (Johnson, 2017). For result related to order, size, and girth of graph see (Chang & Lu, 2011; Erskine & Tuite, 2023; Gani & Begum, 2010; Jajcay et al., 2018; Jannesari, 2015; Potočnik & Vidali, 2019).

In this paper, order and size of graph G written by $|V(G)|$ and $|E(G)|$, respectively. In graph G, the vertices that are adjacent to a vertex x called by neighborhood of x (Chartrand, 2016). Two graphs G_1 and G_2 are isomophorfic or can be written $G_1 \cong G_2$, if there exist a bijective function $\emptyset: V(G_1) \rightarrow V(G_2)$ (Chartrand, 2016). Prime graph over a ring ($PG(R)$) is a graph with the set of vertices constitutes of all elements of rings and two vertices form an edge if it multipled by that rings equal to zero set (Satyanarayana, 2010). Complement graph over a ring ($(PG(R)^c)$) is a graph which the set of vertices constitutes of elements of rings and two vertices are adjacent in $(PG(R)^c)$ if these vertices are not adjacent in $PG(R)$.

This paper study a connection between algebra and graph theory. This concept was introduced by (Beck, 1988). However, that study is only focused on coloring graph of commutative ring. (Anderson & Livingston, 1999) defining the graph in (Beck, 1988) as a zero divisor graph. And then, in 2010, (Satyanarayana, 2010) introduced a new concept about graph theory and algebra that is prime graph of a ring. (Satyanarayana, 2010) discussed $PG(\mathbb{Z}_p)$, for any prime p and some properties. (Rajendra et al., 2019) gives a corrected version of the theorem of (Satyanarayana, 2010) and gives some counter examples. (Pawar & Joshi, 2019) introduced a complement of prime graph over a ring $\mathbb{Z}_p$, for prime p and determined degree and girth of prime graph over a ring and the complement graph of it. In 2013, (Kalita & Patra, 2013) defining a prime graph of cartesian product of rings. (Satyanarayana et al., 2015) compared cartesian product of prime graph over ring and prime graph over cartesian product of rings. (Fatimah et al., 2023; Krisnawati et al., 2023) discussed some characteristics of prime graph of ring $\mathbb{Z}_m \times \mathbb{Z}_n$, for $m = p_1$, $n = p_2$, and $m = p_2$, $n = {p_2}^2$, where p_1 and p_2 are prime. Study of prime graph over a ring are expanded by (Joshi et al., 2018; Joshi & Pawar, 2020; Kalita et al., 2014; Kalita & Patra, 2021; Pawar & Joshi, 2017).

Motivated by the article of (Fatimah et al., 2023; Kalita & Patra, 2013; Krisnawati et al., 2023; Pawar & Joshi, 2019) we obtain some properties of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement, for $m = p_1, n = p_2$, and $m = p_2, n = {p_2}^2$, where p_1 and p_2 are prime numbers. The properties of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ such as order, size, degree, and girth. Also, some examples of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement have been observed. This paper divided into four section. In section B, we explain the method used in this paper. In section C, we discuss the result such as order, size, degree and prime graph over cartesian product over rings and its complement. In the last section, we conclude the results of the discussion in section C.

B. METHODS

This study uses the literature research method with reference to books and journals related to ring graph theory In particular, the analysis of finding the order and size, degree, and girth, it is tried first by studying the construction of small order and size graph, then expanded on the construction with a larger order and size. However, if we still have not found a suitable pattern, repeat the finding for the suitable pattern from the graph construction structure with small to large order and size, so that the pattern can be generalized and formed theorem. The definitions are used in this paper as follows:

Definition 2.1 (Gallian, 2010): Let $R \neq \emptyset$. A set R with two binary operations called by ring. For example, addition (+) and multiplication (.) which satisfy the axioms of commutative group over addition, semigroup over multiplication, and distributive.

Definition 2.2 (Durbin, 2009): Let $a, b \in R, a, b \neq 0$. Element a is a zero divisor in R if $ab = 0$ or $ba = 0$. The set of all zero divisors in R is $Z(R)$.

