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Super (a, d)-$P_2 \odot P_m$-Antimagic Total Labeling of Corona Product of Two Paths

Bela Zainun Yatin[1]*, Zata Yumni Awanis[2], I Gede Adhitya Wisnu Wardhana[1]
[1]Department of Mathematics, Universitas Mataram, Indonesia
[2]Research Center for Computing, National Research and Innovation Agency (BRIN), Indonesia
bellazy4@gmail.com

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ABSTRACT

Graph labeling involves mapping the elements of a graph (edges and vertices) to a set of positive integers. The concept of an anti-magic super outer labeling (a,d)-H pertains to assigning labels to the vertices and edges of a graph using natural numbers $\{1,2,3,\dots,p+q\}$. The weights of the outer labels H form an arithmetic sequence $\{a, a+d, a+2d, \dots, a+(k-1)d\}$, where 'a' represents the first term, 'd' is the common difference, and 'k' denotes the total number of outer labels, with the smallest label assigned to a vertex. This study investigates the super (a, d)-$P_2 \odot P_m$-antimagic total labeling of the corona product $P_n \odot P_m$, where n and m are both greater than or equal to 3. We define the labeling functions for vertices and edges based on the partitioning of labels into three subsets. Using k-balanced and (k, δ)-anti balanced multisets, we demonstrate that for m being odd, $P_n \odot P_m$ is super $(9m^2n+4mn+m-n+3, 1)$-$P_2 \odot P_m$ -antimagic, and for m being even, $P_n \odot P_m$ is super $(9m^2n+4mn+m-2n+5, 3)$-$P_2 \odot P_m$ -antimagic. The labelling scheme is illustrated through examples. For the case when m is odd, an anti-magic total labelling of $P_3 \odot P_3$ forms a super $(282, 1)$- $P_2 \odot P_3$ -antimagic labeling. In the case of even m, an antimagic total labeling of $P_3 \odot P_4$ results in a super (483,3)- $P_2 \odot P_4$ -antimagic labeling. Both of these examples provide insights into the antimagic properties of corona products.

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A. INTRODUCTION

The field of graph theory is a branch of mathematics that centers on the examination of abstract structures referred to as graphs. These graphs have been widely applied across multiple domains, including computer science, social science, and transportation (Bača et al., 2019; Chang et al., 2019; Liu et al., 2020). Their origins can be traced back to Leonard Euler's work in 1736. Within graph theory, a particularly fascinating subject of study revolves around graph labelling, which involves creating mappings from sets of vertices, edges, or both to a set of natural number (Diestel, 2017; Indunil & Perera, 2022; Nurvazly et al., 2022). At present, the investigation of magical and anti-magical labelling continues to be a significant and actively researched area (Gallian, 2022).

The notion of magic labelling was first introduced by Sedlacek in 1963 (J. Sedlacek, 1963). This concept was subsequently expanded upon by Kotzig and Rosa in 1970 (Kotzig & Rosa, 1970) and Enomoto et al. in 1998, who introduced edge-magic total labelling and super edge-magic total labelling (Enomoto et al., 1998). Additionally, Gutiérrez and Lladó in 2005 further extended these ideas to the terms of H-(super) magic total labelling. (Gutiérrez & Lladó, 2005).

This paper focuses on a finite and simple graph G where vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A set of distinct subgraphs $H_1, H_2, \dots, H_k$ of G is defined as an *edge-covering* of G if every edge of G is included in at least one of the H_i. When each subgraph is isomorphic to a specific graph H, the graph G is said to have an H-*covering*. Now let G has an H-covering. If there exists a bijective function $f: V(G) \cup E(G) \to \{1,2,\dots,|V(G)|+|E(G)|\}$ and a positive integer k_f such that for all subgraphs H' that are isomorphic to H, the H-*weight* $\omega(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e) = k_f$, then G is called an H-*magic*. The function f is known as an H-*magic total labeling* of G. Furthermore, the labeling f is known as an H-*supermagic total labelling* of G. if the vertices of G are labeled with $1,2,\dots,|V(G)|$. In this case, graph G is H-*supermagic* (Gutiérrez & Lladó, 2005). When H is an edge, we say that G is *(super) edge-magic* (Enomoto et al., 1998; Kotzig & Rosa, 1970). In their paper, Gutiérrez and Lladó presented star-supermagic and path-supermagic total labelings for certain connected graphs. They also offered constructions for infinite families of H-magic graphs for any given graph H. Furthermore, Llado A & Moragas J (2007) and Ngurah et al. (2010) investigated cycle-supermagic total labellings for certain connected graphs. The readers are referred (Anjaneyulu et al., 2015; Hendy et al., 2018; Hendy et al., 2018; Sandariria et al., 2017; Simanihuruk et al., 2021; Lakshmi & Sagayakavitha, 2018; Ulfatimah et al., 2017; Agustin et al., 2019; Murugan & Chandra Kumar, 2019) for other results about H-(super) magic total labelling.

