Multivariate Control Chart based on Neutrosophic Hotelling T² Statistics and Its Application

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ABSTRACT

Under classical statistics Hotelling T² control chart is applied when the observations of quality characteristics are precise, exact, or crisp data. However, in reality, under uncertain conditions, the observations are not necessarily precise, exact, or indeterminacy. As a consequence, the classical Hotelling T² control chart is not appropriate to monitor the process for this condition. To tackle this situation, we proposed new Hotelling T² monitoring scheme based on a fuzzy neutrosophic concept. Neutrosophic is the generalization of fuzzy. It is used to handle uncertainty using indeterminacy. The combination of statistics based on neutrosophic Hotelling T² and classical Hotelling T² control chart will be proposed to tackle indeterminacy observations. The proposed Hotelling T² statistics, its call neutrosophic Hotelling T² (T²) control chart. This chart involves the indeterminacy of observations, its call neutrosophic data and will be expressed in the indeterminacy interval. T² control charts consist T² lower chart and T² upper chart. In this paper, the neutrosophic Hotelling T² will be applied to individual observations of glass production and will be compared by using classical Hotelling T² control chart. Based on T² control charts of glass production, nine points fall outside of UCL of lower control chart and 24 points outside from UCL of upper control chart. Whereas using classical Hotelling T² control chart, just one point outside from of UCL. From the comparison, it concluded that the neutrosophic Hotelling T² control chart is more suitable for the indeterminacy of observations.

A. INTRODUCTION

A control chart is one of most the famous tools to monitor the process. Generally, there are two categories of control charts based on the types of quality characteristics: variable control chart and attribute control chart. Both of these charts, based on the number of monitored quality characteristics consist of a univariate control chart and multivariate control chart. Univariate control chart used to monitor the single quality characteristics. \( \bar{X} - R \) is one of the most popular univariate control charts. Hotelling \( T^2 \) a control chart is one of the tools that is widely used to monitor the multivariate process (Montgomery, 2020). The multivariate control chart is used to monitor the process simultaneously of two or more interrelated
quality characteristics. The classical Hotelling $T^2$ control chart suitable to monitor when all data observations are crisp and precise.

In manufacture, we often meet the data observations, not price, or vague. In this case classical Hotelling $T^2$ control chart not suitable to use. To handle the condition control chart-based fuzzy logic proposed by Zadeh (1965). Several researchers have developed fuzzy attribute control chart; fuzzy $p$ control chart and the extension (Pandian & Puthiyanyagam, 2013) (Sogandi et al., 2014) (Shabani & Rezayian, n.d.), fuzzy $c$ and $u$ control chart (Darestani et al., 2014) (Fadaei & Pooya, 2018) (Ercan-Teksen & Anagün, 2020) (Ercan-Teksen & Anagün, 2018). Meanwhile, the fuzzy variable control charts; $X$-Individual fuzzy control chart (Gildeh & Shafiee, 2015) (Moraditadi & Avakhdarestani, 2016a) (Alizadeh & Ghomi, 2011) (Moraditadi & Avakhdarestani, 2016b) and fuzzy EWMA and CUSUM and also the performance (Wang & Hryniewicz, 2013) (Göztok et al., 2021) (Erginel & Şentürk, 2016). And the development of fuzzy multivariate control charts: (Ghobadi et al., 2014) (Ghobadi et al., 2015) (Pastuizaca Fernández et al., 2015) (Wibawati, 2020).

The other approach to monitoring the unprecise data neutrosophic can be used. Neutrosophic is an extension of fuzzy logic (Smarandache, 2014). The logic of neutrosophic is considered indeterminacy in the measurement. However, based on literature reviews, the neutrosophic control chart is still limited. The neutrosophic control charts that have been developed are neutrosophic $\bar{X}$ (Aslam & Khan, 2019), neutrosophic $S$ chart (Khan, Gulistan, Chamnam, et al., 2020) (Khan, Gulistan, Hashim, et al., 2020), neutrosophic Exponentially Weighted Moving Average (NEWMA) $X$ (Aslam et al., 2019). Among these charts is the univariate control chart. Meanwhile, we are often interested to monitor multivariate processes that involve the indeterminacy of observations. Recently, Aslam introduced Hotelling $T^2$ statistics under neutrosophic statistics (Aslam & Arif, 2020). The new statistics are the generalization of classical statistics under uncertainty conditions. This procedure is applied to chemical data. Based on comparison with classical Hotelling $T^2$ statistics, the proposed method is more effective. Therefore in this paper proposed new Hotelling $T^2$ monitoring scheme based on fuzzy neutrosophic concept and it call neutrosophic Hotelling $T^2$ ($T^2_N$) control chart. The proposed chart will be applied at specific glass production and will be compared with the classical Hotelling $T^2$ control chart.