Definition 2.3 (Kalita & Patra, 2013): Let $R = R_1 \times \ldots \times R_n$ be a ring. Let $(x_1, x_2), (y_1, y_2) \in R$. Prime graph over a ring R $(PG(R))$ is a graph with set of vertices $V\big(PG(R)\big) = R$ and two vertices are adjacent if $(x_1, x_2)(R)(y_1, y_2) = \{0\}$.

Definition 2.4 (Pawar & Joshi, 2019): A complement of prime graph over a ring R ($\left(PG(R)\right)^c$) defined as a graph with $V\left(\left(PG(R)\right)^c\right) = R$ and the set of edges is $E = \{(x,y)|xRy \neq 0,\ x \neq y\}$.

Definition 2.5 (Pawar & Joshi, 2019)**:** The shortest cycle length in G called by girth ($gr(G)$). If G containing no cycle, then $gr(G) = \infty$.

C. RESULT AND DISCUSSION

We obtain the result related to prime graph over cartesian product of rings and its complement with some examples, order and size, degree, and girth.

1. Prime Graph of Cartesian Product of Rings and Its Complement

We discuss some examples of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement, for $m = p_1$, $n = p_2$ and $m = p_1,\ n = {p_2}^2$, where p_1 and p_2 are primes.

Let us construct $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Based on Definition 2.3, we know that the set of vertices $V\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the set of edges satisfy $(x_1, x_2)(\mathbb{Z}_2 \times \mathbb{Z}_2)(y_1, y_2) = (\bar{0}, \bar{0})$. Therefore, $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is shown in Fig 1.

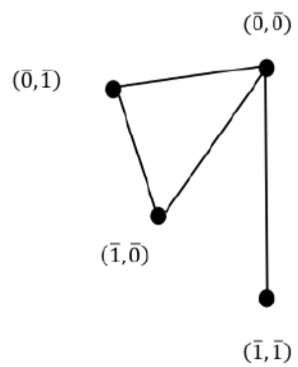

Figure 1. $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Next, we construct the complement of $PG(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Based on Definitiom 2.4, we can write $V\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $E\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \{(x,y)|(x_1, x_2)(\mathbb{Z}_2 \times \mathbb{Z}_2)(y_1, y_2) \neq 0\}$. Therefore, $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$ is shown in Fig 2.

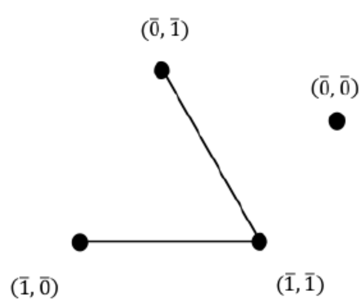

Figure 2. $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$

Similarly, we can construct $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$, $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$, $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$, and $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$. Those graphs are shown in below.