On the other hand, the concept of (a,d)-edge-antimagic total labeling was initially introduced by (Simanjuntak et al., 2000). This concept was later extended into the concept of (super) (a,d)-H-antimagic total labeling by (Inayah et al., 2009). Let a, d, and t be three positive integers, where t denotes the number of subgraphs of G that isomorphic to H. Suppose G has an H-covering. Graph G is called an (a,d)-H-*antimagic* if there exists a bijective function $f: V(G) \cup E(G) \to \{1,2,\dots,|V(G)|+|E(G)|\}$ so that for all subgraphs H' that are isomorphic to H, the H-weight $\omega(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$ forms an arithmetic sequence $a, a+d, a+2d, \dots, a+(t-1)d$. In this case, the labeling f is called an (a,d)-H-*antimagic total labeling* of G. If the vertices of G are labeled with $1,2,\dots,|V(G)|$, then f is called the *super* (a,d)-H-*antimagic total labeling* of G and G is called the *super* (a,d)-H-*antimagic.* The readers are referred to (Martin Bača et al., 2006; Agustin et al., 2019; Hussain et al., 2012.; Inayah et al., 2013; Palanivelu & Neela, 2019; Permata Sari et al., 2019; Prihandini & Adawiyah, 2023; Raheem et al., 2014; Taimur et al., 2018) for other results about super (a,d)-H-antimagic.

In this paper, we study the super (a,d)-H-antimagic total labeling of corona product of two paths. Recall that *corona product* of two graphs G and G', denoted by $G \odot G'$, is a graph obtained by taking one copy of G and $|V(G)|$ copies G' and then connecting the i-th vertex of G to every vertex in the i-th copy G' (Hasni et al., 2022; Permata Sari et al., 2019; Sandariria et al., 2017). We show that the corona product $P_n \odot P_m$, $m,n \geq 3$, is super (a,d)-$P_2 \odot P_m$-antimagic where $d=1$ if m is odd and $d=3$ if m is even.

B. METHODS

The method used in this research is deductive method. This method starts from a review literature about magic and anti-magic labelling, in particular (super) (a,d)-H-anti-magic total labeling (Martin Bača et al., 2006), then continues by constructing the labeling on corona product of two paths. This construction produces several conjectures the hypothesis is that the graph under study has a total labeling of super (a,d)-H-anti-magic then validity will be proven, where the proof process will utilize several techniques in k-balanced multisets and (k,δ)-anti balanced multisets (Maryati et al., 2013). The given conjecture will be stated as a corollary, lemma, or theorem, if it is proven to be true. Otherwise, the research will be repeated by reconsidering the conjecture, the construction of the labelling, or the proof method used (Prihandini & Adawiyah, 2022). The following is a flowchart of the research procedure, as shown in Figure 1.

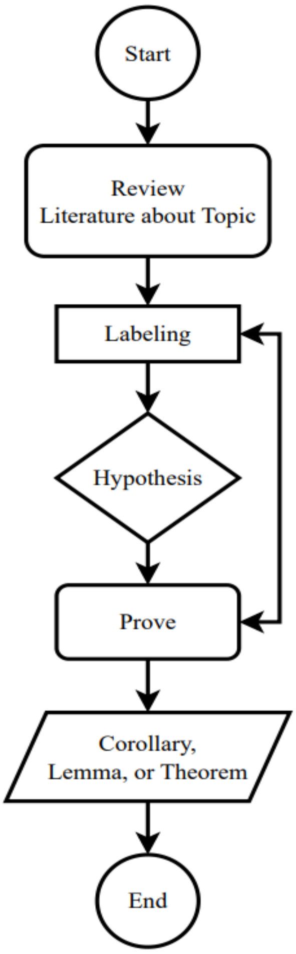

Figure 1. Research Procedure Flowchart

C. RESULT AND DISCUSSION

Let P_n and P_m be two paths of order $n \geq 3$ and $m \geq 3$, respectively. In this paper, we consider corona product $P_n \odot P_m$. Let

$$V(P_n \odot P_m) = \{v_i |\, i = 1,2,\dots,n\} \cup \{v_i^j|\} \cup \{v_i^j|i = 1,2,\dots,n, j = 1,2,\dots,m\},$$

and

$$E(P_n \odot P_m) = \{v_i v_{i+1} \mid i = 1,2,\dots,n-1\} \cup \{v_i v_i^j \mid i = 1,2,\dots,n, j = 1,2,\dots,m-1\} \cup \{v_i^j v_i^{j+1} \mid i = 1,2,\dots,n, j = 1,2,\dots,m-1\}.$$

Hence, we have $|V(P_n \odot P_m)| = mn + n$ and $|E(P_n \odot P_m)| = 2mn - 1$. Figure 2 below illustrates the general form of corona product $P_n \odot P_m$.