B. METHODS
1. Neutrosophic Hotelling $T^2_N$ Statistics
   If $x_{jKN} = [x_{jKL}, x_{jKU}]$ be a random variable of neutrosophic that represents neutrosophic observation for $k$th variable and $j$th observations. This interval expresses the indeterminacy, $x_{jKL}$ show the smallest value and $x_{jKU}$ is the largest value. Based on this form $x_{jKL}$ is a part of determinate and part of indeterminate is $x_{jKU}I_N$, where $I_N = [I_N]^L, I_N]^U$ and it can be stated as if $x_{jKN} = x_{jKL} + x_{jKU}I_N$. If $x_N = [n_L, n_U]$ are observations of neutrosophic from neutrosophic variable $p_N = [p_L, p_U]$.
The form of neutrosophic $X_N\epsilon[X_L, X_U]$ can be shown as

$$X_N = X_L + X_U I_N; I_N \epsilon [I_L, I_U]$$

The neutrosophic sample mean is presented by

$$\bar{x}_{kN} \epsilon \left[ \frac{1}{n_L} \sum_{j=1}^{n_L} x_{jkl}, \frac{1}{n_U} \sum_{j=1}^{n_U} x_{jku} \right]; \bar{x}_{kN} \epsilon [\bar{x}_{kl}, \bar{x}_{ku}],$$

where $\bar{x}_{kN}$ can be presented as $\bar{x}_{kN} = \bar{x}_{kl} + \bar{x}_{ku} I_N; I_N \epsilon [I_L, I_U]$.

The form of the neutrosophic sample variance is

$$s^2_{kN} \epsilon \left[ \frac{1}{n_L} \sum_{j=1}^{n_L} (x_{jkl} - \bar{x}_{kl})^2, \frac{1}{n_U} \sum_{j=1}^{n_U} (x_{jku} - \bar{x}_{ku})^2 \right]; s^2_{kN} \epsilon [s^2_{kl}, s^2_{ku}],$$

where $s^2_{kN} = s^2_{kl} + s^2_{ku} I_N; I_N \epsilon [I_L, I_U]$.

The formula of neutrosophic Covariance is:

$$S_{i kn} \epsilon \left[ \frac{1}{n_L} \sum_{j=1}^{n_L} \left(x_{jil} - \bar{x}_{kl} \right) \left( (x_{jkl} - \bar{x}_{kl}) \right), \frac{1}{n_U} \sum_{j=1}^{n_U} \left(x_{jil} - \bar{x}_{ku} \right) \left( (x_{jku} - \bar{x}_{kl}) \right) \right];$$

where:

$$S_{i kn} \epsilon [S_{i kl}, S_{i ku}],$$

$$S_{i kn} = S_{i kl} + S_{i ku} I_N; I_N \epsilon [I_L, I_U].$$

The Statistics neutrosophic $T^2$ Hotelling is:

$$T^2_N = \left[ (\bar{X}_L - \mu_{0L})' \left( \frac{1}{n_L} S_L \right)^{-1} \right] \left( \bar{X}_L - \mu_{0L} \right), $$

$$\left( \frac{1}{n_U} S_U \right)^{-1} \left( \bar{X}_U - \mu_{0U} \right); T^2_N \epsilon [T^2_L, T^2_U],$$

where

$$T^2_N = T^2_L + T^2_U I_N; I_N \epsilon [I_L, I_U].$$

$$T^2_N \sim \left( \frac{n_L - 1}{(n_N - 1) \text{P}} \right) \frac{F_{P_N(n_N - P_N)}}{T^2_N \epsilon [T^2_L, T^2_U]}. $$

These statistics can be applied for testing hypothesis.

### 2. Classical Hotelling $T^2$ Control Chart

The classical Hotelling $T^2$ control chart widely used to monitor mean processes simultaneously of more than one interrelated quality characteristics observations. This chart was proposed by Harold Hotelling, 1947 (Montgomery, 2020). The classical Hotelling $T^2$ control chart can be used to monitor both of subgroup and individual observation. The statistics of Hotelling $T^2$ chart for individual as follows.

$$T^2 = (x - \bar{x})' S^{-1} (x - \bar{x}).$$
Let \( p \) is the number of quality characteristics, suppose we have \( m \) sample, each sample consist \( n = 1 \). The Control limits of statistic \( T^2 \) the control chart is (Montgomery, 2020)

Upper control limit (UCL)

\[
UCL = \frac{(m-1)^2}{m} \beta_{\alpha/2} \beta_{10/2} (m-p-10/2)
\]

and Lower control limit (LCL),

\[
LCL = 0.
\]

The process is in control if all the points fall between control limits.