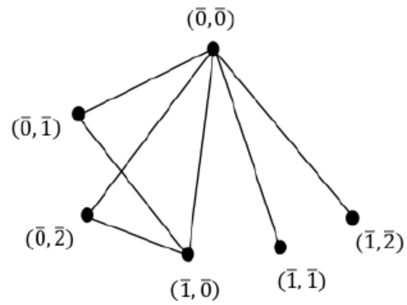

Figure 3. $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$

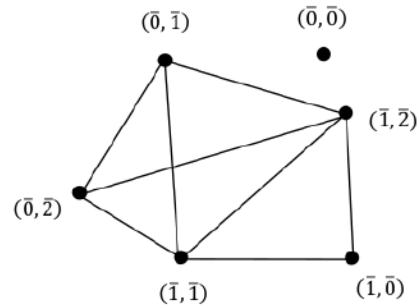

Figure 4. $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_3)\right)^c$

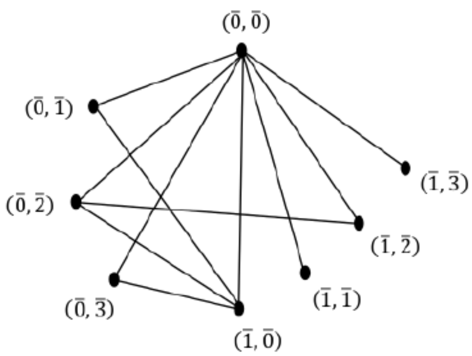

Figure 5. $PG(\mathbb{Z}_2 \times \mathbb{Z}_4)$

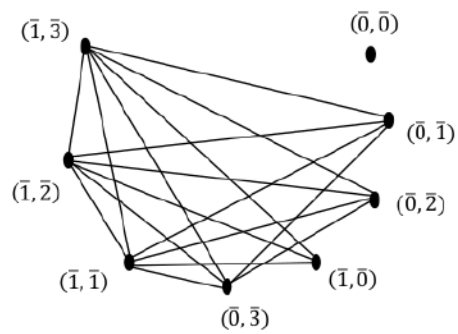

Figure 6. $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_4)\right)^c$

Based on some of the figures above, we can easily write this following lemma.

Lemma 1.1. Let $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ be the complement graph of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$. Construction graph of $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ is two components disconnected graph that is a trivial graph and a simple graph with $(mn-1)$ vertices.

Note. One component of $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ is a trivial graph with vertex $(\bar{0},\bar{0})$. Hence a vertex $(\bar{0},\bar{0})$ is always adjacent to all other vertices in $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$. Therefore, a vertex $(\bar{0},\bar{0})$ is not adjacent to all vertices in $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$. A vertex $(\bar{0},\bar{0})$ refers to isolated vertex.

Lemma 1.2. Let $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ be the complement graph of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$. Two graphs $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ are isomorphic or can be written

$$PG(\mathbb{Z}_m \times \mathbb{Z}_n) \cong \left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$$

Next, we give a correction statement of (Pawar & Joshi, 2019). We begin by giving some counter examples.

Example 1.3. Let construct $PG(\mathbb{Z}_p)$ and $\left(PG(\mathbb{Z}_p)\right)^c$, where $p = 2$ and $p = 4$.

Figure 7. $PG(\mathbb{Z}_2)$

Figure 8. $\left(PG(\mathbb{Z}_2)\right)^c$

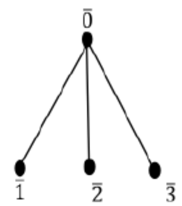

Figure 9. $PG(\mathbb{Z}_4)$

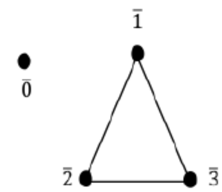

Figure 10. $\left(PG(\mathbb{Z}_4)\right)^c$

We see that $\left(PG(\mathbb{Z}_2)\right)^c$ not containing complete graph but both are trivial graph. Then, $\left(PG(\mathbb{Z}_4)\right)^c$ is a disconnected graph consisting of a trivial graph and $(p-1)$ vertices that form a complete graph.

The statement of the Observation 3.4 (1) in (Pawar & Joshi, 2019) is as follows:

Let $\mathbb{Z}_p$ be a ring and $\left(PG(\mathbb{Z}_p)\right)^c$ be the complement of prime graph. For any prime p, then $\left(PG(\mathbb{Z}_p)\right)^c$ is a disconnected graph with two components in which one component is a trivial graph containing one vertex and other component is always a complete graph with $(p-1)$ vertices.

By our Example 1.3, it follows that the Observation 3.4 (1) is not true for $p = 2$ and also true for $p = 4$. The corrected version of that observation is as follows:

Let $\mathbb{Z}_p$ be a ring and $\left(PG(\mathbb{Z}_p)\right)^c$ be the complement of prime graph of a ring. For prime $p = 3,5,7,..$ and $p = 4$, then $\left(PG(\mathbb{Z}_p)\right)^c$ is a disconnected graph with two components in which one component is a trivial graph containing one vertex and other component is always a complete graph with $(p-1)$ vertices.

2. Order and Size of Prime Graph over Cartesian Product over Rings and Its Complement

In this section, we obtain result about the order and size of $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$, for $m = p_1$, $n = p_2$ and $m = p_1, n = {p_2}^2$, where p_1 and p_2 are primes. Firstly, we review the theorem about order and size of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$.