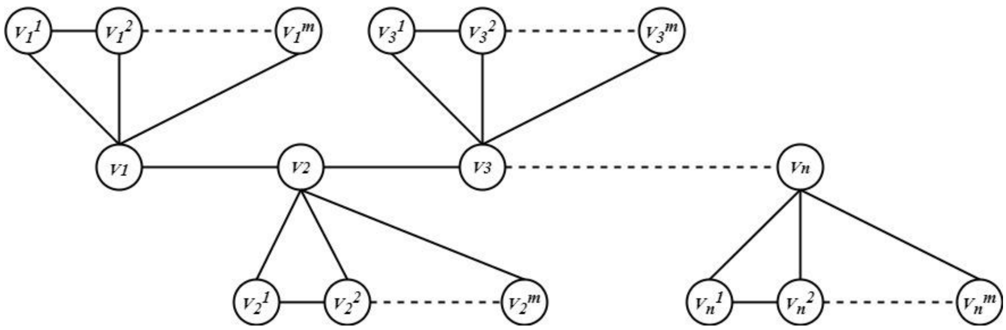

Figure 2. Corona product $P_n \odot P_m$

In this section, we define $\sum X = \sum_{x \in X} x$ and $\{a\} \uplus \{a,b\} = \{a,a,b\}$. This section is divided into three subsections. In Subsection 3.1 and 3.2., we provide the definition of k-balanced multisets and k-anti balanced multisets, respectively, and also some useful lemmas that will be used to prove our main results about the super (a,d)-$P_2 \odot P_m$-antimagic total labeling of $P_n \odot P_m$ in Subsection 3.

1. *k*-Balanced Multisets

A *multiset* is a set that allows the same elements to appear more than once within the set. One technique used to partition a multiset was initially introduced by (Maryati et al., 2013). Let $k \in \mathbb{N}$ and Y be a multiset containing positive integers. We said the multiset Y as *k-balanced* if there exists k subsets of Y, namely $Y_1, Y_2, \dots, Y_k$, so that for each $i = 1,2,\dots,k$, the subset satisfies $|Y_i| = \frac{|Y|}{k}$, $\sum Y_i = \frac{\sum Y}{k} \in \mathbb{N}$, and $\uplus_{i=1}^{k} Y_i = Y$. Such subset Y_i for each $i = 1,2,\dots,k$ is called a *balanced subset* of Y. The following lemma is one of the k-balanced multisets.

Lemma 1. (Maryati et al., 2013) Let $x, y, k \in \mathbb{Z}$ so that $1 \le x \le y$ and $k > 1$. If $X = [x, y]$ and $|X|$ is a multiple of $2k$, then X is k-balanced.

Proof. For each $i = 1,2,\dots,k,$ define $X_i = \left\{\alpha_j^i \mid 1 \le j \le \frac{|X|}{k}\right\}$ where

$$\alpha_j^i = \begin{cases} x - 1 + (j-1)k + i & \text{, for odd } j; \\ x + jk - i & \text{, for even } j. \end{cases}$$

It is not hard to check that for each $i = 1,2,\dots,k$, we have $|X_i| = \frac{|X|}{k}$, $\uplus_{i=1}^{k} X_i = X$, and $\sum X_i = \frac{|X|}{2k}(x+y)$. Therefore, X is k-balanced. ∎

The following two corollaries are an immediate consequence of Lemma 1, which will be used to prove our main results.

Corollary 1. Let $m, n \in \mathbb{Z}$ with $m, n \geq 3$. Let $x = n+1, y = 2mn+n$, and $k = mn$. If $B = [n+1, 2mn+n]$ and $|B|$ is a multiple of $2mn$, then B is mn-balanced.

Proof. Note that $|B| = 2mn$. For each $i = 1,2,\dots,mn$, define $B_i = \left\{b_j^i | 1 \leq j \leq \frac{2mn}{mn}\right\}$ where

$$b_j^i = \begin{cases} n+i & \text{, for } j = 1; \\ n+2mn+1-i, & \text{for } j = 2. \end{cases}$$

it is not hard to check that for each $i = 1,2,\dots,mn$, we have $|B_i| = 2$, $\uplus_{i=1}^{mn} B_i = B$, and $\sum B_i = 2mn + 2n + 1$. Therefore, B is mn-balanced. ■

Corollary 2. Let $m, n \in \mathbb{Z}$ with $m, n \geq 3$ and m is odd. Let $x = 2mn+2n, y = 3mn+n-1$, and $k = n$. If $C = [2mn+2n, 3mn+n-1]$ and $|C|$ is a multiple of $2n$, then C is n-balanced.

Proof. Note that $|C| = (m-1)n$. For each $i = 1,2,\dots,n$, define $C_i = \left\{c_j^i | 1 \leq j \leq m-1\right\}$ where

$$c_j^i = \begin{cases} 2mn+2n-1+(j-1)n+i & \text{, for odd } j; \\ 2mn+2n+jn-i & \text{, for even } j. \end{cases}$$

it is not hard to check that for each $i = 1,2,\dots,n$, we have $|C_i| = m-1$, $\uplus_{i=1}^{n} C_i = C$, and $\sum C_i = \frac{5m^2n-2mn-m-3n+1}{2}$. Therefore, C is n-balanced. ■

2. (k, δ)-anti balanced multisets

Another technique of partitioning a multiset was introduced by Maryati et al. in (Maryati et al., 2013). Let $k \in \mathbb{N}$ and Y be a multiset containing positive integers. The multiset Y is said to be (k, δ)-anti *balanced* if there are k subsets of Y, namely $Y_1, Y_2, \dots, Y_k$, so that for each $i = 1,2,\dots,k$, $|Y_i| = \frac{|Y|}{k}$, $\uplus_{i=1}^{k} Y_i = Y$, and for each $i = 1,2,\dots,k-1$, $\sum Y_{i+1} - \sum Y_i = \delta$. We now provide two lemmas about (k, δ)-anti balanced multisets which will be used to prove our main results.