C. RESULT AND DISCUSSION

1. Neutrosophic Hotelling \( T^2(\mathcal{T}_N^2) \) Control Chart

In this part, we discuss the new Hotelling \( T^2 \) monitoring scheme based on a fuzzy neutrosophic concept. Control charts and hypothesis testing have a close relationship. A point plotting within the control limits is the same as failing to reject the statistical control hypothesis, whereas a point plotting outside the control limits is the same as rejecting the statistical control hypothesis. The statistics in this new chart is obtained by combining classical Hotelling \( T^2 \) control chart and neutrosophic Hotelling \( T^2 \) statistic \( (\mathcal{T}_N^2) \) which was proposed by Aslam (Aslam & Arif, 2020). Let \( p_N \in [p_L, p_U] \) is the number of quality characteristics and the data matrix \( X_N \in \{X_L, X_U\} \), suppose we have \( m \) sample with individual observation. The statistics of neutrosophic Hotelling \( T^2(\mathcal{T}_N^2) \) Control Chart is given by,

\[
\mathcal{T}_N^2 = [(\bar{x}_L - \mu_{0L})(S_L)^{-1}](\bar{x}_L - \mu_{0L}), (\bar{x}_U - \mu_{0U}),(S_U)^{-1}(\bar{x}_U - \mu_{0U}),
\]

\[
\mathcal{T}_N^2 \in [T_L^2, T_U^2],
\]

where \( \mu_{0L} \) and \( \mu_{0U} \) are target.

Based on equation (9), the proposed chart consists of two charts, namely lower \( \mathcal{T}_N^2 \) control chart and upper \( \mathcal{T}_N^2 \) control chart. The control limits of the proposed chart are as follows:

\[
UCL_N = \frac{p_N(m-1)^2}{m} \beta_{\alpha/2} \beta_{10/2} (m-p_N-10/2) \quad \text{and} \quad LCL_N = 0.
\]

The process is in control if all the points fall between control limits \( (UCL_N \text{ and } LCL_N) \).

2. Numerical Example

The application of neutrosophic hotelling \( \mathcal{T}_N^2 \) Control Chart using data from Quality Control division for quality characteristics of glass production. There are two quality characteristics such as Cutter line \( (X_{1N}) \) and Edge Distorsion \( (X_{2N}) \). The target of the cutter is 115 mm and the Edge distortion is 40 mm. The form of the data can be seen in Table 1.

Based on these data we find the neutrosophic mean sample are \( \bar{x}_{1N} \in [132.76, 151.03] \) and \( \bar{x}_{2N} \in [30.61, 37.27] \), and the neutrosophic covariance sample \( (s_N^2) \) is
By using equation (9) and equation (10) we calculate neutrosophic Hotelling $T^2$ statistic (see Table 2) and the control limits are $LCL_N = 2.23$, $UCL_N = 24.63$. The result of the neutrosophic hotelling $T^2_N$ Control Chart as shown in Figure 1.

Table 1. The neutrosophic data of the cutter and Edge distortion

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<th>Edge distortion</th>
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<td>$X_{1NU}$ $X_{2}$ $X_{2NU}$</td>
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<td>115</td>
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<td>149  34  38</td>
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Table 2. The neutrosophic $T^2_N$ statistic of the cutter and Edge distortion

<table>
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<tr>
<th>Subgroup</th>
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<th>$T^2_{NU}$</th>
<th>Subgroup</th>
<th>$T^2_{NL}$</th>
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In Figure 1, the visualization of the neutrosophic Hotelling $T^2_N$ control chart, indicate the process mean of glass production is out of control. Nine points fall outside of $UCL_N$ of lower control chart and in the upper control chart closed to 73% of data outside from $UCL_N$. In the next step, we compare the neutrosophic hotelling $T^2_N$ control chart with the classical Hotelling $T^2$ control chart, see Figure 2. This figure shows the process mean of glass production is also out of control, but by using the classical method, just one point falls outside $UCL_N$. Therefore based on this case the neutrosophic hotelling $T^2_N$ control chart is more sensitive than with the classical hotelling $T^2$ control chart.

The comparison of the case study shows that $T^2_N$ control chart is more suitable for use than classical hotelling $T^2$ control chart. So the impact for the future, monitoring quality in problem in manufacture industries which has data are inaccurate or indeterminacy will be tackle with neutrosophic control chart.
D. CONCLUSION AND SUGGESTIONS

The statistics of new Hotelling $T^2$ monitoring scheme based on the fuzzy neutrosophic concept is obtained from a combination of neutrosophic hotelling $T^2_N$ statistic and classical Hotelling $T^2$ control chart. Hotelling $T^2$ chart based on the fuzzy neutrosophic is appropriate for indeterminacy data. The proposed chart consists of two charts, namely lower $T^2_N$ control chart and upper $T^2_N$ control chart. Based on the case study the proposed chart is more sensitive than the classical Hotelling $T^2$ control chart. The performance evaluation of $T^2_N$ control chart can be considered for future works. Further future research can be developed to the $T^2_N$ control chart that can be designed using a robust estimator or shrinkage estimator.

ACKNOWLEDGEMENT

This paper was supported by Institut Teknologi Sepuluh Nopembar, Indonesia, under Grand No. 1854/PKS/ITS/2021. The authors, gratefully acknowledge the financial support.

REFERENCES