Theorem 2.1 (Krisnawati et al., 2023): Let $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ be a prime graph of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, where p_1 and p_2 are prime numbers. Then, $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ has order and size

$$\left|V\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)\right| = p_1p_2 \text{ , and}$$
$$\left|E\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)\right| = 2p_1p_2 - p_1 - p_2$$

Theorem 2.2 (Fatimah et al., 2023): Let $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})$ be a prime graph $\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2}$, where p_1 and p_2 are prime numbers. Then, $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})$ has order and size

$$\left|V\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)\right| = p_1{p_2}^2 \text{ , and}$$
$$\left|E\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)\right| = \frac{1}{2}\left(p_2(6p_1p_2 - 4p_1 - 3p_2 + 1)\right)$$

The order and size of $\left(PG(\mathbb{Z}_m\times\mathbb{Z}_n)\right)^c$ for $m = p_1, n = p_2$ and $m = p_1, n = {p_2}^2$, where p_1 and p_2 are primes in the next theorem.

Theorem 2.3. Let $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c$ be a complement of prime graph of $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}$, where p_1 and p_2 are prime numbers. Then, $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c$ has order and size

$$\left|V\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c\right)\right| = p_1p_2 \text{ , and}$$
$$\left|E\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c\right)\right| = \frac{1}{2}({p_1}^2{p_2}^2 - 5p_1p_2 + 2p_1 + 2p_2)$$

Proof. Based on Definition 2.4, $V\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c\right) = \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}$, so it is clear that $\left|V\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c\right)\right| = p_1p_2$.
Suppose the size in complete graph with p_1p_2 vertices is $\frac{1}{2}\left(p_1p_2(p_1p_2 - 1)\right)$.
By Theorem 2.1, the number of edges in $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$ is $2p_1p_2 - p_1 - p_2$.
We know that two distinct vertices are adjacent in $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c$ if thhese vertices are not adjacent in $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$. Therefore, the number of edges in $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c$ can be written

$$\frac{1}{2}\left(p_1p_2(p_1p_2 - 1)\right) - (2p_1p_2 - p_1 - p_2)$$
$$= \frac{1}{2}({p_1}^2{p_2}^2 - 5p_1p_2 + 2p_1 + 2p_2)$$

Thus, we obtain $\left|E\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c\right)\right| = \frac{1}{2}({p_1}^2{p_2}^2 - 5p_1p_2 + 2p_1 + 2p_2)$. ■

Theorem 2.4. Let $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)^c$ be the complement of prime graph of $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}$, where p_1 and p_2 are prime numbers. Then, $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)^c$ has order and size

$$\left|V\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)^c\right)\right| = p_1{p_2}^2 \text{ , and}$$
$$\left|E\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)^c\right)\right| = \frac{1}{2}\left(p_2({p_2}^3{p_1}^2 + (3 - 7p_1)p_2 + 4p_1 - 1)\right)$$

Proof. It follows from Definition 2.4, that $\left|V\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)^c\right)\right| = p_1{p_2}^2$.
Suppose the size n complete graph with $p_1{p_2}^2$ vertices is $\frac{1}{2}\left(p_1{p_2}^2(p_1{p_2}^2 - 1)\right)$.
By Theorem 2.2, the number of edges in $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})$ is $\frac{1}{2}\left(p_2(6p_1p_2 - 4p_1 - 3p_2 + 1)\right)$.
Therefore, the number of edges in $\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)^c$ can be written

$$\frac{1}{2}\left(p_1{p_2}^2({p_1 p_2}^2-1)\right)-\left(\frac{1}{2}\left(p_2(6p_1p_2-\ 4p_1-\ 3p_2+1)\right)\right)$$
$$=\frac{1}{2}\left(p_2({p_2}^3{p_1}^2\ +\ (3-7p_1)p_2\ +\ 4p_1\ -\ 1)\right)$$

Thus, we obtain $\left|E\left(\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)^c\right)\right| = \frac{1}{2}\left(p_2({p_2}^3{p_1}^2\ +\ (3-7p_1)p_2\ +\ 4p_1\ -\ 1)\right)$. ■

3. Degree of Prime Graph over Cartesian Product over Rings and Its Complement

We give theorem about degree of a vertex in $PG(\mathbb{Z}_m\times\mathbb{Z}_n)$ and its complement, for $m=p_1, n=p_2$ and $m=p_1, n={p_2}^2$, where p_1 and p_2 are primes.