Lemma 2. Let $m, n \in \mathbb{Z}$ with $m, n \geq 3$. Then $K = [1, n] \uplus [2, n-1] \uplus [2mn+n+1, 2mn+2n-1]$ is $(n-1, 1)$-anti balanced.

Proof. Note that $|K| = 3(n-1)$. For each $i = 1,2,\dots,n-1$, define $K_i = \{a_i, b_i, c_i\}$ where

$$a_i = i;$$
$$b_i = i+1;$$
$$c_i = 2mn+2n-i.$$

It is not hard to check for each $i = 1,2,\dots,n-1$, $|K_i| = 3$, $\uplus_{i=1}^{n-1} K_i = K$, $\sum K_i = 2mn + 2n + 1 + i$, and for each $i = 1,2,\dots,n-2$, $\sum K_{i+1} - \sum K_i = 1$. Therefore, K is $(n-1, 1)$-anti balanced. ■

Lemma 3. Let $m,n \in \mathbb{Z}$ with $m,n \geq 3$ and m is even. Then

$$L = [2mn+2n, 3mn+n-1] \uplus \left\{2mn+2n+2n(j-1)+i \middle| i=1,2,\ \dots,n-2, j=1,2,\dots,\frac{m}{2}\right\} \uplus \left\{2mn+4n+2n(j-1)-i-1 \middle| i=1,2,\ \dots,n-2, j=1,2,\dots,\frac{m}{2}-1\right\}$$

is $(n-1,2)$-anti balanced.

Proof. Note that $|L| = 2(m-1)(n-1)$. For each $i = 1,2,\dots,n-1$, we first define $L_i = \{a_j^i, b_j^i, c_j^i, d_j^i\}$ where

$$\begin{aligned} a_j^i &= 2mn+2n+i-1+n(j-1) &&, \text{for odd } j = 1,2,\dots,m-1;\\ b_j^i &= 2mn+2n+i+n(j-1) &&, \text{for odd } j = 1,2,\dots,m-1;\\ c_j^i &= 2mn+2n+nj-i &&, \text{for even } j = 1,2,\dots,m-1;\\ d_j^i &= 2mn+2n+nj-i-1 &&, \text{for even } j = 1,2,\dots,m-1. \end{aligned}$$

It is not hard to check for each $i = 1,2,\dots,n-1$, $|L_i| = 2(m-1)$, $\uplus_{i=1}^{n-1} L_i = L$, $\sum L_i = 5m^2n - 2mn - m - 4n + 2i + 1$, and for each $i = 1,2,\dots,n-2$, $\sum L_{i+1} - \sum L_i = 2$. Therefore, L is $(n-1, 2)$-anti balanced. ■

3. Super (a,d)-$P_2 \odot P_m$-antimagic total labelling of $P_n \odot P_m$

We now ready to provide our main results. In this subsection, we show that $P_n \odot P_m$ for $m,n \geq 3$ is super (a,d)-$P_2 \odot P_m$-antimagic where $d = 1$ if m is odd and $d = 3$ if m is even.

Theorem 1. Let $m,n \in \mathbb{Z}$ with $m,n \geq 3$ and m is odd. Then the corona product $P_n \odot P_m$ is super $(9m^2n+4mn+m-n+3,1)$-$P_2 \odot P_m$-antimagic.

Proof. Let $\lambda = [1,3mn+n-1]$ denotes a set of labels for all vertices and edges of $P_n \odot P_m$. Partition λ into three subsets, $\lambda = A \uplus B \uplus C$, where $A = [1,n] \uplus [2mn+n+1, 2mn+2n-1]$, $B = [n+1, 2mn+n]$ and $C = [2mn+2n, 3mn+n-1]$. We first define a function $f: V(P_n \odot P_m) \cup E(P_n \odot P_m) \to [1,3mn+n-1]$ as follows.

(i) Define $K = A \uplus [2,n-1]$. According to Lemma 2, set K is $(n-1,1)$-anti balanced where for each $i = 1,2,\dots,n-1$, $K_i = \{a_i, b_i, c_i\}$ and $\sum K_i = 2mn+2n+1+i$. We label vertices v_i and edges $v_i v_{i+1}$ by using the elements of K as follows.

$$f(v_i) = \begin{cases} \min K_i &, \text{for } i = 1,2,\dots,n-1;\\ n &, \text{for } i = n; \end{cases}$$
$$f(v_i v_{i+1}) = \max K_i \ , \text{for } i = 1,2,\dots,n-1.$$

(ii) According to Corollary 1, set B is mn-balanced where for each $i = 1,2,\dots,mn$, $B_i = \{b_j^i | 1 \leq j \leq 2\}$ and $\sum B_i = 2mn+2n+1$. We label vertices $v_i{}^j$ and edges $v_i v_i^j$ by using the elements of B as follows.