Theorem 3.1

Suppose $(\bar{x},\bar{y})$ be a vertex in $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$, then a vertex $(\bar{x},\bar{y})$ has degree

$$\deg(\bar{x},\bar{y})=\begin{cases} p_1p_2-1 & \text{, if } \bar{y}=0 \text{ and } \bar{x}=0\\ p_1 & \text{, if } \bar{y}\neq 0 \text{ and } \bar{x}=0\\ p_2 & \text{, if } \bar{y}=0 \text{ and } \bar{x}\neq 0\\ 1 & \text{, if } \bar{y}\neq 0 \text{ and } \bar{x}\neq 0\end{cases}$$

Proof Degree of of vertex $(\bar{x},\bar{y})\in V\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})\right)$ divided into four cases as follows:

(i) If $\bar{y}=0$ and $\bar{x}=0$.
Since a vertex $(\bar{0},\bar{0})$ is adjacent to all vertices of $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$, then $(\bar{0},\bar{0})$ has degree p_1p_2-1 .

(ii) if $\bar{y}\neq 0$ and $\bar{x}=0$.
Since a vertex $(\bar{0},\bar{y})$ is adjacent to $(\bar{x},\bar{0})$ and $(\bar{0},\bar{0})$, then $(\bar{0},\bar{y})$ has degree $(p_1-1)+1=p_1$.

(iii) if $\bar{y}=0$ and $\bar{x}\neq 0$.
Since a vertex $(\bar{x},\bar{0})$ is adjacent to $(\bar{0},\bar{y})$ and $(\bar{0},\bar{0})$, then $(\bar{x},\bar{0})$ has degree $(p_2-1)+1=p_2$.

(iv) if $\bar{y}\neq 0$ and $\bar{x}\neq 0$.
Since a vertex $(\bar{x},\bar{y})$ is not adjacent to all other vertices in $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$ except $(\bar{0},\bar{0})$, then $(\bar{x},\bar{y})$ has degree 1. ■

Theorem 3.2

Suppose $(\bar{x},\bar{y})$ be a vertex in $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})$, then a vertex $(\bar{x},\bar{y})$ has degree

$$\deg(\bar{x},\bar{y})=\begin{cases} p_1{p_2}^2-1 & \text{, if } \bar{y}=0 \text{ and } \bar{x}=0\\ {p_2}^2 & \text{, if } \bar{y}=0 \text{ and } \bar{x}\neq 0\\ p_1p_2-1 & \text{, if } \bar{y}\neq 0, \bar{x}=0, \text{ and } \bar{y}\in Z(\mathbb{Z}_{{p_2}^2})\\ p_2 & \text{, if }, \bar{y}\neq 0, \bar{x}\neq 0, \text{ and } \bar{y}\in Z(\mathbb{Z}_{{p_2}^2})\\ 1 & \text{, if } \bar{y}\neq 0, \bar{x}\neq 0, \text{and } \bar{y}\notin Z(\mathbb{Z}_{{p_2}^2})\end{cases}$$

Proof Suppose $Z(\mathbb{Z}_{{p_2}^2})=(p_2-1)$. Degree of of vertex $(\bar{x},\bar{y})\in V\left(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{{p_2}^2})\right)$ divided into five cases as follows:

(i) If $\bar{y} = 0$ and $\bar{x} = 0$.
Since a vertex $(\bar{0},\bar{0})$ is adjacent to other vertices in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2})$, then $(\bar{0},\bar{0})$ has degree $p_1 p_2{}^2 - 1$.

(ii) If $\bar{y} = 0$ and $\bar{x} \neq 0$.
Since a vertex $(\bar{x},\bar{0})$ is adjacent to $(\bar{0},\bar{y})$ and $(\bar{0},\bar{0})$, then $(\bar{0},\bar{x})$ has degree $(p_2{}^2 - 1) + 1 = p_2{}^2$.

(iii) If $\bar{y} \neq 0$, $\bar{x} = 0$, and $\bar{y} \in Z(\mathbb{Z}_{p_2{}^2})$
In this case, we consider three cases as follows:
(a) $(\bar{0},\bar{y})$ is adjacent to $(\bar{x},\bar{0})$ and $(\bar{0},\bar{0})$, then the degree of $(\bar{0},\bar{y})$ is $(p_1 - 1) + 1 = p_1$.
(b) $(\bar{0},\bar{y})$ is adjacent to $(\bar{x},\bar{y})$, then the degree of $(\bar{0},\bar{y})$ is $(p_2 - 1)(p_1 - 1)$.
(c) $(\bar{0},\bar{y})$ is adjacent to $(\bar{0},\bar{y})$, except it self, then the degree of $(\bar{0},\bar{y})$ is $(p_2 - 2)$.
By (a), (b), and (c), the degree of $(\bar{0},\bar{y})$ is $p_1 + (p_2 - 1)(p_1 - 1) + (p_2 - 2) = p_1 p_2 - 1$.

(iv) If $\bar{x} \neq 0$, $\bar{y} \neq 0$, and $\bar{y} \in Z(\mathbb{Z}_{p_2{}^2})$
Since a vertex $(\bar{x},\bar{y})$ is adjacent with $(\bar{0},\bar{y})$ and $(\bar{0},\bar{0})$, then the degree of $(\bar{x},\bar{y})$ is $(p_2 - 1) + 1 = p_2$.

(v) If $\bar{x} \neq 0$, $\bar{y} \neq 0$ and $\bar{y} \notin Z(\mathbb{Z}_{p_2{}^2})$
Since a vertex $(\bar{x},\bar{y})$ is only adjacent to $(\bar{0},\bar{0})$, then the degree of $(\bar{x},\bar{y})$ is 1. ■

Theorem 3.3

Suppose $(\bar{x},\bar{y})$ be a vertex in $\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)^c$, then a vertex $(\bar{x},\bar{y})$ has degree

$$\deg(\bar{x},\bar{y}) = \begin{cases} p_1 p_2 - 2 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} \neq 0 \\ p_1 p_2 - p_2 - 1 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} \neq 0 \\ p_1 p_2 - p_1 - 1 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} = 0 \\ 0 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} = 0 \end{cases}$$

Proof Degree of of vertex $(\bar{x},\bar{y}) \in V\left(\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)^c\right)$ divided into four cases as follows:

(i) if $\bar{y} \neq 0$ and $\bar{x} \neq 0$.
Since a vertex $(\bar{x},\bar{y})$ is adjacent to other vertices of $\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)^c$, then $(\bar{x},\bar{y})$ has degree $p_1 p_2 - 2$.

(ii) if $\bar{y} = 0$ and $\bar{x} \neq 0$.
Since a vertex $(\bar{x},\bar{0})$ is adjacent to $(\bar{x},\bar{y})$ and $(\bar{x},\bar{0})$ (except it self), then $(\bar{x},\bar{0})$ has degree $(p_1 - 1)(p_2 - 1) + (p_1 - 1) - 1 = p_1 p_2 - p_2 - 1$.

(iii) if $\bar{y} \neq 0$ and $\bar{x} = 0$.
Since a vertex $(\bar{0},\bar{y})$ is adjacent to $(\bar{0},\bar{y})$ (except it self) and $(\bar{x},\bar{y})$, then $(\bar{0},\bar{y})$ has degree $(p_1 - 1)(p_2 - 1) + (p_2 - 1) - 1 = p_1 p_2 - p_1 - 1$.