$$f(v_i{}^j) = \min B_{j+m(i-1)} \text{ , for } i = 1,2,\dots,n, j = 1,2,\dots,m;$$
$$f(v_i v_i^j) = \max B_{j+m(i-1)} \text{ , for } i = 1,2,\dots,n, j = 1,2,\dots,m.$$

(iii) According to Corollary 2, set C is n-balanced where for each $i = 1, 2, \dots, n$, $C_i = \{c_j^i | 1 \le j \le m-1\}$ and $\sum C_i = \frac{5m^2n-2mn-m-3n+1}{2}$. We label edges $v_i^j v_i^{j+1}$ by using the elements of C as follows.

$$f(v_i^j v_i^{j+1}) = c_j^i \text{ ,} \qquad \text{for } i = 1,2,\dots,n, j = 1,2,\dots,m-1.$$

It is easy to see that the function f defined in (i)-(iii) is a bijective function from $V(P_n \odot P_m) \cup E(P_n \odot P_m)$ to $\{1,2,\dots,3mn+n-1\}$ where $f(V(P_n \odot P_m)) = \{1,2,\dots,mn+n\}$. Now, observe that $P_n \odot P_m$ contains $n-1$ subgraphs that isomorphic to $P_2 \odot P_m$. Then for each $i = 1,2,\dots,n-1$, the $P_2 \odot P_m$ -weight of the i-th subgraph $P_2 \odot P_m$ is

$$\begin{aligned}
\omega(P_2 \odot P_m)_i & \\
&= \sum K_i + 2m\sum B_i + 2\sum C_i \\
&= (2mn + 2n + 1 + i) + 2m\,(2mn + 2n + 1) + 2\left(\frac{5m^2n - 2mn - m - 3n + 1}{2}\right) \\
&= 9m^2n + 4mn + m - n + 2 + i
\end{aligned}$$

Since $\omega(P_2 \odot P_m)_1 = 9m^2n + 4mn + m - n + 3$ and $\omega(P_2 \odot P_m)_{i+1} - \omega(P_2 \odot P_m)_i = 1$ for for each $i = 1,2,\dots,n-2$, we obtain that f is a super $(9m^2n + 4mn + m - n + 3,1)$ - $P_2 \odot P_m$ -antimagic total labeling of $P_n \odot P_m$. Thus, $P_n \odot P_m$ is super $(9m^2n + 4mn + m - n + 3,1)$ - $P_2 \odot P_m$ -antimagic. ■

Figure 3 below illustrates a super $(282, 1)$- $P_2 \odot P_3$ -antimagic total labeling of $P_3 \odot P_3$.

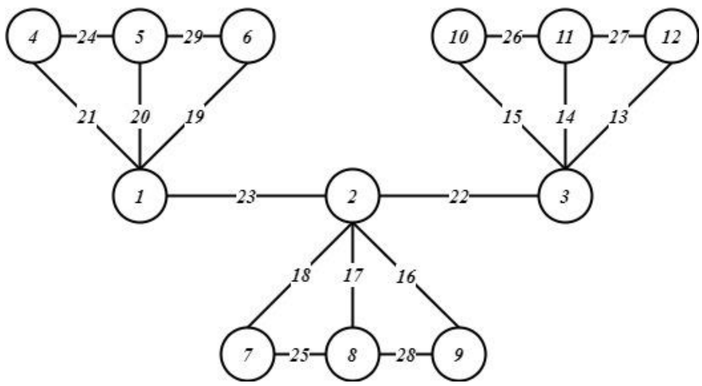

Figure 3. A super $(\mathbf{282, 1})$- $\boldsymbol{P_2 \odot P_3}$ -antimagic total labeling of $\boldsymbol{P_3 \odot P_3}$.

Figure 3 above, is an illustration of Theorem 1 with its simplest example. Graph $P_3 \odot P_3$ proved to be a super $(282, 1)$- $P_2 \odot P_3$ -antimagic total labelling where the values $a = 282$ and $d = 1$.

Theorem 2. Let $m,n \in \mathbb{Z}$ with $m,n \geq 3$ and m is even. Then the corona product $P_n \odot P_m$ is super $(9m^2n + 4mn + m - 2n + 5,3)$-$P_2 \odot P_m$-antimagic.

Proof. Let $\lambda = [1,3mn + n - 1]$ denotes a set of labels for all vertices and edges of $P_n \odot P_m$. In this proof, we also partition λ into three subsets, $\lambda = A \uplus B \uplus C$, where $A = [1,n] \uplus [2mn + n + 1,2mn + 2n - 1]$, $B = [n + 1,2mn + n]$ and $C = [2mn + 2n, 3mn + n - 1]$. We first define a function $f: V(P_n \odot P_m) \cup E(P_n \odot P_m) \to [1,3mn + n - 1]$ as follows.