(iv) if $\bar{y} = 0$ and $\bar{x} = 0$.
Since a vertex $(\bar{0},\bar{0})$ is not adjacent to all vertices in $\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})\right)^c$, then $(\bar{x},\bar{y})$ has degree 0. ■

Theorem 3.4

Suppose $(\bar{x},\bar{y})$ be a vertex in $\left(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2})\right)^c$, then a vertex $(\bar{x},\bar{y})$ has degree

$$\deg(\bar{x},\bar{y}) = \begin{cases} p_1 p_2{}^2 - 2 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} \neq 0 \\ p_1 p_2{}^2 - p_2{}^2 - 1 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} \neq 0 \\ p_1 p_2{}^2 - p_1 - 1 & \text{, if } \bar{y} \neq 0 \text{ and } \bar{x} = 0 \\ 0 & \text{, if } \bar{y} = 0 \text{ and } \bar{x} = 0 \end{cases}$$

Proof Degree of of vertex $(\bar{x},\bar{y}) \in V\left(\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2}\right)\right)^c\right)$ divided into four cases as follows:

(i) if $\bar{y} \neq 0$ and $\bar{x} \neq 0$.
Since a vertex $(\bar{x},\bar{y})$ is adjacent to other vertices in $PG\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2}\right)\right)^c$, then $(\bar{x},\bar{y})$ has degree $p_1 p_2{}^2 - 2$.

(ii) if $\bar{y} = 0$ and $\bar{x} \neq 0$.
Since a vertex $(\bar{x},\bar{0})$ is adjacent to $(\bar{x},\bar{y})$ and $(\bar{x},\bar{0})$ (except it self), then $(\bar{x},\bar{0})$ has degree $(p_1 - 1)(p_2{}^2 - 1) + (p_1 - 1) - 1 = p_1 p_2 - p_2{}^2 - 1$.

(iii) if $\bar{y} \neq 0$ and $\bar{x} = 0$.
Since a vertex $(\bar{0},\bar{y})$ is adjacent to $(\bar{0},\bar{y})$ (except it self) and $(\bar{x},\bar{y})$, then $(\bar{0},\bar{y})$ has degree $(p_1 - 1)(p_2{}^2 - 1) + (p_2{}^2 - 1) - 1 = p_1 p_2 - p_1 - 1$.

(iv) if $\bar{y} = 0$ and $\bar{x} = 0$.
Since a vertex $(\bar{0},\bar{0})$ is not adjacent to other vertices in $\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2{}^2}\right)\right)^c$, then $(\bar{x},\bar{y})$ has degree 0. ■

4. Girth of Prime Graph over Cartesian Product over Rings and Its Complement

In this section, we give theorem about girth of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and its complement, where m and n are integers.

Theorem 4.1

Let $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ be a prime graph over cartesian product of rings and $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ be a complement of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$.

(i) If $m, n \in \mathbb{Z}^+$ and $m, n \geq 2$, then $gr\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right) = 3$.

(ii) For $m, n = 2$, then $gr\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \infty$.

Proof (i) Based on Definiton 2.3, a vertex $(\bar{0},\bar{0})$ is adjacent to other vertices in $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and a vertex $(\bar{x},\bar{0})$ is adjacent with $(\bar{0},\bar{y})$. Moreover, a pairwise verices are adjacent if at least one of the vertex containing zero divisors. Hence, it is clear that in $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ is always exist a subgraph which forms a triangle with one of the vertex in that triangle is $(\bar{0},\bar{0})$ and the probability of two other vertices is $(\bar{x},\bar{0})$ and $(\bar{0},\bar{y})$ or vertices are containing zero divisors. Therefore, it forms a shortest cycle of length 3. In other words, $gr\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right) = 3$. If $m, n < 2$, no triangles are forms. (ii) Based on example on section 1, $\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c$ does not form the shortest cycle. Therefore, $gr\left(\left(PG(\mathbb{Z}_2 \times \mathbb{Z}_2)\right)^c\right) = \infty$. ■

D. CONCLUSION AND SUGGESTIONS

Based on section C, some properties of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ and $\left(PG(\mathbb{Z}_m \times \mathbb{Z}_n)\right)^c$ such as order and size, degree, and girth have been obtained. On the next research, can be studied $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$, where m, n are not prime numbers. Moreover, some properties such as energy, Wiener index, line graph, coloring, and labeling can also studied.

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