(i) Define $K = A \uplus [2, n - 1]$. According to Lemma 2, set K is $(n - 1,1)$-anti balanced where for each $i = 1, 2, \dots, n - 1$, $K_i = \{a_i, b_i, c_i\}$ and $\sum K_i = 2mn + 2n + 1 + i$. We label vertices v_i and edges $v_i v_{i+1}$ by using the elements of K as follows.

$$f(v_i) = \begin{cases} \min K_i, & \text{for } i = 1,2, \dots, n - 1; \\ n, & \text{for } i = n; \end{cases}$$
$$f(v_i v_{i+1}) = \max K_i, \text{ for } i = 1, 2, \dots, n - 1.$$

(ii) According to Corollary 1, set B is mn-balanced where for each $i = 1, 2, \dots, mn$, $B_i = \{b_j^i | 1 \leq j \leq 2\}$ and $\sum B_i = 2mn + 2n + 1$. We label vertices $v_i{}^j$ and edges $v_i v_i^j$ by using the elements of B as follows.

$$f(v_i{}^j) = \min B_{j+m(i-1)}, \text{ for } i = 1,2, \dots, n, j = 1,2, \dots, m;$$
$$f(v_i v_i^j) = \max B_{j+m(i-1)}, \text{ for } i = 1,2, \dots, n, j = 1,2, \dots, m.$$

(iii) Define

$$L = [2mn + 2n, 3mn + n - 1] \uplus \left\{2mn + 2n + 2n(j - 1) + i \,\middle|\, i = 1, 2, \dots, n - 2, j = 1, 2, \dots, \frac{m}{2}\right\} \uplus \left\{2mn + 4n + 2n(j - 1) - i - 1 \,\middle|\, i = 1, 2, \dots, n - 2, j = 1, 2, \dots, \frac{m}{2} - 1\right\}.$$

(iv) According to Lemma 3, set L is n is $(n - 1,2)$-anti balanced where for each $= 1, 2, \dots, n - 1$, $L_i = \{a_j^i, b_j^i, c_j^i, d_j^i\}$ and $\sum L_i = 5m^2n - 2mn - m - 4n + 2i + 1$. We label edges $v_i^j v_i^{j+1}$ by using the elements of L as follows.

$$f(v_i^j v_i^{j+1}) = \begin{cases} 2mn + 2n + i - 1 + n(j - 1), & \text{for odd } j = 1,2, \dots, m - 1; \\ 2mn + 2n + nj - i, & \text{for even } j = 1,2, \dots, m - 1. \end{cases}$$

It is easy to see that the function f defined in (i)-(iii) is a bijective function from $V(P_n \odot P_m) \cup E(P_n \odot P_m)$ to $\{1,2, \dots, 3mn + n - 1\}$ where $f(V(P_n \odot P_m)) = \{1,2, \dots, mn + n\}$. Now, observe that $P_n \odot P_m$ contains $n - 1$ subgraphs that isomorphic to $P_2 \odot P_m$. Then for each $i = 1,2, \dots, n - 1$, the $P_2 \odot P_m$-weight of the i-th subgraph $P_2 \odot P_m$ is

$$\begin{aligned}\omega(P_2 \odot P_m)_i &= \sum K_i + 2m\sum B_i + \sum L_i \\ &= (2mn + 2n + 1 + i) + 2m\,(2mn + 2n + 1) + 5m^2n - 2mn - m - 4n + 2i + 1 \\ &= 9m^2n + 4mn + m - 2n + 3i + 2\end{aligned}$$

Since $\omega(P_2 \odot P_m)_1 = 9m^2n + 4mn + m - 2n + 5$ and $\omega(P_2 \odot P_m)_{i+1} - \omega(P_2 \odot P_m)_i = 3$ for each $i = 1,2,\dots,n-2$, we obtain that f is a super $(9m^2n + 4mn + m - 2n + 5,3)$-$P_2 \odot P_m$-antimagic total labeling of $P_n \odot P_m$. Thus, $P_n \odot P_m$ is super $(9m^2n + 4mn + m - 2n + 5,3)$-$P_2 \odot P_m$-antimagic. ■

Figure 4 below illustrates a super $(483, 3)$- $P_2 \odot P_4$ -antimagic total labeling of $P_3 \odot P_4$.

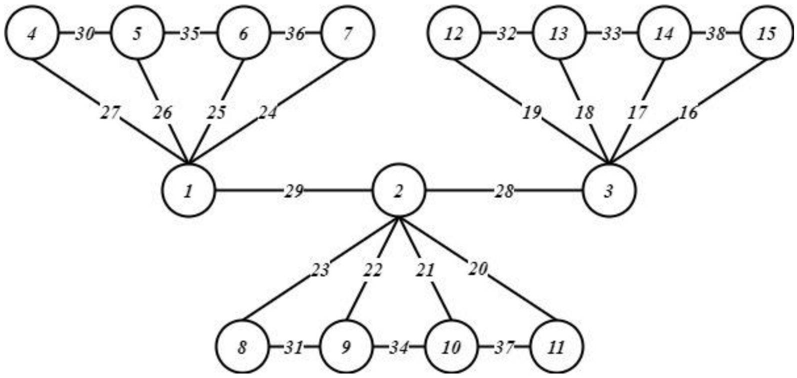

Figure 4. A super $(483, 3)$- $P_2 \odot P_4$ -antimagic total labeling of $P_3 \odot P_4$.

Figure 4 above, is an illustration of Theorem 2 with its simplest example. Graph $P_3 \odot P_4$ proved to be a super $(483, 3)$- $P_2 \odot P_4$-antimagic total labeling where the values $a = 483$ and $d = 3$.

D. CONCLUSION AND SUGGESTIONS

We have shown that the corona product $P_n \odot P_m$ for $m, n \geq 3$ is super $(9m^2n + 4mn + m - n + 3,1)$-$P_2 \odot P_m$ -antimagic if m is odd and super $(9m^2n + 4mn + m - 2n + 5,3)$-$P_2 \odot P_m$ -antimagic if m is even. The values of d obtained in paper are limited, hence it is interesting for the readers to studying the super (a, d)-$P_2 \odot P_m$-antimagic total labeling of $P_n \odot P_m$ for the remaining possible values of d. Further, the readers can also consider to studying the super (a, d)-$P_r \odot P_m$-antimagic total labeling of $P_n \odot P_m$ for other possible values of r. In other words, we have discovered specific patterns within the mathematical structure known as the "corona product". These patterns vary depending on whether we use odd or even numbers. Some values remain unexplored, and readers might be interested in delving further into this topic. Additionally, there is potential to investigate patterns within slightly different structures as well.

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REFERENCES

Agustin, I. H., Prihandini, R. M., & Dafik. (2019). $P_2 \triangleright$ H-super antimagic total labeling of comb product of graphs. *AKCE International Journal of Graphs and Combinatorics*, 16(2), 163–171. https://doi.org/10.1016/j.akcej.2018.01.008

Anjaneyulu, G. S. G. N., Vijayabarathi, A., & Liu, L. (2015). Super edge-magic sequence of maximal outer planer graph and its characteristics. *Cogent Mathematics*, 2(1), 1123340. https://doi.org/10.1080/23311835.2015.1123340

Bača, M., Miller, M., Ryan, J., & Semaničová-Feňovčíková, A. (2019). Magic and Antimagic Graphs (1st ed., Vol. 60). Springer International Publishing. https://doi.org/10.1007/978-3-030-24582-5

Chang, F., Chen, H., & et al. (2019). Shifted-Antimagic Labelings for Graphs. *Graphs and Combinatorics*, 37(2021), 1065–1082, https://doi.org/10.1007/s00373-021-02305-w.

Diestel, R. (2017). Graph Theory (5th ed., Vol. 173). *Springer Berlin Heidelberg*. https://doi.org/10.1007/978-3-662-53622-3

Enomoto, H., Llado, A. S., Nakamigawa, T., & Ringel, G. (1998). Super Edge-Magic Graphs. *SUT Journal of Mathematics*, 34(2). https://doi.org/10.55937/sut/991985322

Gallian, J. A. (2022). A Dynamic Survey of Graph Labeling. *In the electronic journal of combinatorics*, https://www.combinatorics.org/files/Surveys/ds6/ds6v25-2022.pdf.

Gutiérrez, A., & Lladó, A. (2005). Magic coverings. *Ournal of Combinatorial Mathematics and Combinatorial Computing*, 55, 43, https://www.researchgate.net/publication/266257368_Magic_coverings.

Hasni, R., Tarawneh, I., & Husin, M. Na. (2022). A Remark On The Edge Irregularity Strength Of Corona Product Of Two Paths. *Journal of Mathematical Sciences and Informatics*, 2(1), 51–58. https://doi.org/10.46754/jmsi.2022.06.005

Hendy, H., Sugeng, K. A., & Salman, A. N. M. (2018). An H -super magic decompositions of the lexicographic product of graphs. *AIP Conference Proceedings*, 2023. https://doi.org/10.1063/1.5064190

Hendy, H., Sugeng, K. A., Salman, A. N. M., & Ayunda, N. (2018). Another H-super magic decompositions of the lexicographic product of graphs. *Indonesian Journal of Combinatorics*, 2(2), 72. https://doi.org/10.19184/ijc.2018.2.2.2

Hussain, M., Baskoro, E. T., Ali, K., & Hussain, M. (2012). On super antimagic total labeling of Harary graph. https://www.researchgate.net/publication/228458682

Inayah, N., Salman, A. N. M., & Simanjuntak, R. (2009). On (a, d)-H-antimagic coverings of graphs, Paper_Ina_JMCC-libre.pdf (d1wqtxts1xzle7.cloudfront.net).

Inayah, N., Simanjuntak, R., Salman, A. N. M., & Syuhada, K. I. A. (2013). Super (a, d)-H-antimagic total labelings for shackles of a connected graph H. *In Australasian Journal Of Combinatorics* (Vol. 57), 127-138, L175.dvi (uq.edu.au).

Indunil, minusha, & Perera.A.A.I. (2022). Graceful Labeling of Chain Graphs with Pendants. *Nternational Conference On Business Innovations* 2022, 595–598, https://nspace.nsbm.ac.lk/handle/123456789/213.

Kotzig, A., & Rosa, A. (1970). Magic Valuations of Finite Graphs. *Canadian Mathematical Bulletin*, 13(4), 451–461. https://doi.org/10.4153/cmb-1970-084-1

Liu, J. B., Afzal, H. U., & Javaid, M. (2020). Computing Edge Weights of Magic Labeling on Rooted Products of Graphs. *Mathematical Problems in Engineering*, 2020. https://doi.org/10.1155/2020/2160104

Llado A, & Moragas J. (2007). Cycle-Magic Graphs. *DiscreteMath*, 307, 2925–2933, https://doi.org/10.1016/j.disc.2007.03.007.

Martin Bača, Martin Bača, Rinovia Simanjuntak, & Rinovia Simanjuntak. (2006). Super edge-antimagic labelings of the generalized Petersen graph P(n,(n-1)/2). *Utilitas Mathematica*, 70, 119–127, https://www.researchgate.net/publication/262723084_Super_edge-antimagic_labelings_of_the_generalized_Petersen_graph_Pnn-12.

Maryati, T. K., Salman, A. N. M., & Baskoro, E. T. (2013). Supermagic coverings of the disjoint union of graphs and amalgamations. *Discrete Mathematics*, 313(4), 397–405. https://doi.org/10.1016/j.disc.2012.11.005

Ngurah, A. A. G., Salman, A. N. M., & Susilowati, L. (2010). H-supermagic labelings of graphs. *Discrete Mathematics*, 310(8), 1293–1300. https://doi.org/10.1016/j.disc.2009.12.011

Nurvazly, D. E., Chasanah, S. L., & Wiranto, A. R. (2022). Variations of graceful labelling of subgraph of millipede graph. AIP Conference Proceedings, 050018. https://doi.org/10.1063/5.0105447

Palanivelu, C., & Neela, N. (2019). Super (a, d)-Edge Antimagic Total Labeling of Union of Stars. In International Journal of Applied Engineering Research (Vol. 14, Issue 9). 2089-2092 http://www.ripublication.com

Permata Sari, A., Sri Martini, T., & Yugi Kurniawan, V. (2019). Super (a, d)-H-antimagic total labeling of edge corona product on cycle with path graph and cycle with cycle graph. Journal of Physics: Conference Series, 1306(1), 1-6, https://doi.org/10.1088/1742-6596/1306/1/012006

Prihandini, R. M., & Adawiyah, R. (2022). On Super (a,d)-Edge Antimagic Total Labeling of Some Generalized Shackle of Fan Graph. In International Journal of Academic and Applied Research (Vol. 6, issue. 4), 28-32, www.ijeais.org/ijaar.

Prihandini, R. M., & Adawiyah, R. (2023). On Super (3n+5,2)- Edge Antimagic Total Labeling And It's Application To Construct Hill Chiper Algorithm. BAREKENG: Jurnal Ilmu Matematika Dan Terapan, 17(1), 0029–0036. https://doi.org/10.30598/barekengvol17iss1pp0029-0036

Raheem, A., Baig, A. Q., & Javaid, M. (2014). On Super (a, d)-Edge-Antimagic Total Labeling of a Class of Trees. In International J.Math. Combin (Vol. 4), 1-6, https://mathcombin.com/upload/file/20150127/1422320940239094100.pdf#page=52.

Rajeswari, R., Aruna Sakthi, K., & Meenakumari, N. (2021). Product-Set Labeling of Some Zero-Divisor Graphs and its Line Graph. Journal of Physics: Conference Series, 1947(1), 1-7 https://doi.org/10.1088/1742-6596/1947/1/012025

Sandariria, H., Roswitha, M., & Kusmayadi, T. A. (2017). H-Supermagic Labeling on Coronation of Some Classes of Graphs with a Path. Journal of Physics: Conference Series, 855(1), 1-6. https://doi.org/10.1088/1742-6596/855/1/012042

Simanihuruk, M., Kusmayadi, T. A., Swita, B., Romala, M., & Damanik, F. (2021). A Conjecture on Super Edge-Magic Total Labeling of 4-Cycle Books. International Journal of Mathematics and Mathematical Sciences, 2021, 1-8. https://doi.org/10.1155/2021/8483926

Simanjuntak, R., Bertault, F., & Miller, M. (2000). Two new (a, d)-antimagic graph labelings. Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms, 11, 179–189, Two new (a, d)-antimagic graph labelings | Request PDF (researchgate.net).

Lakshmi, S., & Sagayakavitha, D. (2018). Edge-magic total labeling of some graphs. International Journal of Mathematics Trends and Technology (IJMTT), 1, 101–103, https://www.internationaljournalssrg.org/uploads/specialissuepdf/ICRMIT/2018/MTT/ICRMIT-P116.pdf.

Taimur, A., Numan, M., Ali, G., Mumtaz, A., & Semaničová-Feňovčíková, A. (2018). Super (a, d)-H-antimagic labeling of subdivided graphs. Open Mathematics, 16(1), 688–697. https://doi.org/10.1515/math-2018-0062

Ulfatimah, R., Roswitha, M., & Kusmayadi, T. A. (2017). H-Supermagic Labeling on Shrubs Graph and $L_m \odot P_n$. Journal of Physics: Conference Series, 855(1). https://doi.org/10.1088/1742-6596/855/1/012055

Zeen El Deen, M. R. (2019). Edge even graceful labeling of some graphs. Journal of the Egyptian Mathematical Society, 27(1). https://doi.org/10.1186/s42787-019-0025-x